PHYS 3446 – Lecture #7

Wednesday, Sept. 27, 2006
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1. Nature of the Nuclear Force
   • Shape of the Nuclear Potential
   • Yukawa Potential
   • Range of Yukawa Potential

2. Nuclear Models
   • Liquid Drop Model
   • Fermi Gas Model
   • Shell Model
Announcements

• **Workshop**
  – 10am – 5pm, Saturday @CPB303
  – Did all groups purchase what you need?
  – Each group needs to come up with the plans for the day and discuss which group does what in which order
    • Some groups’ activities might interfere with others

• **First term exam**
  – Date and time: 1:00 – 2:30pm, Wednesday, Oct. 4
  – Location: SH105
  – Covers: Appendix A (special relativity) + CH1 – CH3

• **Quiz results**
  – Class Average: 47.4
  – Top score: 71
  – Quizzes account for 10% of the total
Nuclear Potential

- A square well nuclear potential provides the basis of quantum theory with discrete energy levels and corresponding bound state just like in atoms
  - Presence of nuclear quantum states have been confirmed through
    - Scattering experiments
    - Studies of the energies emitted in nuclear radiation

- Studies of mirror nuclei and the scatterings of protons and neutrons demonstrate
  - Without the Coulomb effects, the forces between two neutrons, two protons or a proton and a neutron are the same
    - Nuclear force has nothing to do with electrical charge
    - Protons and neutrons behave the same under the nuclear force
  - Inferred as charge independence of nuclear force.
Nuclear Potential – Iso-spin symmetry

- Strong nuclear force is independent of the electric charge carried by nucleons
  - Concept of strong isotopic-spin symmetry.
    - Proton and neutron are the two different iso-spin state of the same particle called nucleon
  - In other words,
    - If Coulomb effect were turned off, protons and neutrons would be indistinguishable in their nuclear interactions
    - Can you give another case just like this???
  - This is analogous to the indistinguishability of spin up and down states in the absence of a magnetic field!!
- This is called Iso-spin symmetry!!!
Range of the Nuclear Force

• EM force can be understood as a result of a photon exchange
  – Photon propagation is described by the Maxwell’s equation
  – Photons propagate at the speed of light.
  – What does this tell you about the mass of the photon?
    • Massless

• Coulomb potential is $V(r) \propto \frac{1}{r}$

• What does this tell you about the range of the Coulomb force?
  – Long range. Why?
Yukawa Potential

• For massive particle exchanges, the potential takes the form

\[ V(r) \propto \frac{mc}{r} e^{-\frac{\hbar}{r}} \]

– What is the mass, \( m \), in this expression?
  • Mass of the particle exchanged in the interaction
    – The force mediator mass

• This form of potential is called Yukawa Potential
  – Formulated by Hideki Yukawa in 1934

• What does Yukawa potential turn to in the limit \( m \to 0 \)?
  – Coulomb potential
Ranges in Yukawa Potential

- From the form of the Yukawa potential
  \[ V(r) \propto \frac{mc}{\hbar r} e^{-r/\lambda} = \frac{e^{-r/\lambda}}{r} \]

- The range of the interaction is given by some characteristic value of \( r \). What is this?
  - Compton wavelength of the mediator with mass, \( m \): \( \lambda = \frac{\hbar}{mc} \)

- What does this mean?
  - Once the mass of the mediator is known, range can be predicted
  - Once the range is known, the mass can be predicted
Ranges in Yukawa Potential

• Let’s put Yukawa potential to work
• What is the range of the nuclear force?
  – About the same as the typical size of a nucleus
    • \(1.2 \times 10^{-13}\text{cm}\)
  – thus the mediator mass is

\[
mc^2 = \frac{\hbar c}{\lambda} \approx \frac{197\text{MeV} - \text{fm}}{1.2\text{fm}} \approx 164\text{MeV}
\]

• This is close to the mass of a well known \(\pi\) meson (pion)

\[
m_{\pi^+} = m_{\pi^-} = 139.6\text{MeV} / c^2;
m_{\pi^0} = 135\text{MeV} / c^2
\]

• Thus, it was thought that \(\pi\) are the mediators of the nuclear force
Nuclear Models

• Experiments showed very different characteristics of nuclear forces than other forces

• Quantification of nuclear forces and the structure of nucleus were not straightforward
  – Fundamentals of nuclear force were not well understood

• Several phenomenological models (not theories) that describe only limited cases of experimental findings

• Most the models assume central potential, just like Coulomb potential
Nuclear Models: Liquid Droplet Model

- An earliest phenomenological success in describing binding energy of a nucleus
- Nucleus is essentially spherical with radius proportional to $A^{1/3}$.
  - Densities are independent of the number of nucleons
- Led to a model that envisions the nucleus as an incompressible liquid droplet
  - In this model, nucleons are equivalent to molecules
- Quantum properties of individual nucleons are ignored
Nuclear Models: Liquid Droplet Model

- Nucleus is imagined to consist of
  - A stable central core of nucleons where nuclear force is completely saturated
  - A surface layer of nucleons that are not bound tightly
    - This weaker binding at the surface decreases the effective BE per nucleon (B/A)
    - Provides an attraction of the surface nucleons towards the core just as the surface tension to the liquid
Liquid Droplet Model: Binding Energy

• If a constant BE per nucleon is due to the saturation of the nuclear force, the nuclear BE can be written as:

\[ BE = -a_1 A + a_2 A^{2/3} \]

• What do you think each term does?
  – First term: volume energy for uniform saturated binding
  – Second term corrects for weaker surface tension

• This can explain the low BE/nucleon behavior of low A nuclei
  – For low A nuclei, the proportion of the second term is larger.
  – Reflects relatively large number of surface nucleons than the core.
Liquid Droplet Model: Binding Energy

• Small decrease of BE for heavy nuclei can be understood as due to Coulomb repulsion
  – The electrostatic energies of protons have destabilizing effect
• Reflecting this effect, the empirical formula for BE takes the correction term
  \[ BE = -a_1 A + a_2 A^{2/3} + a_3 Z^2 A^{-1/3} \]
• All terms of this formula have classical origin.
• This formula does not explain
  – Lighter nuclei with the equal number of protons and neutrons are stable or have a stronger binding (larger –BE)
  – Natural abundance of stable even-even nuclei or paucity of odd-odd nuclei
• These could mainly arise from quantum effect of spins.
Liquid Droplet Model: Binding Energy

- Additional corrections to compensate the deficiency, give corrections to the empirical formula (again…)

\[ BE = -a_1 A + a_2 A^{2/3} + a_3 Z^2 A^{-1/3} + a_4 \frac{(N-Z)^2}{A} \pm a_5 A^{-3/4} \]

- All parameters are assumed to be positive
- The forth term reflects N=Z stability
- The last term
  - Positive sign is chosen for odd-odd nuclei, reflecting instability
  - Negative sign is chosen for even-even nuclei
  - For odd-A nuclei, \( a_5 \) is chosen to be 0.
Liquid Droplet Model: Binding Energy

- The parameters are determined by fitting experimentally observed BE for a wide range of nuclei:

\[
\begin{align*}
    a_1 &\approx 15.6\text{MeV} & a_2 &\approx 16.8\text{MeV} & a_3 &\approx 0.72\text{MeV} \\
    a_4 &\approx 23.3\text{MeV} & a_5 &\approx 34\text{MeV}; \\
\end{align*}
\]

- Now we can write an empirical formula for masses of nuclei

\[
M(A, Z) = (A - Z)m_n + Zm_p + \frac{BE}{c^2} = (A - Z)m_n + Zm_p
\]

\[
- \frac{a_1}{c^2}A + \frac{a_2}{c^2}A^{2/3} + \frac{a_3}{c^2}Z^2A^{-1/3} + \frac{a_4}{c^2}\left(\frac{N-Z}{A}\right)^2 + \frac{a_5}{c^2}A^{-3/4}
\]

- This is Bethe-Weizsacker semi-empirical mass formula
  - Used to predict stability and masses of unknown nuclei of arbitrary A and Z
Nuclear Models: Fermi Gas Model

• An early attempt to incorporate quantum effects
• Assumes nucleus as a gas of free protons and neutrons confined to the nuclear volume
  – The nucleons occupy quantized (discrete) energy levels
  – Nucleons are moving inside a spherically symmetric well with the range determined by the radius of the nucleus
  – Depth of the well is adjusted to obtain correct binding energy
• Protons carry electric charge \( \Rightarrow \) Senses slightly different potential than neutrons
Nuclear Models: Fermi Gas Model

• Nucleons are Fermions (spin $\frac{1}{2}$ particles) so
  – Obey Pauli exclusion principle
  – Any given energy level can be occupied by at most two identical nucleons – opposite spin projections

• For a greater stability, the energy levels fill up from the bottom to the Fermi level
  – Fermi level: Highest, fully occupied energy level ($E_F$)

• Binding energies are given as follows:
  – BE of the last nucleon= $E_F$ since no Fermions above $E_F$
  – In other words, the level occupied by Fermion reflects the BE of the last nucleon
Nuclear Models: Fermi Gas Model

- Experimental observations show BE is charge independent
- If the well depth is the same for p and n, BE for the last nucleon would be charge dependent for heavy nuclei (Why?)
  - Since there are more neutrons than protons, neutrons sit higher $E_F$
Same Depth Potential Wells

Neutron Well

Proton Well

Nuclear $\beta$-decay

$n \rightarrow e^- + \bar{\nu}_e + p$

$E_F^n \leftarrow$

$E_F^p \leftarrow$
Nuclear Models: Fermi Gas Model

• Experimental observations show BE is charge independent

• If the well depth is the same for p and n, BE for the last nucleon would be charge dependent for heavy nuclei (Why?)
  – Since there are more neutrons than protons, neutrons sit higher $E_F$
  – But experiments observed otherwise

• $E_F$ must be the same for protons and neutrons. How do we make this happen?
  – Make protons move to a shallower potential well

• What happens if this weren’t the case?
  – Nucleus is unstable.
  – All neutrons at higher energy levels would undergo a $\beta$-decay and transition to lower proton levels
Fermi Gas Model: $E_F$ vs $n_F$

- Fermi momentum: $E_F = p_F^2 / 2m \Rightarrow p_F = \sqrt{2mE_F}$
- Volume for momentum space up to Fermi level $V_{p_F} = \frac{4\pi}{3} p_F^3$
- Total volume for the states (kinematic phase space)
  - Proportional to the total number of quantum states in the system
  $$V_{TOT} = V \cdot V_{p_F} = \frac{4\pi}{3} r_0^3 A \cdot \frac{4\pi}{3} p_F^3 = \left( \frac{4\pi}{3} \right)^2 A \left( r_0 p_F \right)^3$$
- Using Heisenberg’s uncertainty principle: $\Delta x \Delta p \geq \hbar / 2$
- The minimum volume associated with a physical system becomes $V_{state} = \left( 2\pi\hbar \right)^3$
- The $n_F$ that can fill up to $E_F$ is
  $$n_F = \frac{V_{TOT}}{(2\pi\hbar)^3} = \frac{2}{(2\pi\hbar)^3} \left( \frac{4\pi}{3} \right)^2 A \left( r_0 p_F \right)^3 = \frac{4}{9\pi} A \left( \frac{r_0 p_F}{\hbar} \right)^3$$

Why?
Fermi Gas Model: $E_F$ vs $n_F$

- Let’s consider a nucleus with $N=Z=A/2$ and assume that all states up to Fermi level are filled

\[ N = Z = \frac{A}{2} = \frac{4}{9\pi} A \left( \frac{r_0 p_F}{\hbar} \right)^3 \quad \text{or} \quad p_F = \frac{\hbar}{r_0} \left( \frac{9\pi}{8} \right)^{1/3} \]

- What do you see about $p_F$ above?
  - Fermi momentum is constant, independent of the number of nucleons
  
  \[ E_F = \frac{p_F^2}{2m} = \frac{1}{2m} \left( \frac{\hbar}{r_0} \right)^2 \left( \frac{9\pi}{8} \right)^{2/3} \approx \frac{2.32}{2 \cdot 940} \left( \frac{197\text{MeV} - \text{fm}}{1.2\text{fm}} \right) \approx 33\text{MeV} \]

- Using the average BE of -8MeV, the depth of potential well ($V_0$) is ~40MeV
  - Consistent with other findings

- This model is a natural way of accounting for a$^4$ term in Bethe-Weizsacker mass formula