1. Symmetries
   • Why do we care about the symmetry?
   • Symmetry in Lagrangian formalism
   • Symmetries in quantum mechanical system
   • Isospin symmetry
   • Local gauge symmetry
Announcements

• No lecture next Monday, Nov. 13 but SH105 is reserved for your discussions concerning the projects
• Quiz next Wednesday, Nov. 15 in class
• 2nd term exam
  – Wednesday, Nov. 22
  – Covers: Ch 4 – whatever we finish on Nov. 20
• Reading assignments
  – 10.3 and 10.4
Quantum Numbers

- We’ve learned about various newly introduced quantum numbers as a patch work to explain experimental observations
  - Lepton numbers
  - Baryon numbers
  - Isospin
  - Strangeness
- Some of these numbers are conserved in certain situation but not in others
  - Very frustrating indeed….
- These are due to lack of quantitative description by an elegant theory
Why symmetry?

• Some quantum numbers are conserved in strong interactions but not in electromagnetic and weak interactions
  – Inherent reflection of underlying forces

• Understanding conservation or violation of quantum numbers in certain situations is important for formulating quantitative theoretical framework
Why symmetry?

• When is a quantum number conserved?
  – When there is an underlying symmetry in the system
  – When the quantum number is not affected (or is conserved) by (under) the changes in the physical system

• **Noether’s theorem**: If there is a conserved quantity associated with a physical system, there exists an underlying invariance or symmetry principle responsible for this conservation.

• Symmetries provide critical restrictions in formulating theories
Symmetries in Lagrangian Formalism

- Symmetry of a system is defined by any set of transformations that keep the equation of motion unchanged or invariant.

- Equations of motion can be obtained through:
  - Lagrangian formalism: $\mathcal{L} = T - V$ where the Equation of motion is what minimizes the Lagrangian $\mathcal{L}$ under changes of coordinates.
  - Hamiltonian formalism: $\mathcal{H} = T + V$ with the equation of motion that minimizes the Hamiltonian under changes of coordinates.

- Both these formalisms can be used to discuss symmetries in non-relativistic (or classical cases) or relativistic cases and quantum mechanical systems.
Symmetries in Lagrangian Formalism?

- Consider an isolated non-relativistic physical system of two particles interacting through a potential that only depends on the relative distance between them
  - EM and gravitational force
- The total kinetic and potential energies of the system are: \( T = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 \) and \( V = V(\mathbf{r}_1 - \mathbf{r}_2) \)
- The equations of motion are then
  \[
  m_1 \ddot{r}_1 = -\nabla_1 V(\mathbf{r}_1 - \mathbf{r}_2) = -\frac{\partial}{\partial \mathbf{r}_1} V(\mathbf{r}_1 - \mathbf{r}_2)
  \]
  \[
  m_2 \ddot{r}_2 = -\nabla_2 V(\mathbf{r}_1 - \mathbf{r}_2) = -\frac{\partial}{\partial \mathbf{r}_2} V(\mathbf{r}_1 - \mathbf{r}_2)
  \]
  where
  \[
  \frac{\partial}{\partial \mathbf{r}_i} V(\mathbf{r}_1 - \mathbf{r}_2) = \hat{x} \frac{\partial}{\partial x_i} V + \hat{y} \frac{\partial}{\partial y_i} V + \hat{z} \frac{\partial}{\partial x_i} V
  \]
Symmetries in Lagrangian Formalism

• If we perform a linear translation of the origin of coordinate system by a constant vector $-\vec{a}$
  – The position vectors of the two particles become
    \[ \vec{r}_1 \rightarrow \vec{r}_1 - \vec{a} \quad \vec{r}_2 \rightarrow \vec{r}_2 - \vec{a} \]
  – But the equations of motion do not change since $-\vec{a}$ is a constant vector
  – This is due to the invariance of the potential $V$ under the translation
    \[ V' = V(\vec{r}'_1 - \vec{r}'_2) = V(\vec{r}_1 - \vec{a} - \vec{r}_2 + \vec{a}) = V(\vec{r}_1 - \vec{r}_2) \]
Symmetries in Lagrangian Formalism?

• This means that the translation of the coordinate system for an isolated two particle system defines a symmetry of the system (remember Noether’s theorem?)

• This particular physical system is invariant under spatial translation

• What is the consequence of this invariance?
  – From the form of the potential, the total force is

  \[
  \vec{F}_{tot} = \vec{F}_1 + \vec{F}_2 = -\nabla_1 V (\vec{r}_1 - \vec{r}_2) - \nabla_2 V (\vec{r}_1 - \vec{r}_2) = 0
  \]

  – Since

  \[
  \frac{\partial V}{\partial \vec{r}_1} = -\frac{\partial V}{\partial \vec{r}_2}
  \]
Symmetries in Lagrangian Formalism?

• What does this mean?
  – Total momentum of the system is invariant under spatial translation

\[ \vec{F}_{tot} = \frac{d\vec{P}_{tot}}{dt} = 0 \]

• In other words, the translational symmetry results in linear momentum conservation

• This holds for multi-particle system as well
Symmetries in Lagrangian Formalism

- For multi-particle system, using Lagrangian $\mathcal{L}=T-V$, the equations of motion can be generalized

$$\frac{d}{dt} \frac{\partial L_i}{\partial \dot{\mathbf{r}}_i} - \frac{\partial L_i}{\partial \mathbf{r}_i} = 0$$

- By construction,

$$\frac{\partial L_i}{\partial \dot{\mathbf{r}}_i} = \frac{\partial T_i}{\partial \dot{\mathbf{r}}_i} = \frac{\partial}{\partial \dot{\mathbf{r}}_i} \left( \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right) = m_i \dot{\mathbf{r}}_i = \mathbf{p}_i$$

- As previously discussed, for the system with a potential that depends on the relative distance between particles, the Lagrangian is independent of particulars of the individual coordinate $\frac{\partial L_i}{\partial r_m} = 0$ and thus $\frac{d\mathbf{p}_i}{dt} = \frac{\partial L_i}{\partial \mathbf{r}_i} = 0$
Symmetries in Lagrangian Formalism

• Momentum $p_i$ can expanded to other kind of momenta for the given spatial translation
  – Rotational translation: Angular momentum
  – Time translation: Energy
  – Rotation in isospin space: Isospin

• The equation $\frac{dp_i}{dt} = \frac{\partial L_i}{\partial r_i} = 0$ says that if the Lagrangian of a physical system does not depend on specifics of a given coordinate, the conjugate momentum is conserved.

• One can turn this around and state that if a Lagrangian does not depend on some particular coordinate, it must be invariant under translations of this coordinate.
Translational Symmetries & Conserved Quantities

- The translational symmetries of a physical system give invariance in the corresponding physical quantities
  - Symmetry under linear translation
    - Linear momentum conservation
  - Symmetry under spatial rotation
    - Angular momentum conservation
  - Symmetry under time translation
    - Energy conservation
  - Symmetry under isospin space rotation
    - Isospin conservation
Symmetries in Quantum Mechanics

• In quantum mechanics, an observable physical quantity corresponds to the expectation value of the Hermitian operator in a given quantum state
  – The expectation value is given as a product of wave function vectors about the physical quantity (operator)
    \[ \langle Q \rangle = \langle \psi | Q | \psi \rangle \]
  – Wave function \( |\psi\rangle \) is the probability distribution function of a quantum state at any given space-time coordinates
  – The observable is invariant or conserved if the operator \( Q \) commutes with Hamiltonian
Types of Symmetry

• All symmetry transformations of the theory can be categorized in
  – Continuous symmetry: Symmetry under continuous transformation
    • Spatial translation
    • Time translation
    • Rotation
  – Discrete symmetry: Symmetry under discrete transformation
    • Transformation in discrete quantum mechanical system
Isospin

• If there is isospin symmetry, proton (isospin up, $I_3 = \frac{1}{2}$) and neutron (isospin down, $I_3 = -\frac{1}{2}$) are indistinguishable

• Let’s define new neutron and proton states as some linear combination of the proton, $|p\rangle$, and neutron, $|n\rangle$, wave functions

• Then the finite rotation of the vectors in isospin space by an arbitrary angle $\theta/2$ about an isospin axis leads to a new set of transformed vectors

$$|p'\rangle = \cos \frac{\theta}{2} |p\rangle - \sin \frac{\theta}{2} |n\rangle$$

$$|n'\rangle = \sin \frac{\theta}{2} |p\rangle + \cos \frac{\theta}{2} |n\rangle$$
Isospin

• What does the isospin invariance mean to nucleon-nucleon interaction?

• Two nucleon quantum states can be written in the following four combinations of quantum states

  – Proton on proton ($I_3=+1$)  
    $$ |\psi_1\rangle = |pp\rangle $$
  
  – Neutron on neutron ($I_3=-1$)  
    $$ |\psi_2\rangle = |nn\rangle $$
  
  – Proton on neutron or neutron on proton for both symmetric or anti-symmetric ($I_3=0$)
    
    $$ |\psi_3\rangle = \frac{1}{\sqrt{2}} (|pn\rangle + |np\rangle) $$
    $$ |\psi_4\rangle = \frac{1}{\sqrt{2}} (|pn\rangle - |np\rangle) $$
Impact of Isospin Transformation

- For $I_3=+1$ wave function with isospin transformation:

$$|\psi_1\rangle = \left( \cos\frac{\theta}{2} p - \sin\frac{\theta}{2} n \right) \left( \cos\frac{\theta}{2} p - \sin\frac{\theta}{2} n \right) =$$

$$= \cos^2\frac{\theta}{2} |pp\rangle - \cos\frac{\theta}{2} \sin\frac{\theta}{2} (|pn\rangle + |np\rangle) + \sin^2 \frac{\theta}{2} |nn\rangle$$

$$= \cos^2\frac{\theta}{2} |\psi_1\rangle - \sqrt{2} \cos\frac{\theta}{2} \sin\frac{\theta}{2} |\psi_3\rangle + \sin^2 \frac{\theta}{2} |\psi_2\rangle$$

Can you do the same for the other two wave functions of $I=1$?
Isospin Trasnformation

• For $I_3=0$ anti-symmetric wave function

\[
|\psi_4\rangle = \frac{1}{\sqrt{2}} \left\{ \left( \cos \frac{\theta}{2} n - \sin \frac{\theta}{2} p \right) \left( \sin \frac{\theta}{2} p + \cos \frac{\theta}{2} n \right) \right\} 
\]

\[
= \frac{1}{\sqrt{2}} \left( \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} \right) (|pn\rangle - |np\rangle) = |\psi_4\rangle
\]

– This state is totally insensitive to isospin rotation\(\rightarrow\) singlet combination of isospins (total isospin 0 state)
Isospin Transformation

- The other three states corresponds to three possible projection state of the total isospin =1 state (triplet state)
  - If there is an isospin symmetry in strong interaction all these three substates are equivalent and indistinguishable

- Based on this, we learn that any two nucleon system can be in an independent singlet or triplet state
  - Singlet state is anti-symmetric under n-p exchange
  - Triplet state is symmetric under n-p exchange