PHYS 1443 – Section 002

Lecture #19

Monday, Nov. 12, 2007

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• Rolling Motion of a Rigid Body
• Torque
• Vector Product
• Moment of Inertia
  – Parallel Axis Theorem

Today’s homework is HW #12, due 9pm, Monday, Nov. 19!!
Announcements

• **Two other problems with homework system**
  – In the middle of last week, the answers you have submitted got reset
    • Most of you were ok since you haven’t submitted HW 11 yet but some of you who submitted earlier need to double check and might have to re-enter your answers
  – On Nov. 7 and over the weekend, another ghost submission problem occurred
    • Will give you full credit as long as you got the answers right
      – Please be honest with the multiple choice problems
    • Please be sure to send me the problem numbers you had ghost submission problems

• **Quiz graded but will be released Wednesday**

• **Today is the evaluation day**
Rolling Motion of a Rigid Body

What is a rolling motion?
A more generalized case of a motion where the rotational axis moves together with an object

A rotational motion about a moving axis

To simplify the discussion, let's make a few assumptions

1. Limit our discussion on very symmetric objects, such as cylinders, spheres, etc
2. The object rolls on a flat surface

Let's consider a cylinder rolling on a flat surface, without slipping.

Under what condition does this “Pure Rolling” happen?

The total linear distance the CM of the cylinder moved is

\[ s = R\theta \]

Thus the linear speed of the CM is

\[ v_{CM} = \frac{ds}{dt} = R \frac{d\theta}{dt} = R\omega \]

The condition for a “Pure Rolling motion”
More Rolling Motion of a Rigid Body

As we learned in rotational motion, all points in a rigid body moves at the same angular speed but at different linear speeds. CM is moving at the same speed at all times.

At any given time, the point that comes to P has 0 linear speed while the point at P' has twice the speed of CM.

Why??

A rolling motion can be interpreted as the sum of Translation and Rotation.

The magnitude of the linear acceleration of the CM is

\[ a_{CM} = \frac{dv_{CM}}{dt} = R \frac{d\omega}{dt} = R\alpha \]
Torque

Torque is the tendency of a force to rotate an object about an axis. Torque, \( \tau \), is a vector quantity.

Consider an object pivoting about the point \( P \) by the force \( \mathbf{F} \) being exerted at a distance \( r \) from \( P \).

The line that extends out of the tail of the force vector is called the line of action.

The perpendicular distance from the pivoting point \( P \) to the line of action is called the moment arm.

Magnitude of torque is defined as the product of the force exerted on the object to rotate it and the moment arm.

When there are more than one force being exerted on certain points of the object, one can sum up the torque generated by each force vectorially. The convention for sign of the torque is **positive** if rotation is in counter-clockwise and **negative if clockwise**.

\[
\tau \equiv rF\sin\phi = Fd
\]

\[
\sum \tau = \tau_1 + \tau_2 = F_1d_1 - F_2d_2
\]
Example for Torque

A one piece cylinder is shaped as in the figure with core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the picture. A rope wrapped around the drum whose radius is \( R_1 \) exerts force \( F_1 \) to the right on the cylinder, and another force exerts \( F_2 \) on the core whose radius is \( R_2 \) downward on the cylinder. A) What is the net torque acting on the cylinder about the rotation axis?

The torque due to \( F_1 \) \( \tau_1 = -R_1 F_1 \) and due to \( F_2 \) \( \tau_2 = R_2 F_2 \)

So the total torque acting on the system by the forces is \( \sum \tau = \tau_1 + \tau_2 = -R_1 F_1 + R_2 F_2 \)

Suppose \( F_1 = 5.0 \) N, \( R_1 = 1.0 \) m, \( F_2 = 15.0 \) N, and \( R_2 = 0.50 \) m. What is the net torque about the rotation axis and which way does the cylinder rotate from the rest?

Using the above result

\[
\sum \tau = -R_1 F_1 + R_2 F_2 = -5.0 \times 1.0 + 15.0 \times 0.50 = 2.5 \text{ N} \cdot \text{m}
\]

The cylinder rotates in counter-clockwise.
Torque and Vector Product

Let's consider a disk fixed onto the origin \( O \) and the force \( F \) exerts on the point \( p \). What happens?

- The disk will start rotating counter clockwise about the Z axis.
- The magnitude of torque given to the disk by the force \( F \) is
  \[
  \tau = Fr \sin \theta
  \]

But torque is a vector quantity, what is the direction?

How is torque expressed mathematically?

- The direction of the torque follows the right-hand rule!!

What is the direction?

The above operation is called the Vector product or Cross product

\[
\vec{C} \equiv \vec{A} \times \vec{B}
\]

\[
|\vec{C}| = |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta
\]

What is another vector operation we’ve learned?

- Scalar product
  \[
  C \equiv \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta
  \]
  Result? A scalar

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Properties of Vector Product

Vector Product is Non-commutative
What does this mean?

If the order of operation changes the result changes

Following the right-hand rule, the direction changes

Vector Product of two parallel vectors is 0.

\[ |\vec{C}| = |\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin \theta = |\vec{A}||\vec{B}| \sin 0 = 0 \]

Thus,

\[ \vec{A} \times \vec{A} = 0 \]

If two vectors are perpendicular to each other

\[ |\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin \theta = |\vec{A}||\vec{B}| \sin 90^\circ = |\vec{A}||\vec{B}| = AB \]

Vector product follows distribution law

\[ \vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \]

The derivative of a Vector product with respect to a scalar variable is

\[ \frac{d}{dt} (\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt} \]
More Properties of Vector Product

The relationship between unit vectors, $\hat{i}$, $\hat{j}$ and $\hat{k}$

\[ \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \]
\[ \hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k} \]
\[ \hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i} \]
\[ \hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j} \]

Vector product of two vectors can be expressed in the following determinant form

\[
\mathbf{A} \times \mathbf{B} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
A_x & A_y & A_z \\
B_x & B_y & B_z \\
\end{vmatrix} = \hat{i} \begin{vmatrix}
A_y & A_z \\
B_y & B_z \\
\end{vmatrix} - \hat{j} \begin{vmatrix}
A_x & A_z \\
B_x & B_z \\
\end{vmatrix} + \hat{k} \begin{vmatrix}
A_x & A_y \\
B_x & B_y \\
\end{vmatrix}
\]

\[= (A_y B_z - A_z B_y) \hat{i} - (A_x B_z - A_z B_x) \hat{j} + (A_x B_y - A_y B_x) \hat{k}\]
Moment of Inertia

Rotational Inertia:
Measure of resistance of an object to changes in its rotational motion. Equivalent to mass in linear motion.

For a group of particles
\[ I = \sum_i m_i r_i^2 \]

For a rigid body
\[ I = \int r^2 \, dm \]

What are the dimension and unit of Moment of Inertia?
\[ [ML^2] \text{ kg} \cdot m^2 \]

Determining Moment of Inertia is extremely important for computing equilibrium of a rigid body, such as a building.

Dependent on the axis of rotation!!!
Example for Moment of Inertia

In a system of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at angular speed $\omega$.

Since the rotation is about y axis, the moment of inertia about y axis, $I_y$, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m \cdot 0^2 + m \cdot 0^2 = 2Ml^2$$

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2} I \omega^2 = \frac{1}{2} (2Ml^2) \omega^2 = Ml^2 \omega^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2(Ml^2 + mb^2) \quad K_R = \frac{1}{2} I \omega^2 = \frac{1}{2} (2Ml^2 + 2mb^2) \omega^2 = (Ml^2 + mb^2) \omega^2$$