PHYS 1443 – Section 002
Lecture #21

Monday, Nov. 19, 2007
Dr. Jae Yu

- Work, Power and Energy in Rotation
- Angular Momentum
- Conservation of Angular Momentum
- Similarity between Linear and Angular Quantities

Today's homework is HW #13, due 9pm, Monday, Nov. 26!!

Happy Thanksgiving!!!

Reminder for the special project

- Prove that $\int x' dm = 0$ and $\int y' dm = 0$ if x' and y' are the distance from the center of mass
- Due by the start of the class Monday, Nov. 26
- Total score is 10 points.

Physics Department
The University of Texas at Arlington

COLLOQUIUM

Physics-Based Computer Modeling and Simulation or What DaVinci and other Polymaths* would be Doing if They Were Alive Today

Dr. Russell Torti
Lockheed Martin Aeronautics Company

4:00 pm Wednesday, November 21, 2007
Room 101 SH

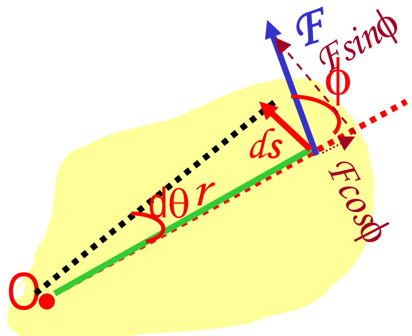
Abstract

Abstract: The application of computer modeling and simulation finds it roots in von Neumann's Monte Carlo hydrodynamics simulations which were conducted for the Manhattan Project during the 1940's. Continuing evolutionary development of von Neumann's computer architecture has yielded Multiprocessor and Graphical Processor capabilities that are opening up a new world of disruptive technologies and complex, real-time physics-based simulations. A multiplicity of scientific and artistic physical modeling is now possible on desktop machines which would make even the most enthusiastic Polymath excited. An overview of scientific, industrial, and artistic simulation technologies will be described, developed, and discussed. Current practical examples and applications will be presented along with the prospects for exciting future development.

* Polymath – a person who excels in a wide variety of subjects or fields

Refreshments will be served in the Physics Library at 3:30 pm

Work, Power, and Energy in Rotation



Let's consider the motion of a rigid body with a single external force $\mathcal{F}$ exerting on the point P, moving the object by ds.

The work done by the force $\mathcal{F}$ as the object rotates through the infinitesimal distance $ds=rd\theta$ is

$$dW = \vec{F}\cdot d\vec{s} = \left(F\cos(\pi/2-\phi)\right)rd\theta = \left(F\sin\phi\right)rd\theta$$

What is $\mathcal{F}\sin\phi$? The tangential component of the force $\mathcal{F}$.

What is the work done by radial component $\mathcal{F}\cos\phi$? Zero, because it is perpendicular to the displacement.

Since the magnitude of torque is $r\mathcal{F}\sin\phi$,

$$dW = \left(rF\sin\phi\right)d\theta = \tau d\theta$$

The rate of work, or power, becomes

$$P = \frac{dW}{dt} = \frac{\tau d\theta}{dt} = \tau\omega$$

How was the power defined in linear motion?

The rotational work done by an external force equals the change in rotational Kinetic energy.

$$\sum\tau = I\alpha = I\left(\frac{d\omega}{dt}\right) = I\left(\frac{d\omega}{d\theta}\right)\left(\frac{d\theta}{dt}\right) = I\omega\left(\frac{d\omega}{d\theta}\right)$$

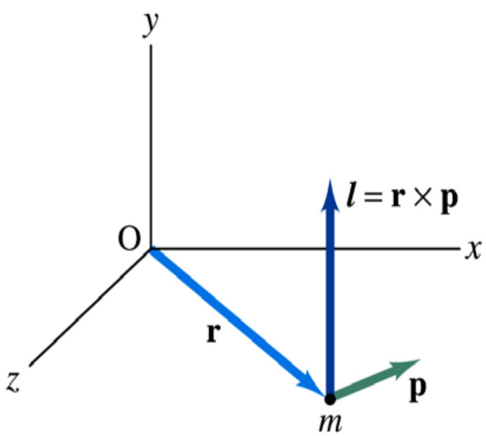
$$\sum\tau d\theta = I\omega d\omega \Rightarrow dW = \sum\tau d\theta = I\omega d\omega$$

The work put in by the external force then

$$W = \int_{\theta_i}^{\theta_f}\sum\tau d\theta = \int_{\omega_i}^{\omega_f} I\omega d\omega = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

Angular Momentum of a Particle

If you grab onto a pole while running, your body will rotate about the pole, gaining angular momentum. We've used the linear momentum to solve physical problems with linear motions, the angular momentum will do the same for rotational motions.



Let's consider a point-like object (particle) with mass m located at the vector location r and moving with linear velocity v

The angular momentum L of this particle relative to the origin O is

$$\boxed{\vec{L} \equiv \vec{r} \times \vec{p}}$$

What is the unit and dimension of angular momentum? $kg \cdot m^2/s$ $[ML^2T^{-1}]$

Note that L depends on origin O. Why? Because r changes

What else do you learn? The direction of L is +z

Since p is mv, the magnitude of L becomes $L = mvr\sin\phi$

What do you learn from this?

If the direction of linear velocity points to the origin of rotation, the particle does not have any angular momentum.

If the linear velocity is perpendicular to position vector, the particle moves exactly the same way as a point on a rim.

Angular Momentum and Torque

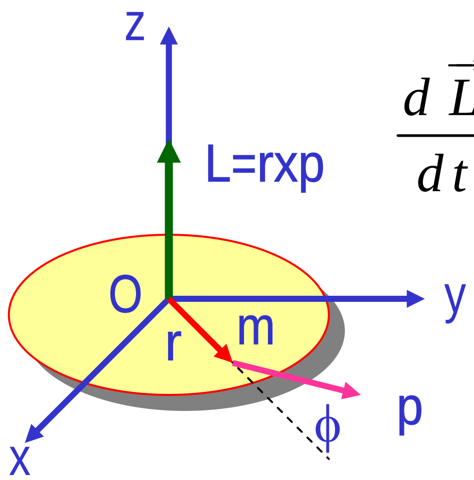
Can you remember how net force exerting on a particle and the change of its linear momentum are related?

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

Total external forces exerting on a particle is the same as the change of its linear momentum.

The same analogy works in rotational motion between torque and angular momentum.

Net torque acting on the particle is $\sum \vec{\tau} = \vec{r} \times \sum \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$



$$\frac{d\vec{L}}{dt} = \frac{d\left(\vec{r} \times \vec{p}\right)}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = 0 + \vec{r} \times \frac{d\vec{p}}{dt} = \sum \vec{\tau}$$

Why does this work?

Because v is parallel to the linear momentum

Thus the torque-angular momentum relationship

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt}$$

The net torque acting on a particle is the same as the time rate change of its angular momentum

Angular Momentum of a System of Particles

The total angular momentum of a system of particles about some point is the vector sum of the angular momenta of the individual particles

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \ldots\ldots + \vec{L}_n = \sum \vec{L}_i$$

Since the individual angular momentum can change, the total angular momentum of the system can change.

Both internal and external forces can provide torque to individual particles. However, the internal forces do not generate net torque due to Newton's third law.

Let's consider a two particle system where the two exert forces on each other.

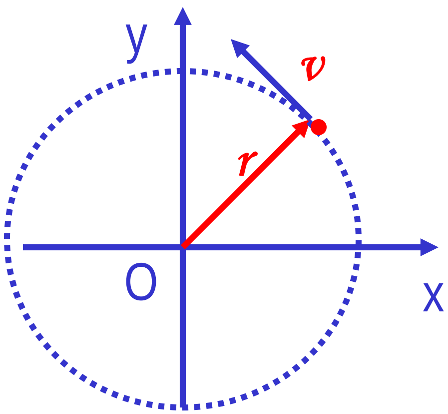
Since these forces are the action and reaction forces with directions lie on the line connecting the two particles, the vector sum of the torque from these two becomes 0.

Thus the time rate change of the angular momentum of a system of particles is equal to <u>only the net external torque</u> acting on the system

$$\boxed{\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}}$$

Example for Angular Momentum

A particle of mass m is moving on the xy plane in a circular path of radius r and linear velocity v about the origin O. Find the magnitude and the direction of the angular momentum with respect to O.



Using the definition of angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m\vec{r} \times \vec{v}$$

Since both the vectors, r and v, are on x-y plane and using right-hand rule, the direction of the angular momentum vector is +z (coming out of the screen)

The magnitude of the angular momentum is $\left|\vec{L}\right| = \left|m\vec{r} \times \vec{v}\right| = mrv\sin\phi = mrv\sin 90^\circ = mrv$

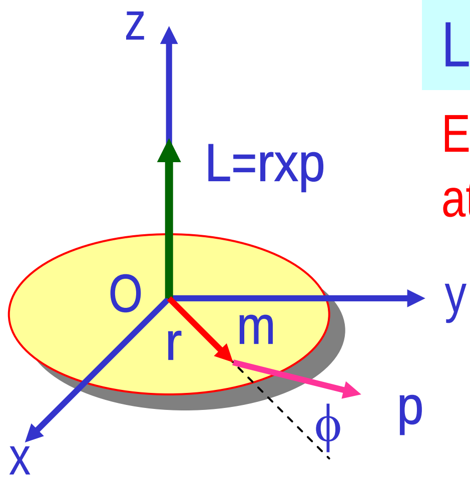
So the angular momentum vector can be expressed as $\vec{L} = mrv\vec{k}$

Find the angular momentum in terms of angular velocity ω.

Using the relationship between linear and angular speed

$$\vec{L} = mrv\vec{k} = mr^2\omega\vec{k} = mr^2\vec{\omega} = I\vec{\omega}$$

Angular Momentum of a Rotating Rigid Body



Let's consider a rigid body rotating about a fixed axis

Each particle of the object rotates in the xy plane about the z-axis at the same angular speed, ω

Magnitude of the angular momentum of a particle of mass m_i about origin O is $m_i v_i r_i$

$$L_i = m_i r_i v_i = m_i r_i^2 \omega$$

Summing over all particle's angular momentum about z axis

$$L_z = \sum_i L_i = \sum_i \left(m_i r_i^2 \omega \right)$$

What do you see?

$$L_z = \sum_i \left(m_i r_i^2 \right) \omega = I\omega$$

Since I is constant for a rigid body

$$\frac{dL_z}{dt} = I \frac{d\omega}{dt} = I\alpha$$

α is angular acceleration

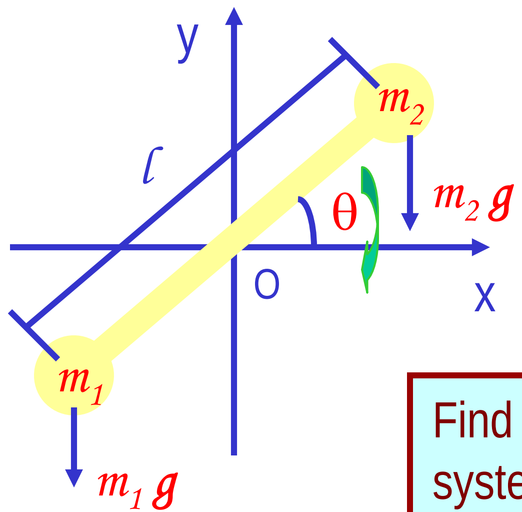
Thus the torque-angular momentum relationship becomes

$$\boxed{\sum \tau_{ext} = \frac{dL_z}{dt} = I\alpha}$$

Thus the net external torque acting on a rigid body rotating about a fixed axis is equal to the moment of inertia about that axis multiplied by the object's angular acceleration with respect to that axis.

Example for Rigid Body Angular Momentum

A rigid rod of mass $\mathcal{M}$ and length ℓ is pivoted without friction at its center. Two particles of mass m_1 and m_2 are attached to either end of the rod. The combination rotates on a vertical plane with an angular speed of ω. Find an expression for the magnitude of the angular momentum.



The moment of inertia of this system is

$$I = I_{rod} + I_{m_1} + I_{m_2} = \frac{1}{12}Ml^2 + \frac{1}{4}m_1 l^2 + \frac{1}{4}m_2 l^2$$

$$= \frac{l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)$$

$$\boxed{L = I\omega = \frac{\omega l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)}$$

Find an expression for the magnitude of the angular acceleration of the system when the rod makes an angle θ with the horizon.

First compute the net external torque

$$\tau_1 = m_1 g \frac{l}{2}\cos\theta \qquad \tau_2 = -m_2 g \frac{l}{2}\cos\theta$$

$$\tau_{ext} = \tau_1 + \tau_2 = \frac{gl\cos\theta(m_1 - m_2)}{2}$$

Thus α becomes

$$\alpha = \frac{\sum \tau_{ext}}{I} = \frac{\frac{1}{2}(m_1 - m_2)gl\cos\theta}{\frac{l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)} = \frac{2(m_1 - m_2)\cos\theta}{\left(\frac{1}{3}M + m_1 + m_2\right)}\frac{g}{l}$$

If $m_1 = m_2$, no angular momentum because the net torque is 0.

If $\theta = +/-\pi/2$, at equilibrium so no angular momentum.

Conservation of Angular Momentum

Remember under what condition the linear momentum is conserved?

Linear momentum is conserved when the net external force is 0.

$$\sum \vec{F} = 0 = \frac{d\vec{p}}{dt}$$

$$\vec{p} = const$$

By the same token, the angular momentum of a system is constant in both magnitude and direction, if the resultant external torque acting on the system is 0.

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt} = 0$$

$$\boxed{\vec{L} = const}$$

What does this mean?

Angular momentum of the system before and after a certain change is the same.

$$\vec{L}_i = \vec{L}_f = \text{constant}$$

Three important conservation laws for isolated system that does not get affected by external forces

$K_i + U_i = K_f + U_f$	Mechanical Energy
$\vec{p}_i = \vec{p}_f$	Linear Momentum
$\vec{L}_i = \vec{L}_f$	Angular Momentum

Example for Angular Momentum Conservation

A star rotates with a period of 30 days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0×10^4km, collapses into a neutron star of radius 3.0km. Determine the period of rotation of the neutron star.

What is your guess about the answer?

The period will be significantly shorter, because its radius got smaller.

Let's make some assumptions:

1. There is no external torque acting on it
2. The shape remains spherical
3. Its mass remains constant

Using angular momentum conservation

$$L_i = L_f$$

$$I_i\omega_i = I_f\omega_f$$

The angular speed of the star with the period T is

$$\omega = \frac{2\pi}{T}$$

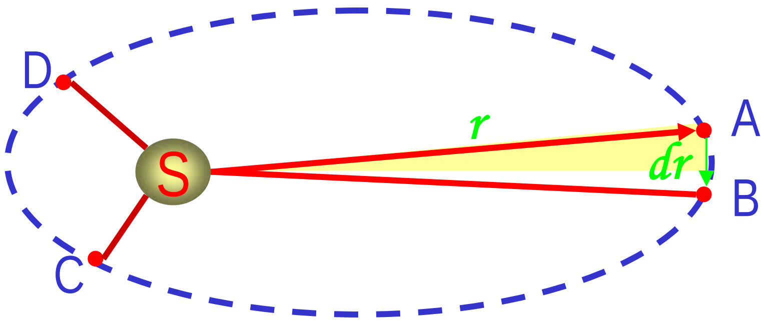
Thus

$$\omega_f = \frac{I_i\omega_i}{I_f} = \frac{mr_i^2}{mr_f^2}\frac{2\pi}{T_i}$$

$$T_f = \frac{2\pi}{\omega_f} = \left(\frac{r_f^2}{r_i^2}\right)T_i = \left(\frac{3.0}{1.0\times10^4}\right)^2 \times 30\ days = 2.7\times10^{-6}\ days = 0.23\ s$$

Kepler's Second Law and Angular Momentum Conservation

Consider a planet of mass M_p moving around the Sun in an elliptical orbit.



Since the gravitational force acting on the planet is always toward radial direction, it is a *central force*

Therefore the torque acting on the planet by this force is always 0.

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times F\hat{r} = 0$$

Since torque is the time rate change of angular momentum *L*, the angular momentum is constant.

$$\vec{\tau} = \frac{d\vec{L}}{dt} = 0 \qquad \vec{L} = const$$

Because the gravitational force exerted on a planet by the Sun results in no torque, the angular momentum L of the planet is constant.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times M_p \vec{v} = M_p \vec{r} \times \vec{v} = const$$

Since the area swept by the motion of the planet is

$$dA = \frac{1}{2}\left|\vec{r} \times d\vec{r}\right| = \frac{1}{2}\left|\vec{r} \times \vec{v}dt\right| = \frac{L}{2M_p}dt \Rightarrow \frac{dA}{dt} = \frac{L}{2M_p} = const$$

This is Keper's second law which states that the radius vector from the Sun to a planet sweeps out equal areas in equal time intervals.

Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Quantities	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = mr^2$
Length of motion	Distance L	Angle θ (Radian)
Speed	$v = \frac{\Delta r}{\Delta t}$	$\omega = \frac{\Delta \theta}{\Delta t}$
Acceleration	$a = \frac{\Delta v}{\Delta t}$	$\alpha = \frac{\Delta \omega}{\Delta t}$
Force	Force $\vec{F} = m\vec{a}$	Torque $\vec{\tau} = I\vec{\alpha}$
Work	Work $W = \vec{F} \cdot \vec{d}$	Work $W = \tau\theta$
Power	$P = \vec{F} \cdot \vec{v}$	$P = \tau\omega$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{\omega}$
Kinetic Energy	Kinetic $K = \frac{1}{2}mv^2$	Rotational $K_R = \frac{1}{2}I\omega^2$