# PHYS 1443 – Section 002
# Lecture #18

*Wednesday, Nov. 12, 2008*
*Dr. Jae Yu*

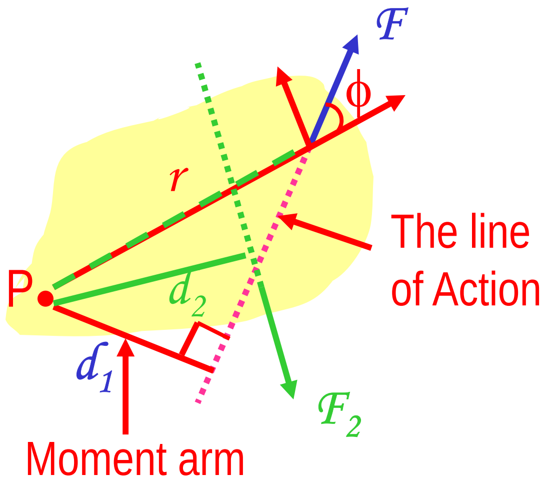
- Rotational Dynamics
  - Torque
  - Vector Product
  - Moment of Inertia
- Rotational Kinetic Energy
- Work, Power and Energy in Rotation
- Rolling Motion of a Rigid Body
- Relationship between angular and linear quantities

# Announcements

- Quiz results
  - Class average: 3.9/6
    - Equivalent to 65/100
    - Previous quizzes: 53/100, 30/100, 56/100
  - Top score: 6/6
- Reading assignment: CH 10.10
- Third term exam
  - 1 – 2:20pm, Wednesday, Nov. 19, in SH103
  - Covers CH 9 – CH10
  - Jason will do a summary session on Monday, Nov. 17
- No colloquium today…

# Torque

Torque is the tendency of a force to rotate an object about an axis. Torque, $\tau$, is a vector quantity.




Consider an object pivoting about the point P by the force $\mathcal{F}$ being exerted at a distance r from P.

The line that extends out of the tail of the force vector is called the line of action.

The perpendicular distance from the pivoting point P to the line of action is called the moment arm.

Magnitude of torque is defined as the product of the force exerted on the object to rotate it and the moment arm.

$$\tau \equiv rF\sin\phi = Fd_1$$

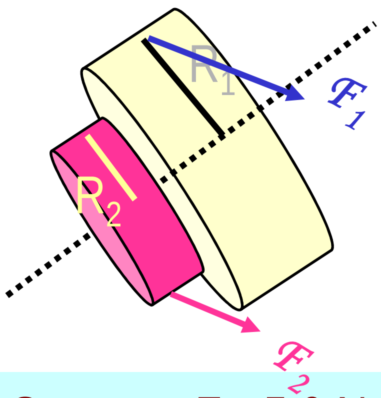
When there are more than one force being exerted on certain points of the object, one can sum up the torque generated by each force vectorially. The convention for sign of the torque is positive if rotation is in counter-clockwise and negative if clockwise.

$$\sum \tau = \tau_1 + \tau_2$$

$$= F_1 d_1 - F_2 d_2$$

# Example for Torque

A one piece cylinder is shaped as shown in the figure with core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the picture. A rope wrapped around the drum whose radius is $R_1$ exerts force $F_1$ to the right on the cylinder, and another force exerts $F_2$ on the core whose radius is $R_2$ downward on the cylinder. A) What is the net torque acting on the cylinder about the rotation axis?




The torque due to $F_1$ $\tau_1 = -R_1 F_1$ and due to $F_2$ $\tau_2 = R_2 F_2$

So the total torque acting on the system by the forces is

$$\sum \tau = \tau_1 + \tau_2 = -R_1 F_1 + R_2 F_2$$

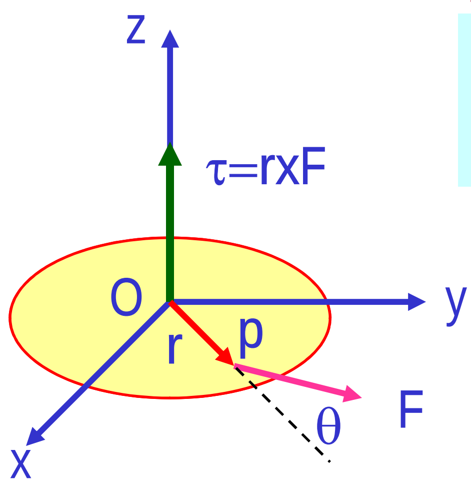
Suppose $F_1$=5.0 N, $R_1$=1.0 m, $F_2$= 15.0 N, and $R_2$=0.50 m. What is the net torque about the rotation axis and which way does the cylinder rotate from the rest?

Using the above result

$$\sum \tau = -R_1 F_1 + R_2 F_2$$
$$= -5.0 \times 1.0 + 15.0 \times 0.50 = 2.5\ N \cdot m$$

The cylinder rotates in counter-clockwise.

# Torque and Vector Product




Let's consider a disk fixed onto the origin O and the force $\mathcal{F}$ exerts on the point p. What happens?

The disk will start rotating counter clockwise about the Z axis

The magnitude of torque given to the disk by the force $\mathcal{F}$ is

$$\tau = Fr\sin\theta$$

But torque is a vector quantity, what is the direction?
How is torque expressed mathematically?

$$\vec{\tau} \equiv \vec{r} \times \vec{F}$$

What is the direction? The direction of the torque follows the right-hand rule!!

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The above operation is called the Vector product or Cross product

$$\vec{C} \equiv \vec{A} \times \vec{B}$$

$$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$$

What is the result of a vector product?

Another vector

What is another vector operation we've learned?

Scalar product

$$C \equiv \vec{A} \cdot \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$$

Result? A scalar

# Properties of Vector Product

Vector Product is Non-commutative

What does this mean?

If the order of operation changes the result changes $\vec{A}\times\vec{B}\neq\vec{B}\times\vec{A}$

Following the right-hand rule, the direction changes $\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$

Vector Product of two parallel vectors is 0.

$$\left|\vec{C}\right| = \left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta = \left|\vec{A}\right|\left|\vec{B}\right|\sin 0 = 0$$

Thus, $\vec{A}\times\vec{A}=0$

If two vectors are perpendicular to each other

$$\left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta = \left|\vec{A}\right|\left|\vec{B}\right|\sin 90^{\circ} = \left|\vec{A}\right|\left|\vec{B}\right| = AB$$

Vector product follows distribution law

$$\vec{A}\times\left(\vec{B}+\vec{C}\right) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

The derivative of a Vector product with respect to a scalar variable is

$$\frac{d\left(\vec{A}\times\vec{B}\right)}{dt} = \frac{d\vec{A}}{dt}\times\vec{B}+\vec{A}\times\frac{d\vec{B}}{dt}$$

# More Properties of Vector Product

The relationship between unit vectors, $\vec{i}, \vec{j}$ and $\vec{k}$

$$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = 0$$

$$\vec{i}\times\vec{j} = -\vec{j}\times\vec{i} = \vec{k}$$

$$\vec{j}\times\vec{k} = -\vec{k}\times\vec{j} = \vec{i}$$

$$\vec{k}\times\vec{i} = -\vec{i}\times\vec{k} = \vec{j}$$

Vector product of two vectors can be expressed in the following determinant form

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \vec{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \vec{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= \left(A_y B_z - A_z B_y\right)\vec{i} - \left(A_x B_z - A_z B_x\right)\vec{j} + \left(A_x B_y - A_y B_x\right)\vec{k}$$

# Moment of Inertia

Rotational Inertia: Measure of resistance of an object against changes in its rotational motion. Equivalent to MASS in linear motion.

For a group of particles

$$I \equiv \sum_i m_i r_i^2$$

For a rigid body

$$I \equiv \int r^2 dm$$

What are the dimension and unit of Moment of Inertia?

$$\left[ML^2\right] \quad kg \cdot m^2$$

Determining Moment of Inertia is extremely important for computing equilibrium of a rigid body, such as a building.

Dependent on the axis of rotation!!!

# Calculation of Moments of Inertia

Moments of inertia for large objects can be computed, if we assume that the object consists of small volume elements with mass, $\Delta m_i$.

The moment of inertia for the large rigid object is

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$

It is sometimes easier to compute moments of inertia in terms of volume of the elements rather than their mass

How do we do this?

Using the volume density, ρ, replace d$m$ in the above equation with dV.

$$\rho = \frac{dm}{dV} \Rightarrow dm = \rho dV$$

The moments of inertia becomes

$$I = \int \rho r^2 dV$$

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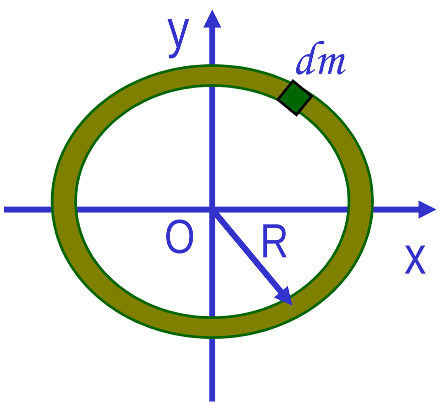
Example: Find the moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.




The moment of inertia is

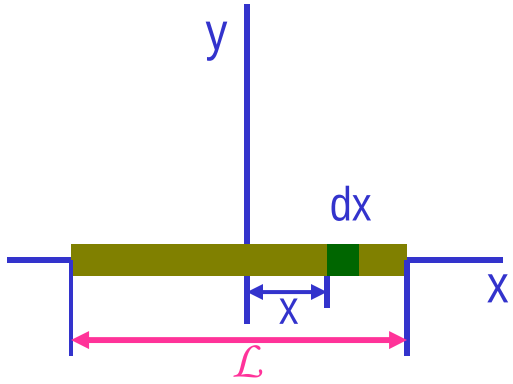
$$I = \int r^2 dm = R^2 \int dm = MR^2$$

What do you notice from this result?

The moment of inertia for this object is the same as that of a point of mass M at the distance R.

# Example for Rigid Body Moment of Inertia

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing through its center of mass.




The line density of the rod is $\lambda = \frac{M}{L}$

so the masslet is $dm = \lambda dx = \frac{M}{L} dx$

The moment of inertia is

$$I = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

What is the moment of inertia when the rotational axis is at one end of the rod.

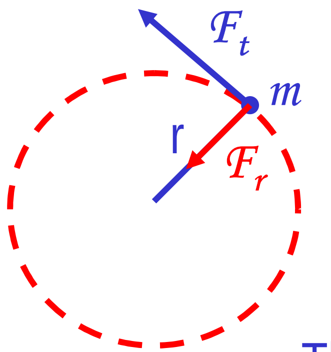
$$I = \int r^2 dm = \int_0^L \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_0^L$$

$$= \frac{M}{3L}\left[(L)^3 - 0\right] = \frac{M}{3L}\left(L^3\right) = \frac{ML^2}{3}$$

Will this be the same as the above. Why or why not?

Since the moment of inertia is resistance to motion, it makes perfect sense for it to be harder to move when it is rotating about the axis at one end.

# Torque & Angular Acceleration




Let's consider a point object with mass $m$ rotating on a circle.

What forces do you see in this motion?

The tangential force $\mathcal{F}_t$ and the radial force $\mathcal{F}_r$

The tangential force $\mathcal{F}_t$ is $\quad F_t = ma_t = mr\alpha$

The torque due to tangential force $\mathcal{F}_t$ is $\quad \tau = F_t r = ma_t r = mr^2\alpha = I\alpha$

What do you see from the above relationship? $\quad \tau = I\alpha$

What does this mean?

Torque acting on a particle is proportional to the angular acceleration.

What law do you see from this relationship?

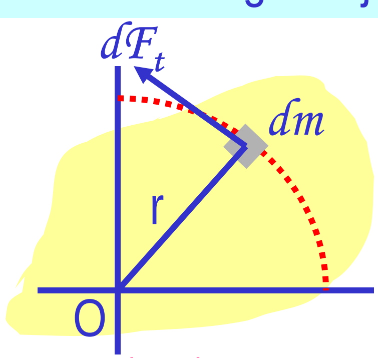
Analogs to Newton's 2nd law of motion in rotation.

How about a rigid object?

The external tangential force $d\mathcal{F}_t$ is $\quad dF_t = dma_t = dmr\alpha$

The torque due to tangential force $\mathcal{F}_t$ is $\quad d\tau = dF_t r = \left(r^2 dm\right)\alpha$

The total torque is $\quad \sum\tau = \alpha\int r^2 dm = I\alpha$

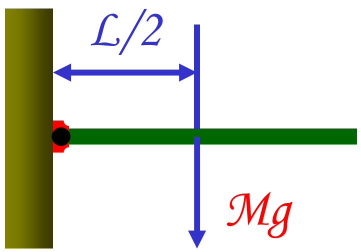



What is the contribution due to radial force and why?

Contribution from radial force is 0, because its line of action passes through the pivoting point, making the moment arm 0.

# Example for Torque and Angular Acceleration

A uniform rod of length $\mathcal{L}$ and mass $\mathcal{M}$ is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane. The rod is released from rest in the horizontal position. What are the initial angular acceleration of the rod and the initial linear acceleration of its right end?




The only force generating torque is the gravitational force $\mathcal{M}g$

$$\tau = Fd = F\frac{L}{2} = Mg\frac{L}{2} = I\alpha$$

Since the moment of inertia of the rod when it rotates about one end

$$I = \int_0^L r^2 dm = \int_0^L x^2 \lambda dx = \left(\frac{M}{L}\right)\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2}{3}$$

We obtain $$\alpha = \frac{MgL}{2I} = \frac{MgL}{\frac{2ML^2}{3}} = \frac{3g}{2L}$$

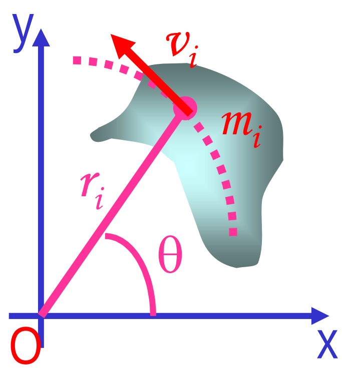
Using the relationship between tangential and angular acceleration

$$a_t = L\alpha = \frac{3g}{2}$$

What does this mean?

The tip of the rod falls faster than an object undergoing a free fall.

# Rotational Kinetic Energy




What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, $m_i$, moving at a tangential speed, $v_i$, is

$$K_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i r_i^2 \omega^2$$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

$$K_R = \sum_i K_i = \frac{1}{2}\sum_i m_i r_i^2 \omega^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

Since moment of Inertia, I, is defined as

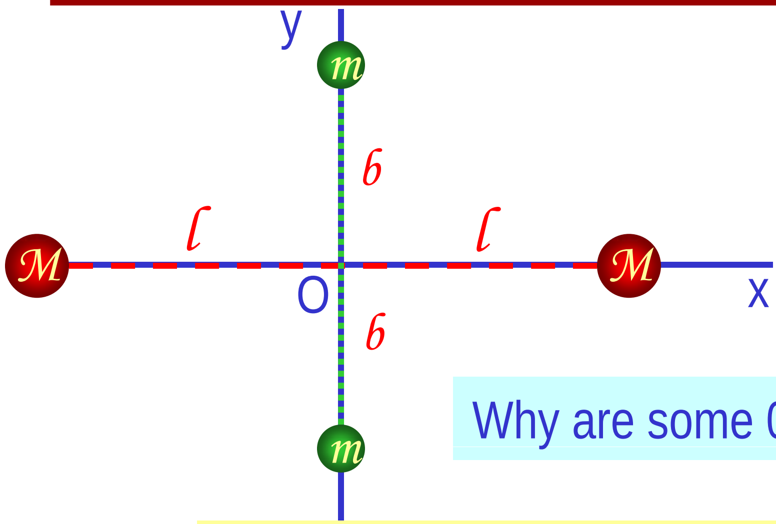
$$I = \sum_i m_i r_i^2$$

The above expression is simplified as

$$K_R = \frac{1}{2}I\omega^2$$

# Example for Moment of Inertia

In a system of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at angular speed $\omega$.




Since the rotation is about y axis, the moment of inertia about y axis, $I_y$, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m\cdot 0^2 + m\cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

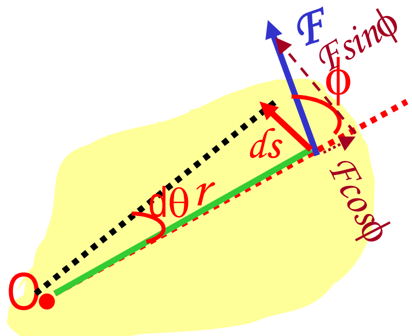
The rotational kinetic energy is

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2\right)\omega^2 = Ml^2\omega^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2\left(Ml^2 + mb^2\right) \qquad K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2 + 2mb^2\right)\omega^2 = \left(Ml^2 + mb^2\right)\omega^2$$

# Work, Power, and Energy in Rotation




Let's consider the motion of a rigid body with a single external force *F* exerting on the point P, moving the object by *ds*.
The work done by the force *F* as the object rotates through the infinitesimal distance ds=rdθ is

$$dW = \vec{F} \cdot d\vec{s} = \left(F\cos(\pi/2 - \phi)\right) r d\theta = \left(F \sin\phi\right) r d\theta$$

What is *F*sinϕ? The tangential component of the force *F*.

What is the work done by radial component *F*cosϕ? Zero, because it is perpendicular to the displacement.

Since the magnitude of torque is r*F*sinϕ,

$$dW = \left(rF\sin\phi\right)d\theta = \tau d\theta$$

The rate of work, or power, becomes

$$P = \frac{dW}{dt} = \frac{\tau\, d\theta}{dt} = \tau\omega$$

How was the power defined in linear motion?

The rotational work done by an external force equals the change in rotational Kinetic energy.

$$\sum \tau = I\alpha = I\left(\frac{d\omega}{dt}\right) = I\left(\frac{d\omega}{d\theta}\right)\left(\frac{d\theta}{dt}\right) = I\omega\left(\frac{d\omega}{d\theta}\right)$$

$$\sum \tau d\theta = I\omega d\omega \quad \Rightarrow \quad dW = \sum \tau d\theta = I\omega d\omega$$

The work put in by the external force then

$$W = \int_{\theta_i}^{\theta_f} \sum \tau d\theta = \int_{\omega_i}^{\omega_f} I\omega d\omega = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2$$