# PHYS 1444 – Section 003
# Lecture #23

*Thursday, Dec. 1, 2011*

*Dr. Jaehoon Yu*

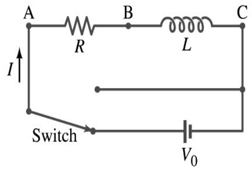
- LR circuit
- LC Circuit and EM Oscillation
- LRC circuit
- AC Circuit w/ Resistance only
- AC Circuit w/ Inductance only
- AC Circuit w/ Capacitance only

# Announcements

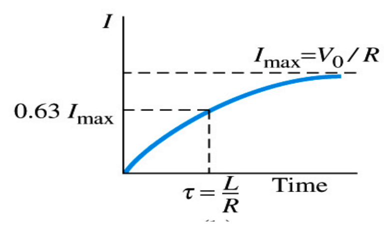
- Term exam results
  - Class average: 68.5/101
    - Equivalent to 67.8/100
    - Previous exams: 59/100 and 66/100
  - Top score: 95/101
- Your planetarium extra credit
  - Please bring your planetarium extra credit sheet by the beginning of the class next Tuesday, Dec. 6
  - Be sure to tape one edge of the ticket stub with the title of the show on top
  - Be sure to write your name onto the sheet
- Quiz #4
  - Coming Tuesday, Dec. 6
  - Covers CH30.1 through CH30.11
- Reading Assignments
  - CH30.7 – CH30.11
- Final comprehensive exam
  - Date and time: 11am, Thursday, Dec. 15, in SH103
  - Covers CH1.1 – what we cover coming Tuesday, Dec. 6 + Appendices A and B

# LR Circuits

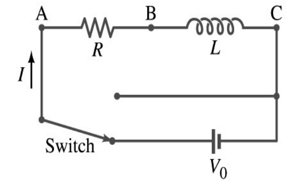
- What happens when an emf is applied to an inductor?
  - An inductor has some resistance, however negligible




    - So an inductor can be drawn as a circuit of separate resistance and coil. What is the name this kind of circuit? **LR Circuit**
  - What happens at the instance the switch is thrown to apply emf to the circuit?
    - The current starts to flow, gradually increasing from 0
    - This change is opposed by the induced emf in the inductor ➔ the emf at point B is higher than point C
    - However there is a voltage drop at the resistance which reduces the voltage across inductance
    - Thus the current increases less rapidly
    - The overall behavior of the current is a gradual increase, reaching to the maximum current $I_{max}=V_0/R$.



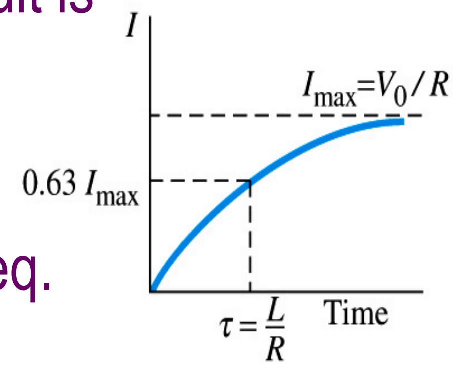
# LR Circuits



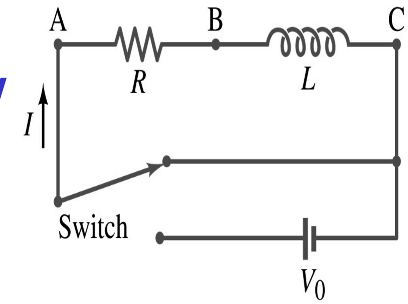

- This can be shown w/ Kirchhoff rule loop rules
  - The emfs in the circuit are the battery voltage $V_0$ and the emf $\varepsilon = -\mathcal{L}(dI/dt)$ in the inductor opposing the current increase
  - The sum of the potential changes through the circuit is

$$V_0 + \varepsilon - IR = V_0 - L\,dI/dt - IR = 0$$

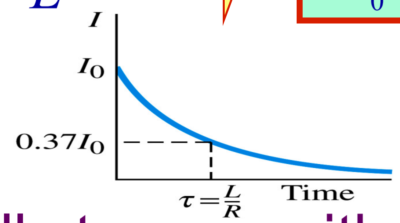
  - Where $I$ is the current at any instance
  - By rearranging the terms, we obtain a differential eq.
  - $L\,dI/dt + IR = V_0$
  - We can integrate just as in RC circuit $\int_{I=0}^{I} \frac{dI}{V_0 - IR} = \int_{t=0}^{t} \frac{dt}{L}$
  - So the solution is $-\frac{1}{R}\ln\left(\frac{V_0 - IR}{V_0}\right) = \frac{t}{L}$ $\boxed{I = V_0\left(1 - e^{-t/\tau}\right)/R = I_{max}\left(1 - e^{-t/\tau}\right)}$
  - Where $\tau$=L/R
    - This is the time constant $\tau$ of the LR circuit and is the time required for the current $I$ to reach 0.63 of the maximum



# Discharge of LR Circuits



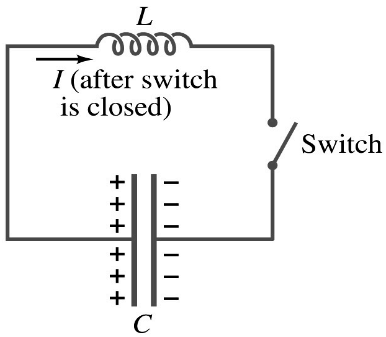

- If the switch is flipped away from the battery
  - The differential equation becomes
  - $L\,dI/dt + IR = 0$
  - So the integration is $\int_{I_0}^{I} \frac{dI}{IR} = \int_{t=0}^{t} \frac{dt}{L}$ ⇒ $\boxed{\ln \frac{I}{I_0} = -\frac{R}{L}t}$
  - Which results in the solution




  - $\boxed{I = I_0 e^{-\frac{R}{L}t} = I_0 e^{-t/\tau}}$
  - The current decays exponentially to zero with the time constant τ=L/R
  - So there always is a reaction time when a system with a coil, such as an electromagnet, is turned on or off.
  - The current in LR circuit behaves almost the same as that in RC circuit but the time constant is inversely proportional to R in LR circuit unlike the RC circuit

# LC Circuit and EM Oscillations

- What's an LC circuit?
  - A circuit that contains only an inductor and a capacitor
    - How is this possible? There is no source of emf!!
    - Well, you can imagine a circuit with a fully charged capacitor
    - In this circuit, we assume the inductor does not have any resistance
- Let's assume that the capacitor originally has $+Q_0$ on one plate and $-Q_0$ on the other
  - Suppose the switch is closed at t=0
  - The capacitor starts discharging
  - The current flowing through the inductor increases
  - Applying Kirchhoff's loop rule, we obtain $-L\,dI/dt + Q/C = 0$
  - Since the current flows out of the plate with positive charge, the charge on the plate reduces, so $I=-dQ/dt$. Thus the differential equation can be rewritten




$$\boxed{\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0}$$

# LC Circuit and EM Oscillations

- This equation looks the same as that of the harmonic oscillation
  - So the solution for this second order differential equation is
  - $Q = Q_0 \cos(\varpi t + \phi)$ The charge on the capacitor oscillates sinusoidally
  - Inserting the solution back into the differential equation
  - $\frac{d^2Q}{dt^2} + \frac{Q}{LC} = -\varpi^2 Q_0 \cos(\varpi t + \phi) + Q_0 \cos(\varpi t + \phi)/LC = 0$
  - Solving this equation for ω, we obtain $\varpi = 2\pi f = 1/\sqrt{LC}$
  - The current in the inductor is
  - $I = -dQ/dt = \varpi Q_0 \sin(\varpi t + \phi) = I_0 \sin(\varpi t + \phi)$
  - So the current also is sinusoidal with the maximum value

$$I_0 = \varpi Q_0 = Q_0/\sqrt{LC}$$

# Energies in LC Circuit & EM Oscillation

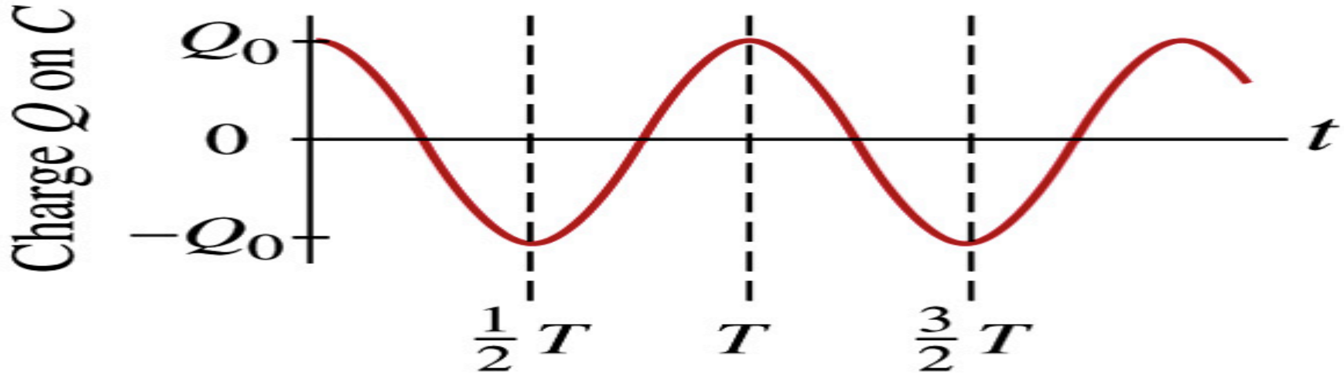
- The energy stored in the electric field of the capacitor at any time t is $U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{Q_0^2}{2C}\cos^2(\varpi t + \phi)$
- The energy stored in the magnetic field in the inductor at the same instant is $U_B = \frac{1}{2}LI^2 = \frac{Q_0^2}{2C}\sin^2(\varpi t + \phi)$
- Thus, the total energy in LC circuit at any instant is

$$U = U_E + U_B = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2 = \frac{Q_0^2}{2C}\left[\cos^2(\varpi t + \phi) + \sin^2(\varpi t + \phi)\right] = \frac{Q_0^2}{2C}$$

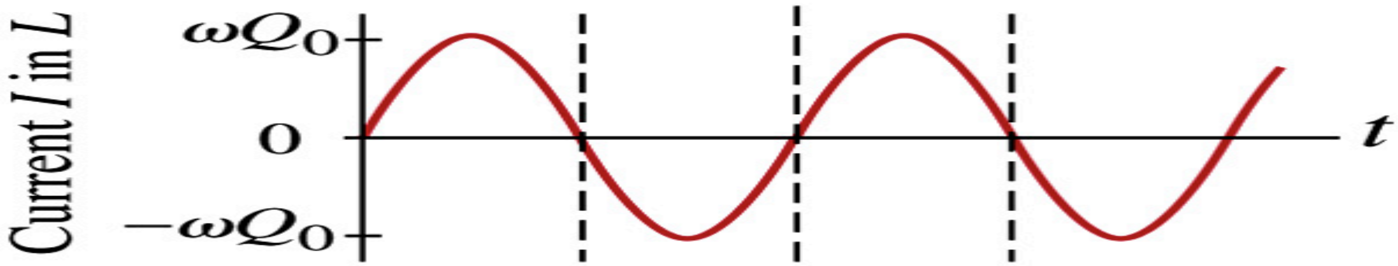
- So the total EM energy is constant and is conserved.
- This LC circuit is an LC oscillator or EM oscillator
  - The charge Q oscillates back and forth, from one plate of the capacitor to the other
  - The current also oscillates back and forth as well

# LC Circuit Behaviors

$$Q = Q_0 \cos(\varpi t)$$

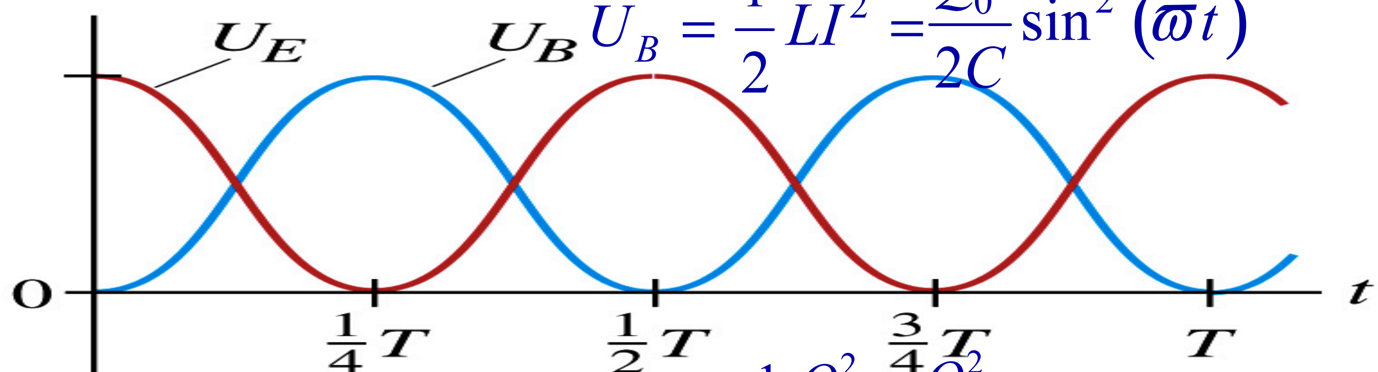



$$I = \varpi Q_0 \sin(\varpi t)$$




$$U_B = \frac{1}{2} L I^2 = \frac{Q_0^2}{2C} \sin^2(\varpi t)$$

$$U = \frac{Q_0^2}{2C}$$




$$U_E = \frac{1}{2} \frac{Q^2}{C} = \frac{Q_0^2}{2C} \cos^2(\varpi t)$$

# Example 30 – 7

**LC Circuit.** A 1200-pF capacitor is fully charged by a 500-V dc power supply. It is disconnected from the power supply and is connected, at t=0, to a 75-mH inductor. Determine: (a) The initial charge on the capacitor, (b) the maximum current, (c) the frequency $f$ and period T of oscillation; and (d) the total energy oscillating in the system.

(a) The 500-V power supply, charges the capacitor to

$$Q = CV = \left(1200\times10^{-12}F\right)\cdot 500V = 6.0\times10^{-7}C$$

(b) The maximum current is

$$I_{\max} = \varpi Q_0 = \frac{Q_0}{\sqrt{LC}} = \frac{6.0\times10^{-7}C}{\sqrt{75\times10^{-3}H\times1.2\times10^{-9}F}} = 63mA$$

(c) The frequency is

$$f = \frac{\varpi}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{75\times10^{-3}H\cdot1.2\times10^{-9}F}} = 1.7\times10^{4}Hz$$

The period is

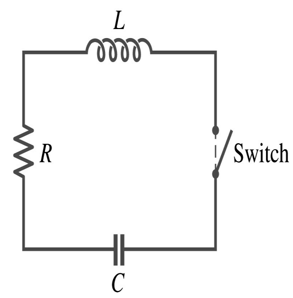
$$T = \frac{1}{f} = 6.0\times10^{-5}S$$

(d) The total energy in the system

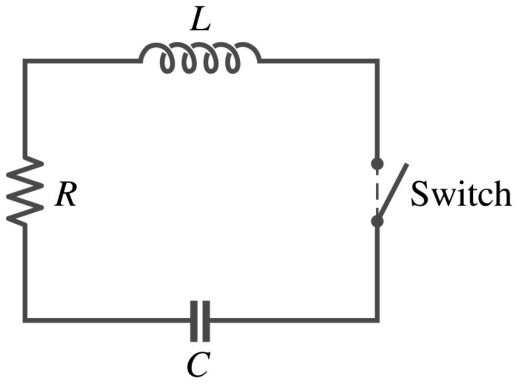
$$U = \frac{Q_0^2}{2C} = \frac{\left(6.0\times10^{-7}C\right)^2}{2\cdot1.2\times10^{-9}F} = 1.5\times10^{-4}J$$

# LC Oscillations w/ Resistance (LRC circuit)

- There is no such thing as zero resistance coil so all LC circuits have some resistance
  - So to be more realistic, the effect of the resistance should be taken into account
  - Suppose the capacitor is charged up to $Q_0$ initially and the switch is closed in the circuit at t=0
  - What do you expect to happen to the energy in the circuit?
    - Well, due to the resistance we expect some energy will be lost through the resister via a thermal conversion
  - What about the oscillation? Will it look the same as the ideal LC circuit we dealt with?
  - No? OK then how would it be different?
  - The oscillation would be damped due to the energy loss.



# LC Oscillations w/ Resistance (LRC circuit)



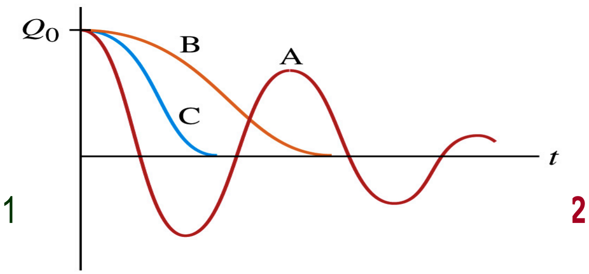

- Now let's do some analysis
- From Kirchhoff's loop rule, we obtain

$$-L\,dI/dt\ -IR+\frac{Q}{C}=0$$

- Since $I$=dQ/dt, the equation becomes

$$-L\frac{d^2Q}{dt^2}\ -R\frac{dQ}{dt}+\frac{Q}{C}=0$$

  - Which is identical to that of a damped oscillator
- The solution of the equation is $Q=Q_0e^{-\frac{R}{2L}t}\cos\left(\varpi' t+\phi\right)$
  - Where the angular frequency is $\varpi'=\sqrt{1/LC-R^2/4L^2}$
    - $R^2$<4L/C: Underdamped
    - $R^2$>4L/C: Overdampled



# Why do we care about circuits on AC?

- The circuits we've learned so far contain resistors, capacitors and inductors and have been connected to a DC source or a fully charged capacitor
  - What? This does not make sense.
  - The inductor does not work as an impedance unless the current is changing. So an inductor in a circuit with DC source does not make sense.
  - Well, actually it does. When does it impede?
    - Immediately after the circuit is connected to the source so the current is still changing. So?
      - It causes the change of magnetic flux.
  - Now does it make sense?
- Anyhow, learning the responses of resistors, capacitors and inductors in a circuit connected to an AC emf source is important. Why is this?
  - Since most the generators produce sinusoidal current
  - Any voltage that varies over time can be expressed in the superposition of sine and cosine functions

# AC Circuits – the preamble

- Do you remember how the rms and peak values for current and voltage are related?

$$V_{rms} = \frac{V_0}{\sqrt{2}} \qquad\qquad I_{rms} = \frac{I_0}{\sqrt{2}}$$

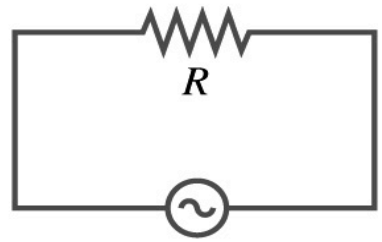
- The symbol for an AC power source is

- We assume that the voltage gives rise to current

$$I = I_0 \sin 2\pi ft = I_0 \sin \varpi t$$

  – where $\varpi = 2\pi f$

# AC Circuit w/ Resistance only




- What do you think will happen when an AC source is connected to a resistor?
- From Kirchhoff's loop rule, we obtain

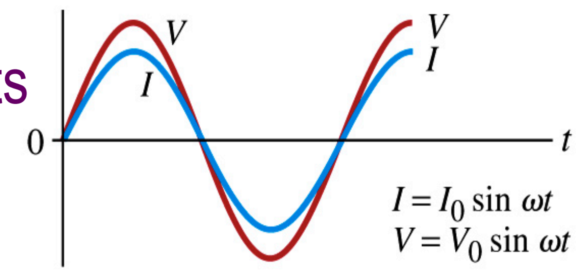
$$V - IR = 0$$

- Thus

$$V = I_0 R \sin \varpi t = V_0 \sin \varpi t$$

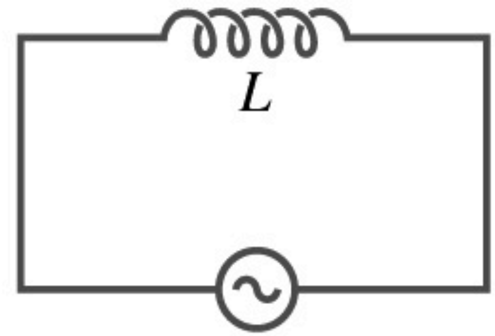
  – where $V_0 = I_0 R$

- What does this mean?
  – Current is 0 when voltage is 0 and current is in its peak when voltage is in its peak.
  – Current and voltage are "in phase"




- Energy is lost via the transformation into heat at an average rate $\overline{P} = \overline{I}\,\overline{V} = I_{rms}^2 R = V_{rms}^2 / R$

# AC Circuit w/ Inductance only




- From Kirchhoff's loop rule, we obtain

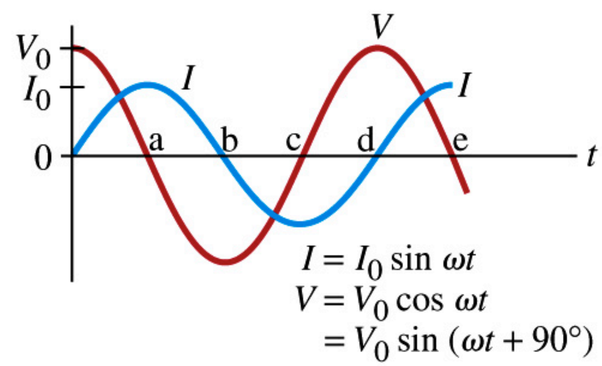
$$V - L\frac{dI}{dt} = 0$$

- Thus

$$V = L\frac{dI}{dt} = L\frac{d\left(I_0 \sin \varpi t\right)}{dt} = \varpi L I_0 \cos \varpi t$$

  – Using the identity $\cos\theta = \sin\left(\theta + 90^\circ\right)$

- $V = \varpi L I_0 \sin\left(\varpi t + 90^\circ\right) = V_0 \sin\left(\varpi t + 90^\circ\right)$
  – where $V_0 = \varpi L I_0$
- What does this mean?
  – Current and voltage are "out of phase by π/2 or 90°" in other words the current reaches its peak ¼ cycle after the voltage
- What happens to the energy?
  – No energy is dissipated
  – The average power is 0 at all times
  – The energy is stored temporarily in the magnetic field
  – Then released back to the source



# AC Circuit w/ Inductance only

- How are the resistor and inductor different in terms of energy?
  - Inductor — Stores the energy temporarily in the magnetic field and then releases it back to the emf source
  - Resistor — Does not store energy but transforms it to thermal energy, getting it lost to the environment
- How are they the same?
  - They both impede the flow of charge
  - For a resistance R, the peak voltage and current are related to $V_0 = I_0 R$
  - Similarly, for an inductor we may write $\boxed{V_0 = I_0 X_L}$
    - Where $X_L$ is the inductive reactance of the inductor $\boxed{X_L = \varpi L}$ **0 when ω=0.**
    - What do you think is the unit of the reactance? $\Omega$
    - The relationship $V_0 = I_0 X_L$ is not valid at a particular instance. Why not?
      - Since $V_0$ and $I_0$ do not occur at the same time

$\boxed{V_{rms} = I_{rms} X_L}$ is valid!

# Example 30 – 9

**Reactance of a coil.** A coil has a resistance R=1.00Ω and an inductance of 0.300H. Determine the current in the coil if (a) 120 V dc is applied to it; (b) 120 V AC (rms) at 60.0Hz is applied.

Is there a reactance for DC? Nope. Why not? Since ω=0, $X_L = \varpi L = 0$

So for DC power, the current is from Kirchhoff's rule $V - IR = 0$

$$I_0 = \frac{V_0}{R} = \frac{120V}{1.00\Omega} = 120A$$

For an AC power with $f$=60Hz, the reactance is

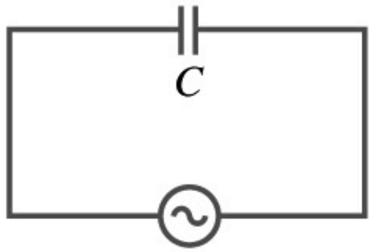
$$X_L = \varpi L = 2\pi f L = 2\pi \cdot \left(60.0s^{-1}\right) \cdot 0.300H = 113\Omega$$

Since the resistance can be ignored compared to the reactance, the rms current is

$$I_{rms} \approx \frac{V_{rms}}{X_L} = \frac{120V}{113\Omega} = 1.06A$$

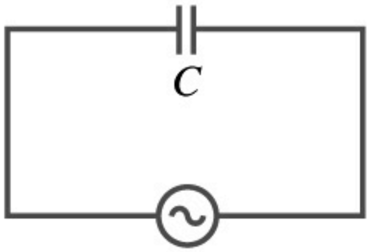
# AC Circuit w/ Capacitance only

- What happens when a capacitor is connected to a DC power source?
  - The capacitor quickly charges up.
  - There is no steady current flow in the circuit
    - Since a capacitor prevents the flow of a DC current




- What do you think will happen if it is connected to an AC power source?
  - The current flows continuously. Why?
  - When the AC power turns on, charge begins to flow one direction, charging up the plates
  - When the direction of the power reverses, the charge flows in the opposite direction

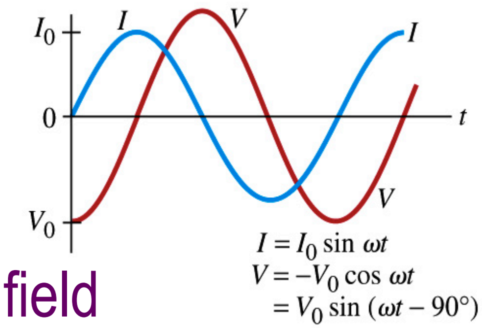
# AC Circuit w/ Capacitance only




- From Kirchhoff's loop rule, we obtain
  $$V = \frac{Q}{C}$$
- The current at any instance is $I = \frac{dQ}{dt} = I_0 \sin \varpi t$
- The charge Q on the plate at any instance is
  $$Q = \int_{Q=0}^{Q} dQ = \int_{t=0}^{t} I_0 \sin \varpi t dt = -\frac{I_0}{\varpi} \cos \varpi t$$
- Thus the voltage across the capacitor is
  $$V = \frac{Q}{C} = -I_0 \frac{1}{\varpi C} \cos \varpi t$$
  – Using the identity $\cos\theta = -\sin\left(\theta - 90^\circ\right)$
  $$V = I_0 \frac{1}{\varpi C} \sin\left(\varpi t - 90^\circ\right) = V_0 \sin\left(\varpi t - 90^\circ\right)$$
  – Where
  – $$V_0 = \frac{I_0}{\varpi C}$$

# AC Circuit w/ Capacitance only

- So the voltage is $V = V_0 \sin\left(\varpi t - 90^\circ\right)$
- What does this mean?
  - Current and voltage are "out of phase by π/2 or 90º" but in this case, the voltage reaches its peak ¼ cycle after the current
- What happens to the energy?
  - No energy is dissipated
  - The average power is 0 at all times
  - The energy is stored temporarily in the electric field
  - Then released back to the source

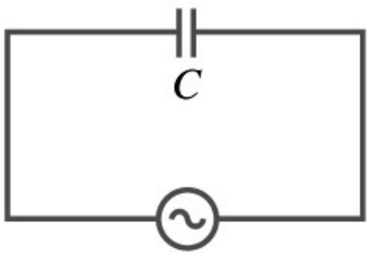



- Applied voltage and the current in the capacitor can be written as $\boxed{V_0 = I_0 X_C}$
  - Where the capacitive reactance $X_C$ is defined as $\boxed{X_C = \frac{1}{\varpi C}}$
  - Again, this relationship is only valid for rms quantities $\boxed{V_{rms} = I_{rms} X_C}$

Infinite when ω=0.

# Example 30 – 10

**Capacitor reactance.** What are the peak and rms current in the circuit in the figure if C=1.0μF and $V_{rms}$=120V? Calculate for (a) $f$=60Hz, and then for (b) $f$=6.0x10$^5$Hz.




The peak voltage is $V_0 = \sqrt{2}V_{rms} = 120V \cdot \sqrt{2} = 170V$

The capacitance reactance is

$$X_C = \frac{1}{\varpi C} = \frac{1}{2\pi f\, C} = \frac{1}{2\pi \cdot \left(60s^{-1}\right) \cdot 1.0\times 10^{-6} F} = 2.7k\Omega$$

Thus the peak current is

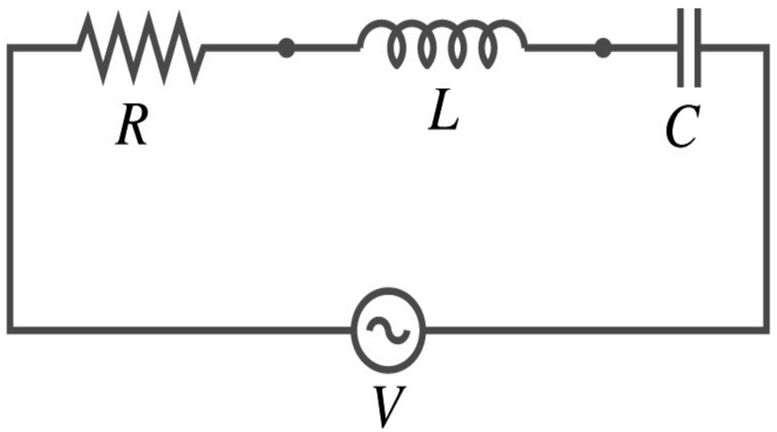
$$I_0 = \frac{V_0}{X_C} = \frac{170V}{2.7k\Omega} = 63mA$$

The rms current is

$$I_{rms} = \frac{V_{rms}}{X_C} = \frac{120V}{2.7k\Omega} = 44mA$$

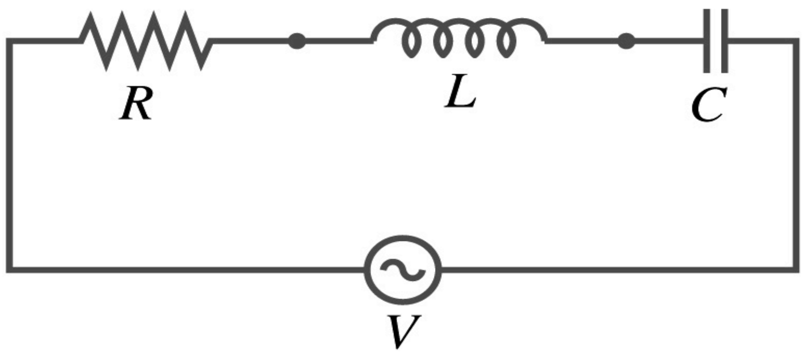
# AC Circuit w/ LRC

- The voltage across each element is
  - $V_R$ is in phase with the current
  - $V_L$ leads the current by 90°
  - $V_C$ lags the current by 90°
- From Kirchhoff's loop rule
- $V=V_R+V_L+V_C$
  - However since they do not reach the peak voltage at the same time, the peak voltage of the source $V_0$ will not equal $V_{R0}+V_{L0}+V_{C0}$
  - The rms voltage also will not be the simple sum of the three
- Let's try to find the total impedance, peak current $I_0$ and the phase difference between $I_0$ and $V_0$.



# AC Circuit w/ LRC

- The current at any instance is the same at all point in the circuit
  - The currents in each elements are in phase
  - Why?
    - Since the elements are in series
  - How about the voltage?
    - They are not in phase.



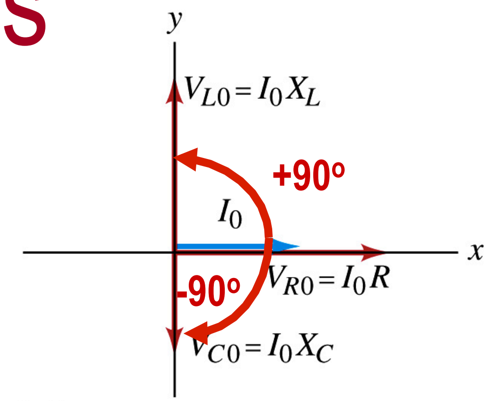

- The current at any given time is

$$I = I_0 \sin \varpi t$$

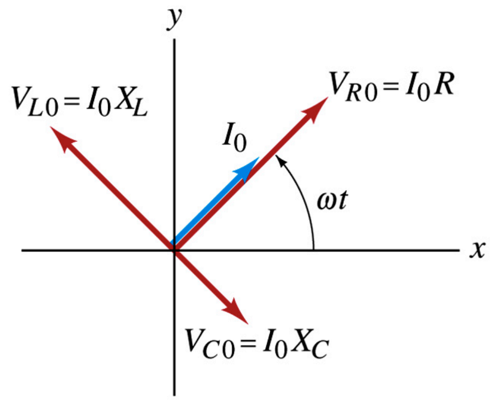
- The analysis of LRC circuit is done using the "phasor" diagram in which arrows are drawn in an xy plane to represent the amplitude of each voltage, just like vectors
  - The lengths of the arrows represent the magnitudes of the peak voltages across each element; $V_{R0}=I_0R$, $V_{L0}=I_0X_L$ and $V_{C0}=I_0X_C$
  - The angle of each arrow represents the phase of each voltage relative to the current, and the arrows rotate at angular frequency w to take into account the time dependence.
    - The projection of each arrow on y axis represents voltage across each element at any given time

# Phasor Diagrams

- At t=0, I=0.
  - Thus $V_{R0}=0$, $V_{L0}=I_0X_L$, $V_{C0}=I_0X_C$
- At t=t, $I = I_0 \sin \varpi t$



(a)



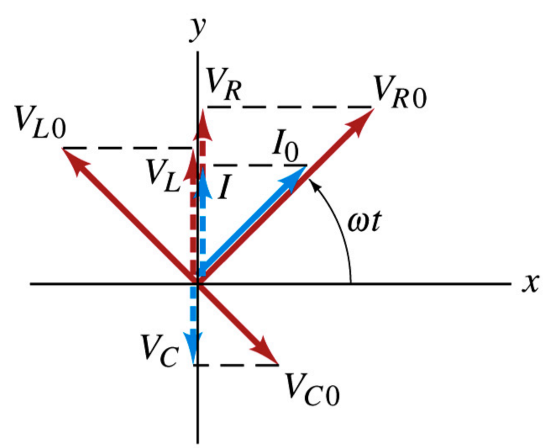
(b)

- Thus, the voltages (y-projections) are

$$V_R = V_{R0} \sin \varpi t$$

$$V_L = V_{L0} \sin\left(\varpi t + 90^\circ\right)$$

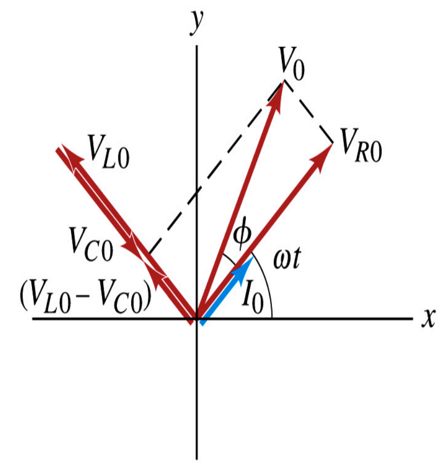
$$V_C = V_{C0} \sin\left(\varpi t - 90^\circ\right)$$



(c)

# AC Circuit w/ LRC

- Since the sum of the projections of the three vectors on the y axis is equal to the projection of their sum.
  - The sum of the projections represents the instantaneous voltage across the whole circuit which is the source voltage
  - So we can use the sum of all vectors as the representation of the peak source voltage $V_0$.




- $V_0$ forms an angle f to $V_{R0}$ and rotates together with the other vectors as a function of time, $V = V_0 \sin(\varpi t + \phi)$
- We determine the total impedance Z of the circuit defined by the relationship $V_{rms} = I_{rms} Z$ or $V_0 = I_0 Z$
- From Pythagorean theorem, we obtain

$$V_0 = \sqrt{V_{R0}^2 + (V_{L0} - V_{C0})^2} = \sqrt{I_0^2 R^2 + I_0^2 (X_L - X_C)^2} = I_0 \sqrt{R^2 + (X_L - X_C)^2} = I_0 Z$$

- Thus the total impedance is $Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(\varpi L - \frac{1}{\varpi C}\right)^2}$

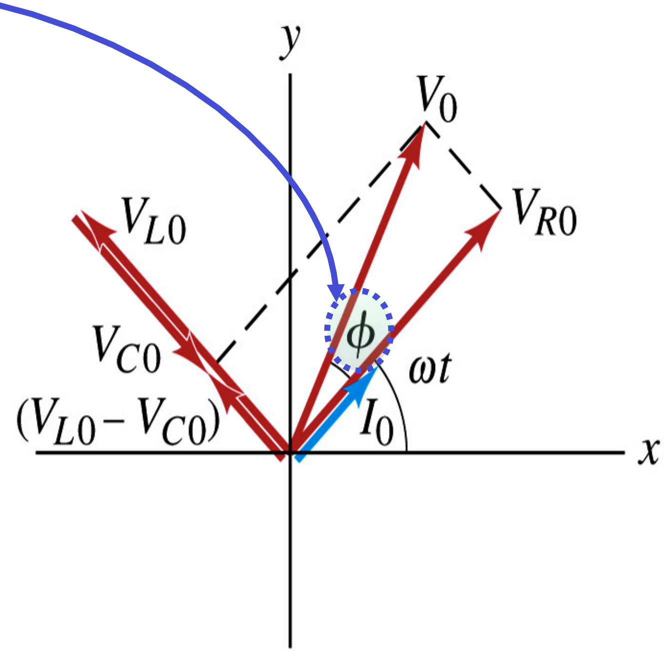
# AC Circuit w/ LRC

- The phase angle $\phi$ is

$$\tan\phi = \frac{V_{L0} - V_{C0}}{V_{R0}} = \frac{I_0\left(X_L - X_C\right)}{I_0 R} = \frac{\left(X_L - X_C\right)}{R}$$

- or

$$\cos\phi = \frac{V_{R0}}{V_0} = \frac{I_0 R}{I_0 Z} = \frac{R}{Z}$$




- What is the power dissipated in the circuit?
  - Which element dissipates the power?
  - Only the resistor
- The average power is $\overline{P} = I_{rms}^2 R$
  - Since $R = Z\cos\phi$
  - We obtain $$\overline{P} = I_{rms}^2 Z\cos\phi = I_{rms} V_{rms} \cos\phi$$
  - The factor $\cos\phi$ is referred as the power factor of the circuit
  - For a pure resistor, $\cos\phi=1$ and $\overline{P} = I_{rms} V_{rms}$
  - For a capacitor or inductor alone $\phi=-90^o$ or $+90^o$, so $\cos\phi=0$ and $\overline{P} = 0.$