PHYS 3313 – Section 001
Lecture #5

Wednesday, Sept. 12, 2012
Dr. Jaehoon Yu

- Spacetime Diagram& Invariants
- The Doppler Effect
- Relativistic Momentum and Energy
- Relationship between relativistic quantities
- Binding energy

Announcements

- Reading assignments: CH 2.10 (special topic), 2.13 and 2.14
  - Please go through eq. 2.45 through eq. 2.49 and example 2.9
- Reminder for homework #1
  - chapter 2 end of the chapter problems
  - 17, 21, 23, 24, 32, 59, 61, 66, 68, 81 and 96
  - Due is by the beginning of the class, Monday, Sept. 17
  - Work in study groups together with other students but PLEASE do write your answer in your own way!
- Colloquium today: Physics faculty research expo
  - Drs. Brandt, Deng, Farbin, Liu, Veervatina and Zhang

Special Project #2

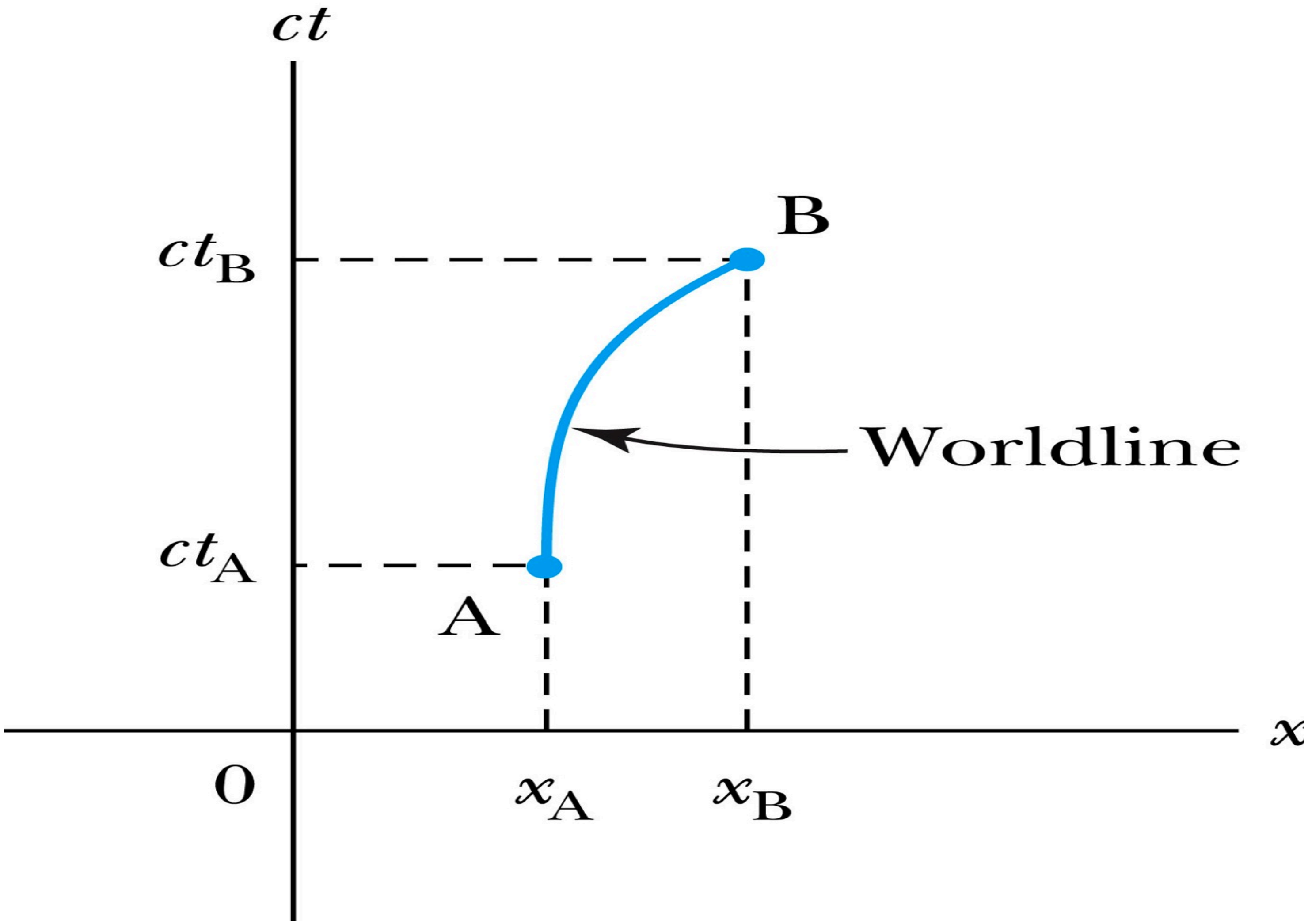
1. Derive the three Lorentz velocity transformation equations. (10 points)
2. Derive the three reverse Lorentz velocity transformation equations. (10 points)
3. Prove that the spacetime invariant quantity $s^2=x^2-(ct)^2$ is indeed invariant, i.e. $s^2=s'^2$, in Lorentz Transformation. (5 points)
4. You must derive each one separately starting from the Lorentz spatial coordinate transformation equations to obtain any credit.
   - Just switching the signs and primes will not cut!
   - Must take the simplest form of the equations, using β and γ.
5. You MUST have your own, independent answers to the above three questions even if you worked together with others. All those who share the answers will get 0 credit if copied.

- Due for the submission is Wednesday, Sept. 19!

Spacetime

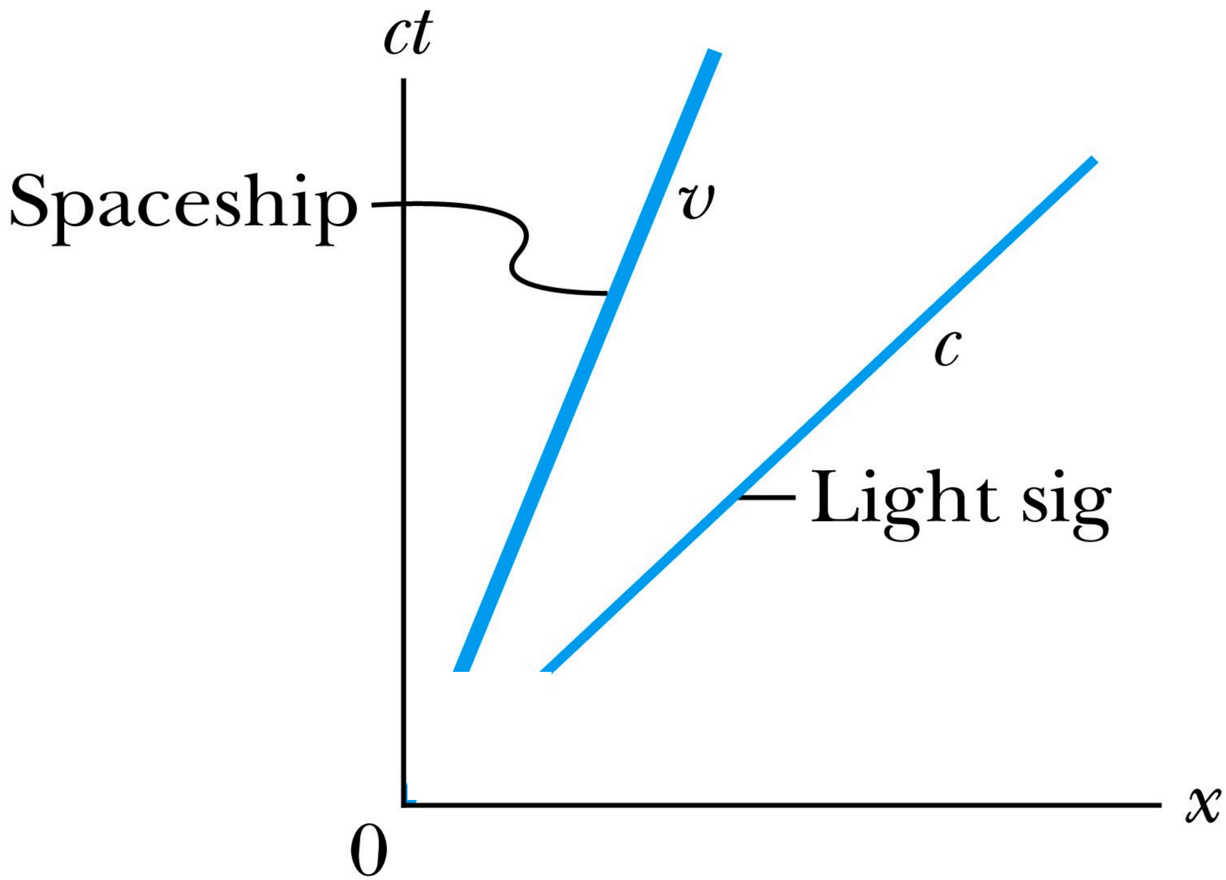
- When describing events in relativity, it is convenient to represent events on a **spacetime** diagram.
- In this diagram one spatial coordinate *x* specifies position and instead of time *t*, *ct* is used as the other coordinate so that both coordinates will have dimensions of length.
- Spacetime diagrams were first used by H. Minkowski in 1908 and are often called **Minkowski diagrams**. Paths in Minkowski spacetime are called **worldlines**.

Spacetime Diagram

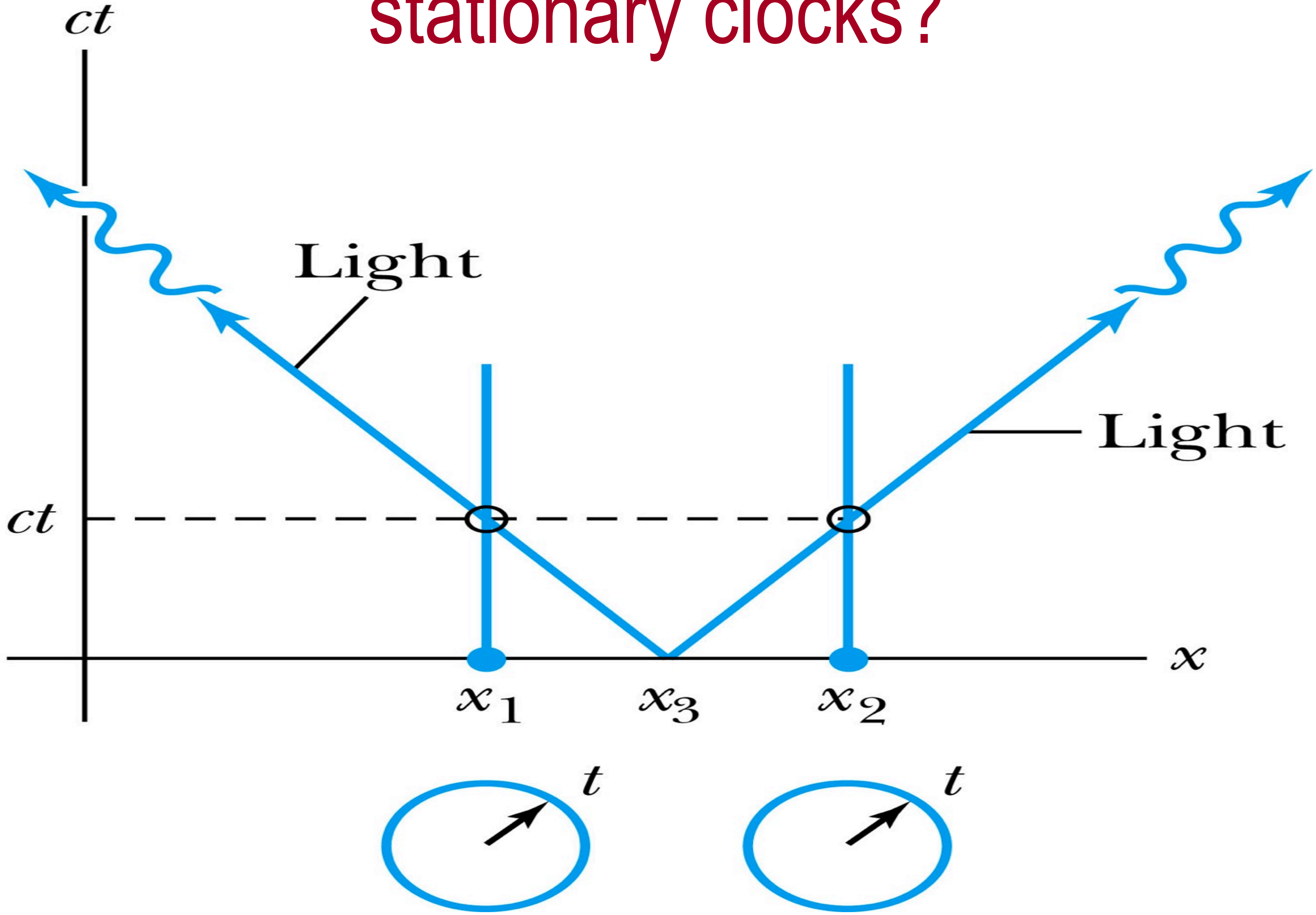


Particular Worldlines

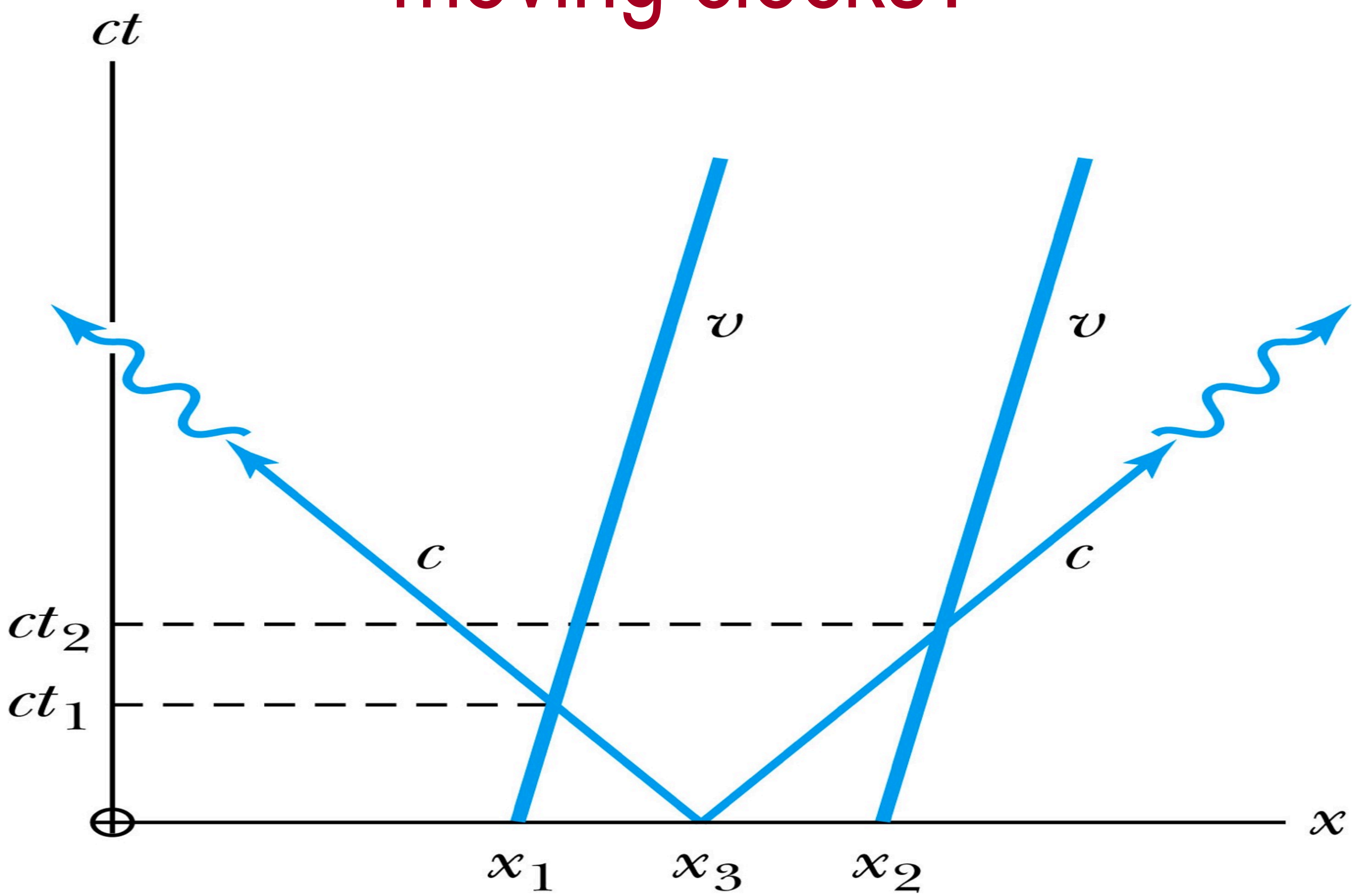
- How does the worldline for a spaceship running at the velocity v(<c) look?
- How does the worldline for light signal look?



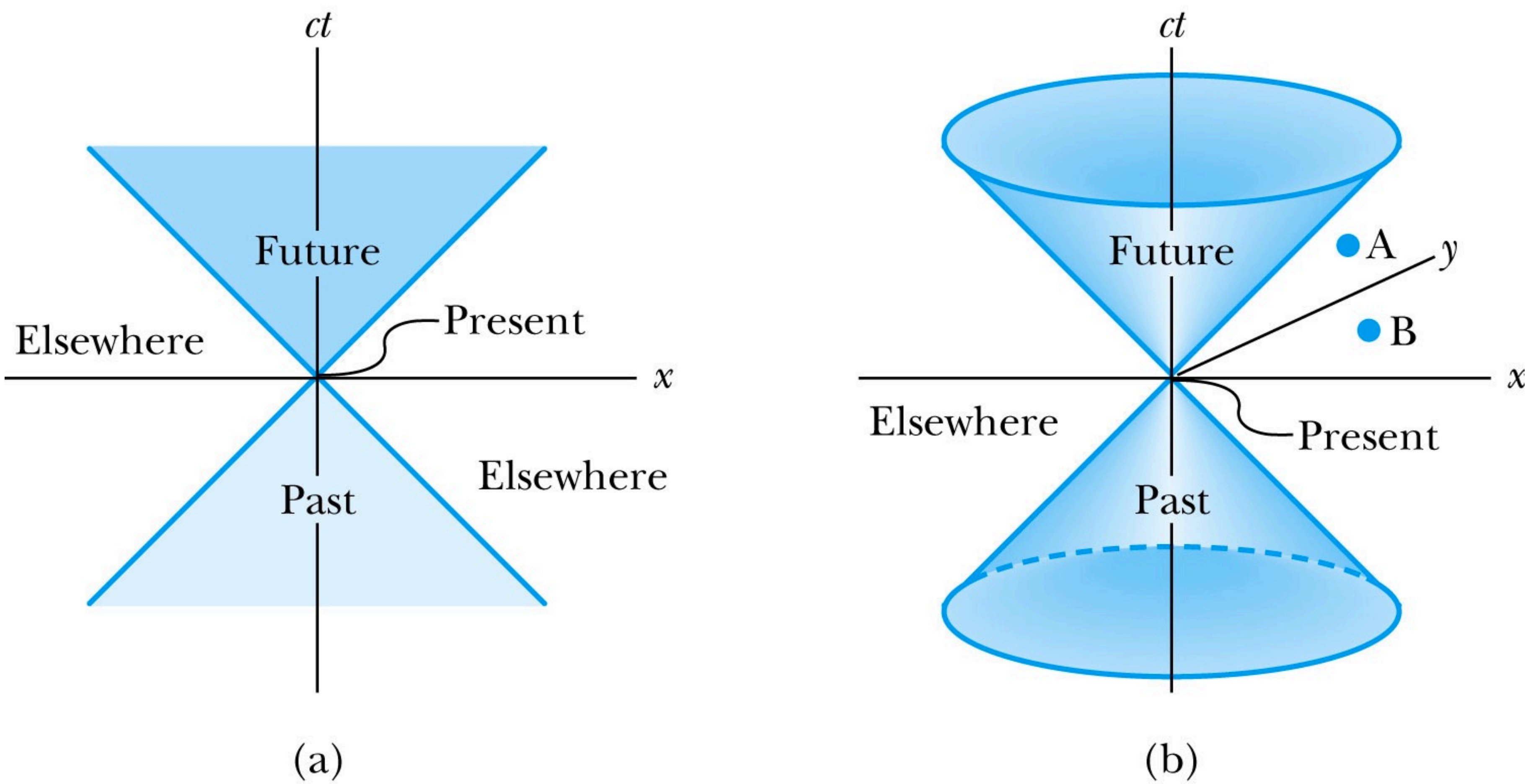
How about time measured by two stationary clocks?



How about time measured by moving clocks?



The Light Cone



Invariant Quantities: The Spacetime Interval

- Since all observers "see" the same speed of light, then all observers, regardless of their velocities, must see spherical wave fronts. Thus the quantity:

$$s^2 = x^2 - c^2t^2 = (x')^2 - c^2(t')^2 = (s')^2$$

  is invariant.

- For any two events, the spacetime interval $\Delta s^2 = \Delta x^2 - \Delta(ct)^2$ between the two events is **invariant** in any inertial frame.

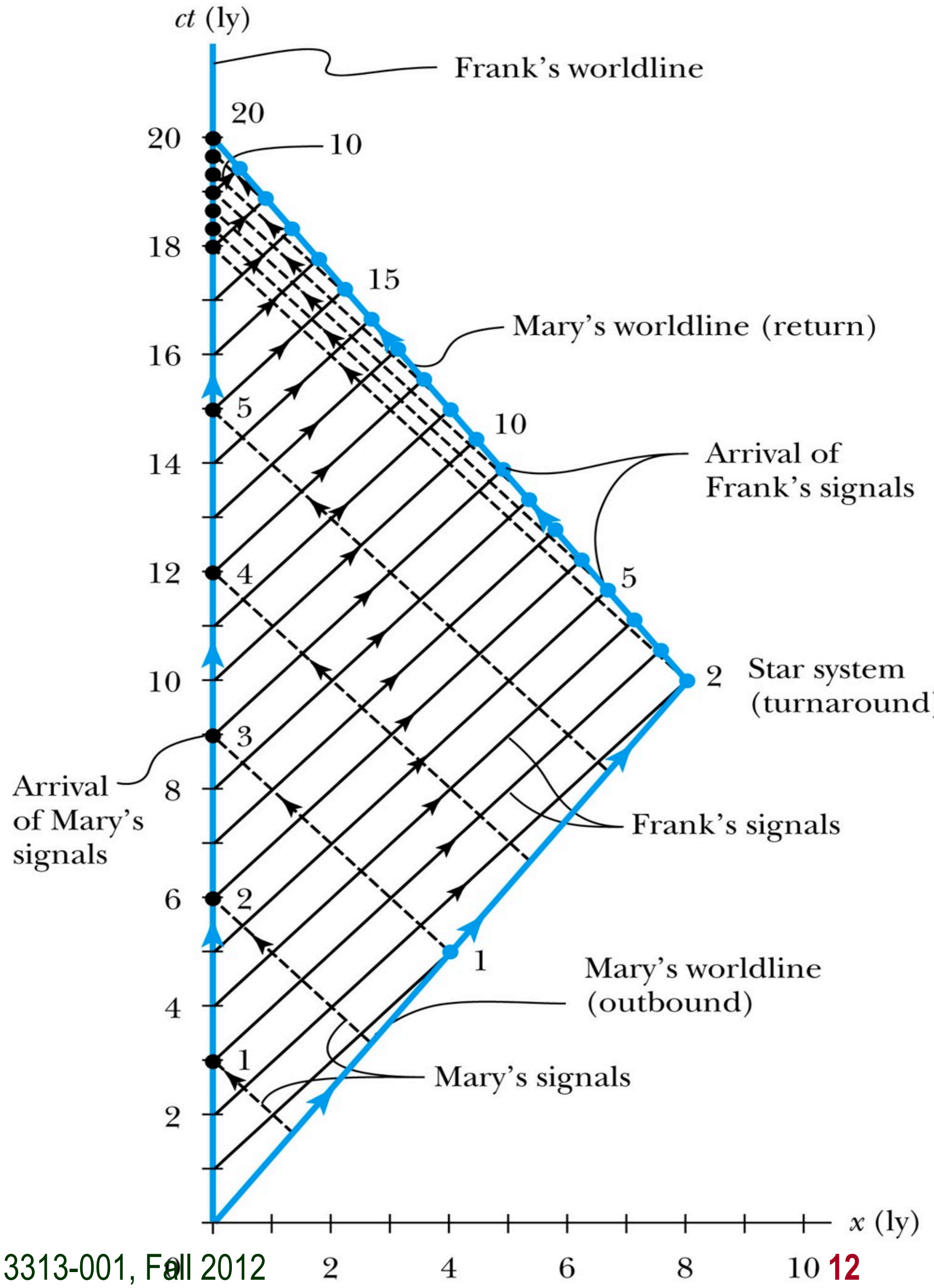
Spacetime Invariants

There are three possibilities for the invariant quantity Δs^2:

$$\Delta s^2=\Delta x^2-\Delta(ct)^2$$

1) $\Delta s^2 = 0$: $\Delta x^2 = c^2\Delta t^2$: **lightlike** separation
   – Two events can be connected only by a light signal.
2) $\Delta s^2 > 0$: $\Delta x^2 > c^2\Delta t^2$: **spacelike** separation
   – No signal can travel fast enough to connect the two events. The events are not causally connected!!
3) $\Delta s^2 < 0$: $\Delta x^2 < c^2\Delta t^2$: **timelike** separation
   – Two events can be causally connected.
   – These two events cannot occur simultaneously!

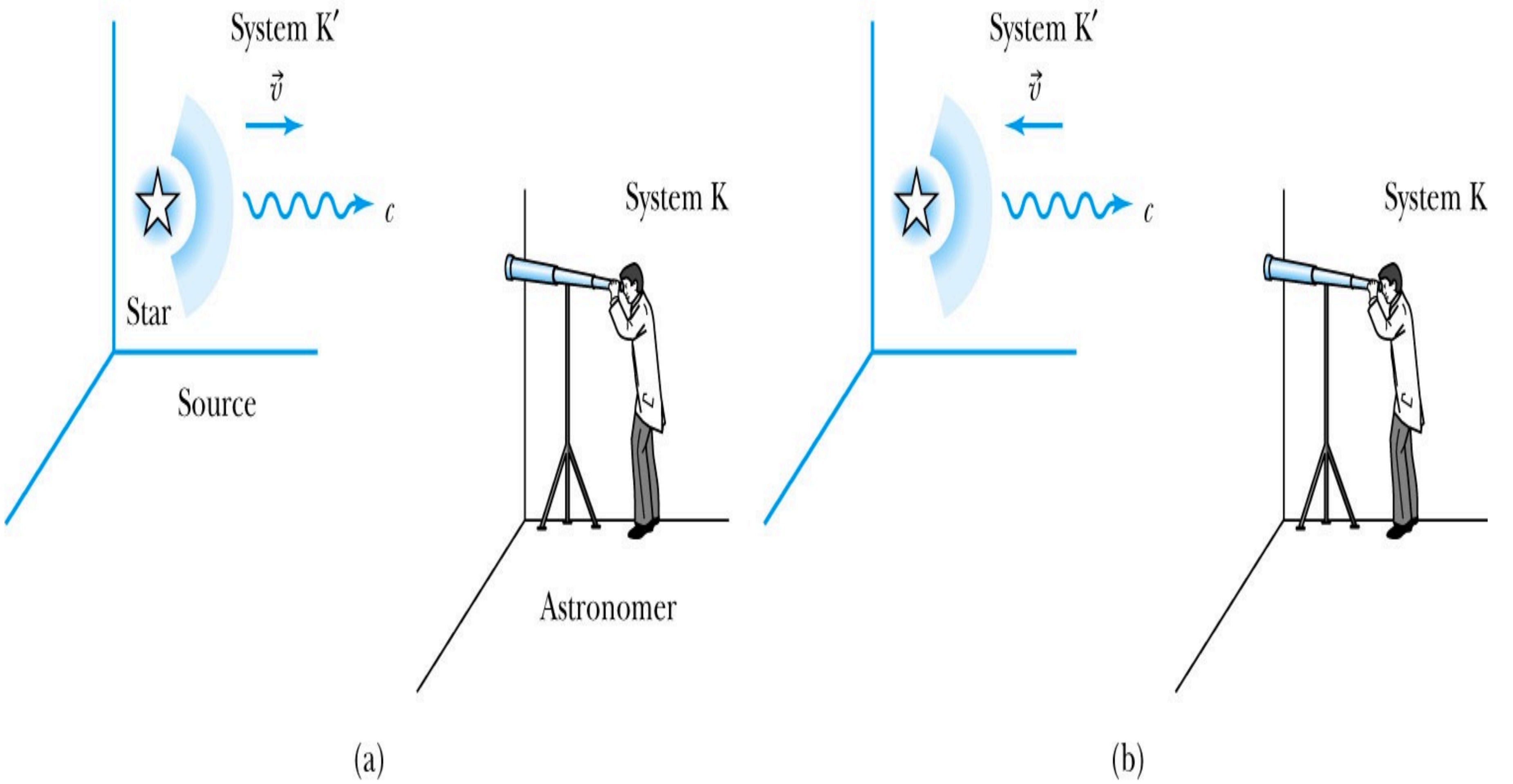
The Twin Paradox in Space-Time



The Doppler Effect

- The Doppler effect of sound
  - *increased frequency* of sound as a source approaches a receiver
  - *decreased frequency* as the source recedes.
- Also, the same change in sound frequency occurs when the source is fixed and the receiver is moving.
  - The change in frequency of the sound wave depends on whether the source or receiver is moving.
- Does this violate the principle of relativity?
  - No
  - Why not?
  - Sounds wave is in a special frame of media such as air, water, or a steel plate in order to propagate;
- Light does not need a medium to propagate!

Recall the Doppler Effect



The Relativistic Doppler Effect

Consider a source of light (a star) and a receiver (an astronomer) approaching one another with a relative velocity *v*.

1) Consider the receiver in system K and the light source in system K' moving toward the receiver with velocity *v*.
2) The source emits *n* waves during the time interval *T*.
3) Because the speed of light is always *c* and the source is moving with velocity *v*, the total distance between the front and rear of the wave transmitted during the time interval *T* is:

$$\text{Length of wave train} = cT - vT$$

The Relativistic Doppler Effect (con't)

Because there are *n* waves emitted by the source in time period T, the wavelength measured by the stationary receiver is

$$\lambda = \frac{cT - vT}{n}$$

And the resulting frequency measured by the receiver is

$$f = \frac{cn}{cT - vT}$$

The number of waves emitted in the moving frame of the source is $n = f_0 T'_0$ with the proper time $T'_0 = T/\gamma$ we obtain the measured frequency by the receiver as

$$f = \frac{cf_0 T/\gamma}{cT - vT} = \frac{1}{1 - v/c}\frac{f_0}{\gamma} = \frac{\sqrt{1-\beta^2}}{1-\beta} f_0 = \sqrt{\frac{(1-\beta)(1+\beta)}{(1-\beta)^2}} f_0 = \sqrt{\frac{1+\beta}{1-\beta}} f_0$$

Results of Relativistic Doppler Effect

When source/receiver is approaching with $\beta = v/c$ the resulting frequency is

$$f = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} f_0$$

Higher than the actual source's frequency, blue shift!!

When source/receiver is receding with $\beta = v/c$ the resulting frequency is

$$f = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} f_0$$

Lower than the actual source's frequency, red shift!!

If we use $+\beta$ for approaching source/receiver and $-\beta$ for receding source/receiver, relativistic Doppler Effect can be expressed

$$\boxed{f = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} f_0}$$

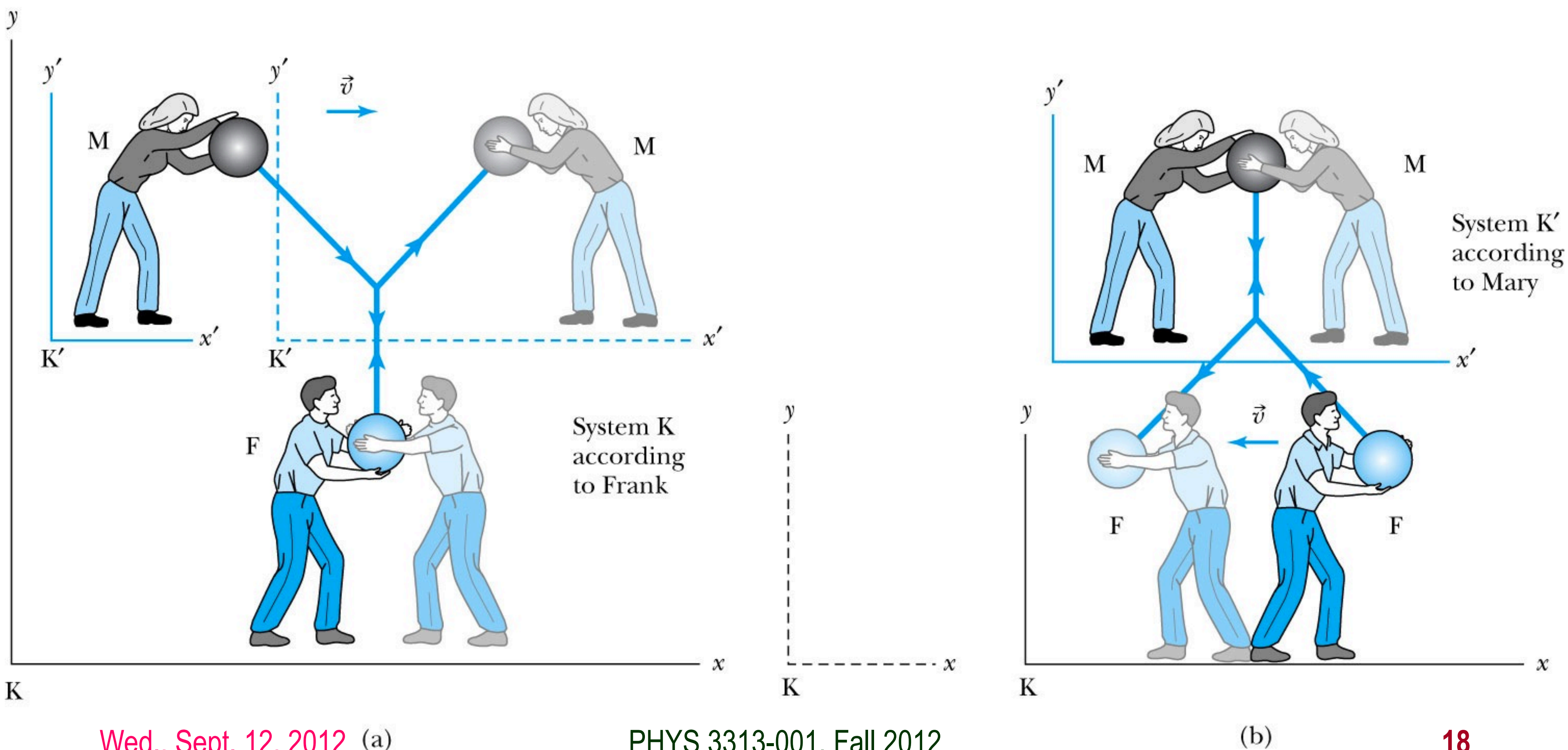
What can we use this for?

Relativistic Momentum

Most fundamental principle used here is the momentum conservation!

Frank is at rest in system K holding a ball of mass *m*.

Mary holds a similar ball in system K' that is moving in the *x* direction with velocity *v* with respect to system K.



Relativistic Momentum

- If we use the definition of momentum, the momentum of the ball thrown by Frank is entirely in the *y* direction

$$p_{Fy} = mu_0$$

- The change of momentum as observed by Frank is

$$\Delta p_F = \Delta p_{Fy} = -2mu_0$$

- Mary measures the initial velocity of her own ball to be

$$u'_{Mx} = 0 \text{ and } u'_{My} = -u_0.$$

- In order to determine the velocity of Mary's ball as measured by Frank we use the velocity transformation equations:

$$u_{Mx} = v \qquad u_{My} = -u_0\sqrt{1 - v^2/c^2}$$

Relativistic Momentum

Before the collision, the momentum of Mary's ball as measured by Frank (the **F**ixed frame) with the Lorentz velocity X-formation becomes

$$p_{Mx} = mv \qquad p_{My} = -mu_0\sqrt{1 - v^2/c^2}$$

For a perfectly elastic collision, the momentum after the collision is

$$p_{Mx} = mv \qquad p_{My} = +mu_0\sqrt{1 - v^2/c^2}$$

Thus the change in momentum of Mary's ball according to Frank is

$$\Delta p_M = \Delta p_{My} = 2mu_0\sqrt{1-\beta^2} \neq -\Delta p_{Fy}$$

OMG! The linear momentum is not conserved even w/o external force!!

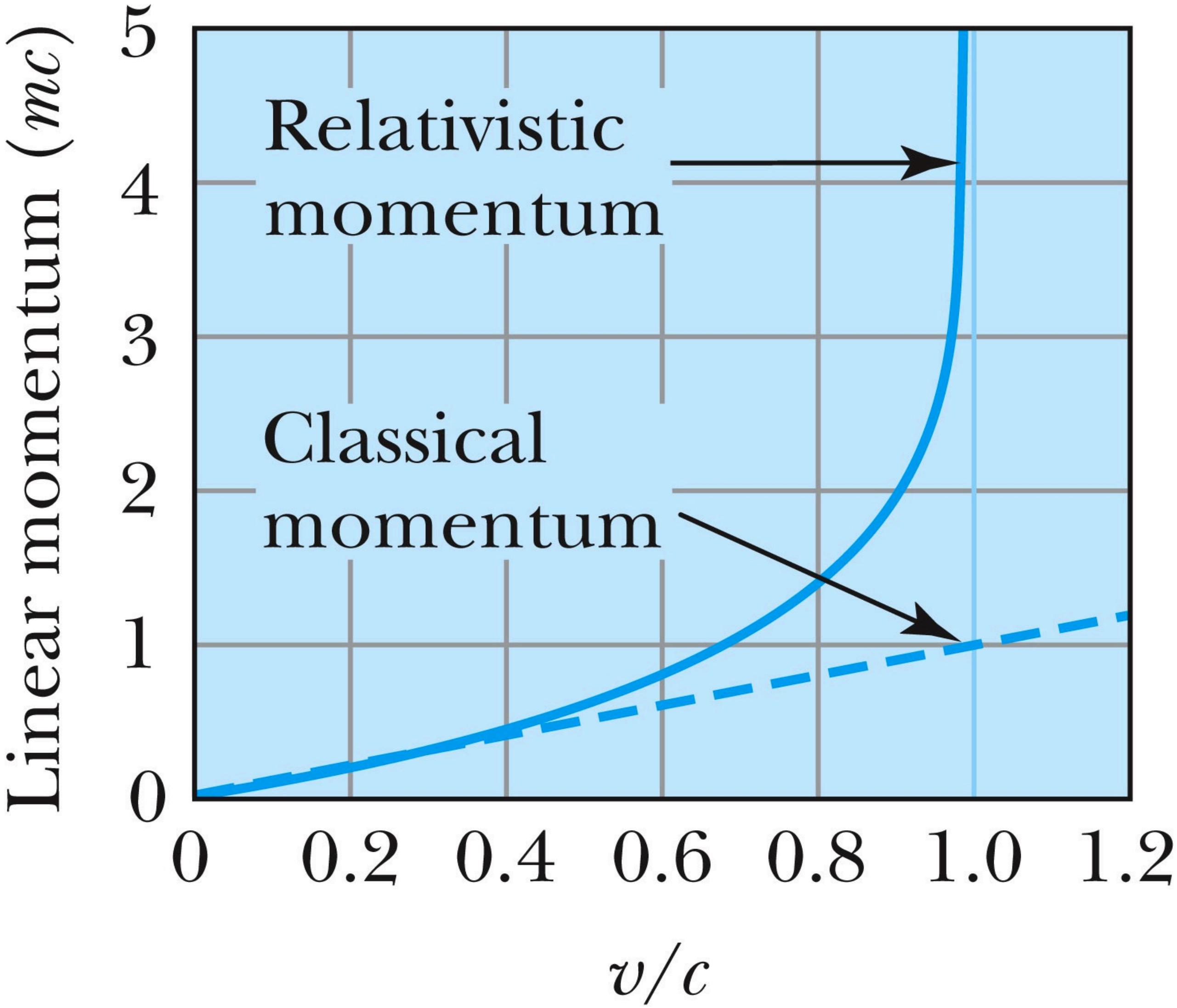
What do we do?

➔Redefine the momentum in a fashion $\quad \vec{p} = m\vec{u} = m\dfrac{d\left(\gamma_u \vec{r}\right)}{dt} = m\gamma_u \vec{u}$

➔Something else has changed. Mass is now, mγ!! The relativistic mass!!

➔Mass as the fundamental property of matter is called the "rest mass", m_0!

Relativistic and Classical Linear Momentum



How do we keep momentum conserved in a relativistic case?

Redefine the classical momentum in the form:

$$\vec{p} = \Gamma(u) m\vec{u} = \frac{1}{\sqrt{1 - u^2/c^2}} m\vec{u}$$

This $\Gamma(u)$ is different than the γ factor since it uses the particle's speed u

➔ What? How does this make sense?

➔ Well the particle itself is moving with relativistic speed, thus that must impact the measurements by the observer in rest frame!!

Now, the agreed value of the momentum in all frames is:

$$\vec{p} = m\frac{d\vec{r}}{d\tau} = m\frac{d\vec{r}}{dt}\frac{dt}{d\tau} = m\vec{u}\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} m\vec{u}$$

Resulting in the new relativistic definition of the momentum:

$$\vec{p} = m\gamma\vec{u}$$

Relativistic Energy

- Due to the new idea of relativistic mass, we must now redefine the concepts of work and energy.
  - Modify Newton's second law to include our new definition of linear momentum, and force becomes:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\gamma m\vec{u}\right) = \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1-u^2/c^2}}\right)$$

- The work *W* done by a force **F** to move a particle from rest to a certain kinetic energy is

$$W = K = \int \frac{d}{dt}\left(\gamma m\vec{u}\right) \cdot \vec{u}dt$$

- Resulting relativistic kinetic energy becomes

$$K = \int_0^{\gamma u} ud\left(\gamma u\right) = \gamma mc^2 - mc^2 = \left(\gamma - 1\right)mc^2$$

- Why doesn't this look anything like the classical KE?

Big note on Relativistic KE

- Only $K = (\gamma - 1)mc^2$ is right!
- $K = \frac{1}{2}mu^2$ and $K = \frac{1}{2}\gamma mu^2$ are wrong!

Total Energy and Rest Energy

Rewriting the relativistic kinetic energy:

$$\gamma mc^2 = \frac{mc^2}{\sqrt{1-u^2/c^2}} = K + mc^2$$

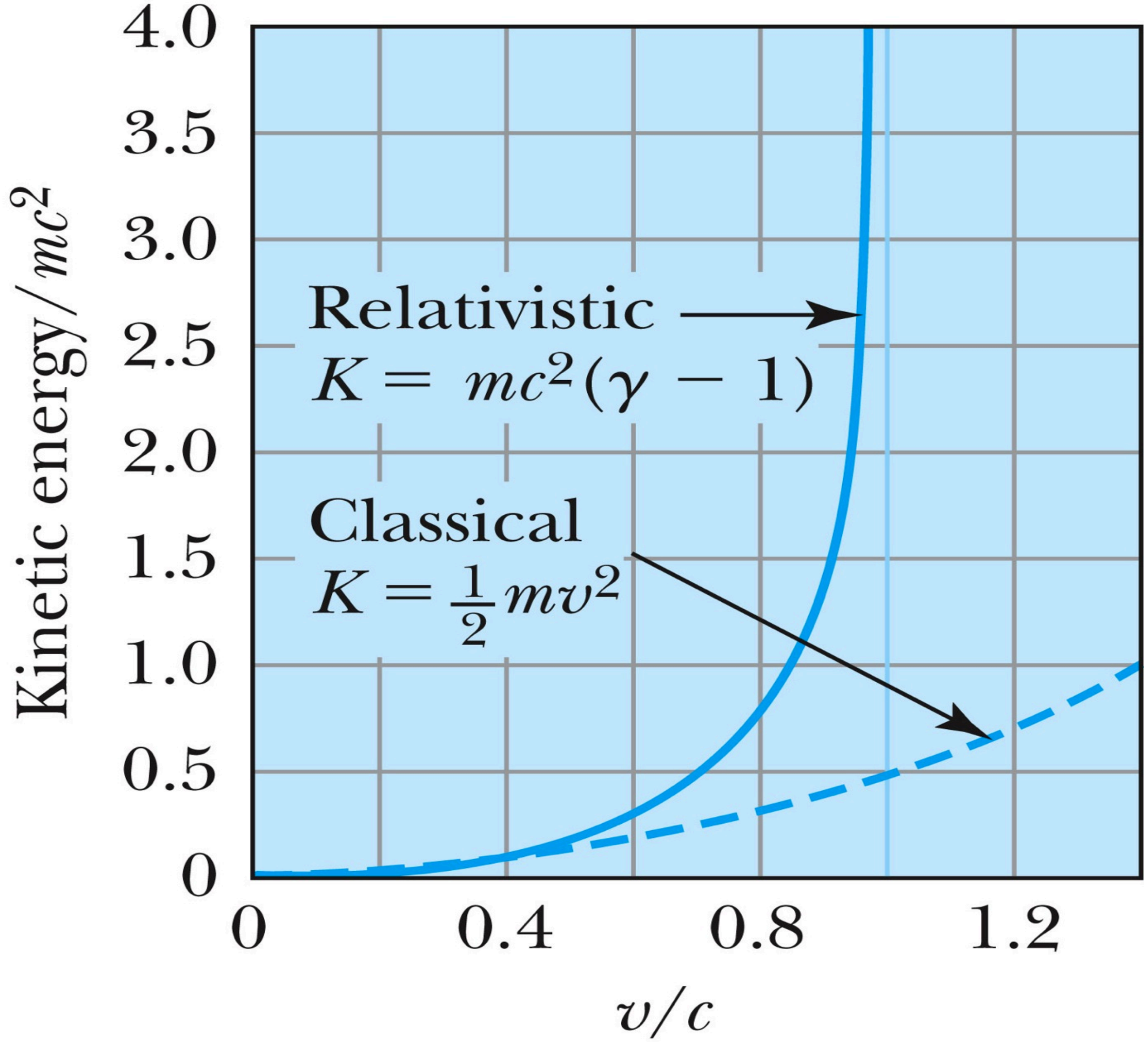
The term mc^2 is called the rest energy and is denoted by E_0.

$$E_0 = mc^2$$

The sum of the kinetic energy and rest energy is interpreted as the total energy of the particle.

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-u^2/c^2}} = \frac{E_0}{\sqrt{1-u^2/c^2}} = K + E_0$$

Relativistic and Classical Kinetic Energies



Relationship of Energy and Momentum

$$p = \gamma m u = \frac{mu}{\sqrt{1 - u^2/c^2}}$$

We square this result, multiply by c^2, and rearrange the result.

$$p^2c^2 = \gamma^2 m^2 u^2 c^2 = \gamma^2 m^2 c^4 \left(\frac{u^2}{c^2}\right) = \gamma^2 m^2 c^4 \beta^2$$

$$\beta^2 = 1 - \frac{1}{\gamma^2} \Rightarrow p^2c^2 = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right) = \gamma^2 m^2 c^4 - m^2 c^4$$

Rewrite

$$p^2c^2 = E^2 - E_0^2$$

Rewrite

$$E^2 = p^2c^2 + E_0^2 = p^2c^2 + m^2c^4$$

Massless Particles have a speed equal to the speed of light c

- Recall that a photon has "zero" rest mass and the equation from the last slide reduces to: E = pc and we may conclude that:

$$E = \gamma mc^2 = pc = \gamma muc$$

- Thus the velocity, u, of a massless particle must be c since, as $m \to 0, \gamma \to \infty$ and it follows that: u = c.

Units of Work, Energy and Mass

- The work done in accelerating a charge through a potential difference is $W = qV$.
  - For a proton, with the charge $e = 1.602 \times 10^{-19}$ C being accelerated across a potential difference of 1 V, the work done is

$$1\ \text{eV} = 1.602 \times 10^{-19}\ \text{J}$$

$$W = (1.602 \times 10^{-19})(1\ \text{V}) = 1.602 \times 10^{-19}\ \text{J}$$

- eV is also used as a unit of energy.

Other Units

1) Rest energy of a particle:
   Example: Rest energy, E_0, of proton

$$E_0(\text{proton}) = (1.67\times10^{-27}\text{ kg})(3\times10^{8}\text{ m/s})^2 = 1.50\times10^{-10}\text{ J}$$

$$= 1.50\times10^{-10}\text{ J}\,\frac{1\text{ eV}}{1.602\times10^{-19}\text{ J}} = 9.38\times10^{8}\text{ eV}$$

2) **Atomic mass unit** (amu): Example: carbon-12

$$\text{Mass }(^{12}\text{C atom}) = \frac{12\text{ g/mol}}{6.02\times10^{23}\text{ atoms/mol}}$$

$$= 1.99\times10^{-23}\text{ g/atom}$$

$$\text{Mass }(^{12}\text{C atom}) = 1.99\times10^{-26}\text{ kg} = 12\text{ u/atom}$$

Binding Energy

- The potential energy associated with the force keeping the system together ➔ E_B.
- The difference between the rest energy of the individual particles and the rest energy of the combined bound system.

$$M_{\text{bound system}}c^2 + E_B = \sum_i m_i c^2$$

$$E_B = \sum_i m_i c^2 - M_{\text{bound system}}c^2$$