

PHYS 3313 – Section 001

Lecture #12

Monday, Oct. 15, 2012

Dr. Jaehoon Yu

- The Schrödinger Wave Equation
- Time-Independent Schrödinger Wave Equation
- Probability Density
- Wave Function Normalization
- Expectation Values
- Operators – Position, Momentum and Energy



Announcements

- Reminder Homework #4
 - End of chapter problems on CH5: 8, 10, 16, 24, 26, 36 and 47
 - Due: This Wednesday, Oct. 17
- Reading assignments
 - CH6.1 – 6.7 + the special topic
- Colloquium this week
 - 4pm, Wednesday, Oct. 17, SH101
 - Drs. Musielak and Fry of UTA



**Physics Department
The University of Texas at Arlington
COLLOQUIUM**

***Fundamental Theories of Physics in
Flat and Curved Space-Time***

Dr. John Fry and Dr. Zdzislaw Musielak

Department of Physics

University of Texas at Arlington

4:00 pm Wednesday October 17, 2012 room 101 SH

Abstract:

As of today, there is no commonly accepted theory that explains the origin and nature of Dark Matter and Dark Energy, and also accounts for the physical effects near the Big Bang singularity. To search for such theory, we developed a novel method that allows us to formulate fundamental theories of elementary particles in flat and curved space-time. In this talk, we shall present our method and use it obtain the original Klein-Gordon equation for scalar state functions as well as its generalization to higher derivatives and to spinor state functions. Possible applications of the generalized Klein-Gordon equations to Dark Matter and Dark Energy will be discussed. We shall also describe our recently developed extension of the method to curved space-time of General Relativity. The corresponding Klein-Gordon equation in space-time of a given curvature will be presented and its possible applications will be discussed. The talk will be concluded with a progress report on our attempts to formulate a new physical theory that is required to be valid in the vicinity of the Big Bang singularity.

Refreshments will be served at 3:30p.m in the Physics Lounge

Special project #5

- Prove that the wave function $\Psi = A[\sin(kx - \omega t) + i\cos(kx - \omega t)]$ is a good solution for the time-dependent Schrödinger wave equation. Do NOT use the exponential expression of the wave function. (10 points)
- Determine whether or not the wave function $\Psi = Ae^{-\alpha|x|}$ satisfy the time-dependent Schrödinger wave equation. (10 points)
- Due for this special project is Monday, Oct. 22.
- You MUST have your own answers!



The Schrödinger Wave Equation

- The Schrödinger wave equation in its time-dependent form for a particle of energy E moving in a potential V in one dimension is

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V\Psi(x,t)$$

- The extension into three dimensions is

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} \right) + V\Psi(x,y,z,t)$$

- where $i = \sqrt{-1}$ is an imaginary number

Ex 6.1: Wave equation and Superposition

The wave equation must be linear so that we can use the superposition principle to. Prove that the wave function in Schrodinger equation is linear by showing that it is satisfied for the wave equation $\Psi(x,t) = a\Psi_1(x,t) + b\Psi_2(x,t)$ where a and b are constants and $\Psi_1(x,t)$ and $\Psi_2(x,t)$ describe two waves each satisfying the Schrodinger Eq.

$$\Psi = a\Psi_1 + b\Psi_2 \quad i\hbar \frac{\partial \Psi_1}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_1}{\partial x^2} + V\Psi_1 \quad i\hbar \frac{\partial \Psi_2}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_2}{\partial x^2} + V\Psi_2$$

$$\frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial t}(a\Psi_1 + b\Psi_2) = a \frac{\partial \Psi_1}{\partial t} + b \frac{\partial \Psi_2}{\partial t}$$

$$\frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial x}(a\Psi_1 + b\Psi_2) = a \frac{\partial \Psi_1}{\partial x} + b \frac{\partial \Psi_2}{\partial x} \quad \Rightarrow \quad \frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial}{\partial x} \left(a \frac{\partial \Psi_1}{\partial x} + b \frac{\partial \Psi_2}{\partial x} \right) = a \frac{\partial^2 \Psi_1}{\partial x^2} + b \frac{\partial^2 \Psi_2}{\partial x^2}$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi \quad \xrightarrow{\text{Rearrange terms}} \quad i\hbar \frac{\partial \Psi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} - V\Psi = \left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V \right) \Psi = 0$$

$$i\hbar \frac{\partial \Psi}{\partial t} = i\hbar \left(a \frac{\partial \Psi_1}{\partial t} + b \frac{\partial \Psi_2}{\partial t} \right) = -\frac{\hbar^2}{2m} \left(a \frac{\partial^2 \Psi_1}{\partial x^2} + b \frac{\partial^2 \Psi_2}{\partial x^2} \right) + V(a\Psi_1 + b\Psi_2)$$

$$a \left(i\hbar \frac{\partial \Psi_1}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \Psi_1}{\partial x^2} - V\Psi_1 \right) = -b \left(i\hbar \frac{\partial \Psi_2}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \Psi_2}{\partial x^2} - V\Psi_2 \right) = 0$$

General Solution of the Schrödinger Wave Equation

- The general form of the solution of the Schrödinger wave equation is given by:

$$\Psi(x, t) = Ae^{i(kx - \omega t)} = A[\cos(kx - \omega t) + i \sin(kx - \omega t)]$$

- which also describes a wave propagating in the x direction. In general the amplitude may also be complex. *This is called the wave function of the particle.*
- The wave function is also **not** restricted to being real. Only the physically measurable quantities (or observables) must be real. These include the probability, momentum and energy.

Ex 6.2: Solution for wave equation

Show that $Ae^{i(kx-\omega t)}$ satisfies the time-dependent Schrodinger wave Eq.

$$\Psi = Ae^{i(kx-\omega t)} \quad \Rightarrow \quad \frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial t} \left(Ae^{i(kx-\omega t)} \right) = -iA\omega e^{i(kx-\omega t)} = -i\omega\Psi$$

$$\frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial x} \left(Ae^{i(kx-\omega t)} \right) = iAke^{i(kx-\omega t)} = ik\Psi$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial}{\partial x} (ik\Psi) = ik \frac{\partial}{\partial x} (\Psi) = ik \left(iAke^{i(kx-\omega t)} \right) = -Ak^2 e^{i(kx-\omega t)} = -k^2\Psi$$

$$i\hbar \frac{\partial \Psi}{\partial t} = i\hbar(-i\omega\Psi) = \hbar\omega\Psi = -\frac{\hbar^2}{2m}(-k^2\Psi) + V\Psi \quad \left(\hbar\omega - \frac{\hbar^2 k^2}{2m} - V \right) \Psi = 0$$

$$\text{The Energy: } E = hf = h \left(\frac{\omega}{2\pi} \right) = \hbar\omega \quad = \left(E - \frac{p^2}{2m} - V \right) = 0$$

$$\text{The wave number: } k = \frac{2\pi}{\lambda} = \frac{2\pi}{h/p} = \frac{2\pi p}{h} = \frac{p}{\hbar} \quad \Rightarrow \quad \text{The momentum: } p = \hbar k$$

$$\text{From the energy conservation: } E = K + V = \frac{p^2}{2m} + V \quad \Rightarrow \quad E - \frac{p^2}{2m} - V = 0$$

So $Ae^{i(kx-\omega t)}$ is a good solution and satisfies Schrodinger Eq.

Ex 6.3: Bad solution for wave equation

Determine $\Psi(x,t) = A \sin(kx - \omega t)$ is an acceptable solution for the time-dependent Schrodinger wave Eq.

$$\Psi = A \sin(kx - \omega t) \quad \Rightarrow \quad \frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial t} (A \sin(kx - \omega t)) = -A\omega \cos(kx - \omega t)$$

$$\frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial x} (A \sin(kx - \omega t)) = kA \cos(kx - \omega t)$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial}{\partial x} (kA \cos(kx - \omega t)) = -k^2 A \sin(kx - \omega t) = -k^2 \Psi$$

$$i\hbar \left(-\omega \cos(kx - \omega t) \right) = -\frac{\hbar^2}{2m} \left(-k^2 \sin(kx - \omega t) \right) + V \sin(kx - \omega t)$$

$$-i\hbar\omega \cos(kx - \omega t) = \left(\frac{\hbar^2 k^2}{2m} + V \right) \sin(kx - \omega t)$$
$$\quad \Rightarrow \quad -iE \cos(kx - \omega t) = \left(\frac{p^2}{2m} + V \right) \sin(kx - \omega t)$$

This is not true in all x and t . So $\Psi(x,t) = A \sin(kx - \omega t)$ is not an acceptable solution for Schrodinger Eq.

Normalization and Probability

- The probability $P(x) dx$ of a particle being between x and $x + dx$ was given in the equation

$$P(x)dx = \Psi^*(x,t)\Psi(x,t)dx$$

- Here Ψ^* denotes the complex conjugate of Ψ
- The probability of the particle being between x_1 and x_2 is given by

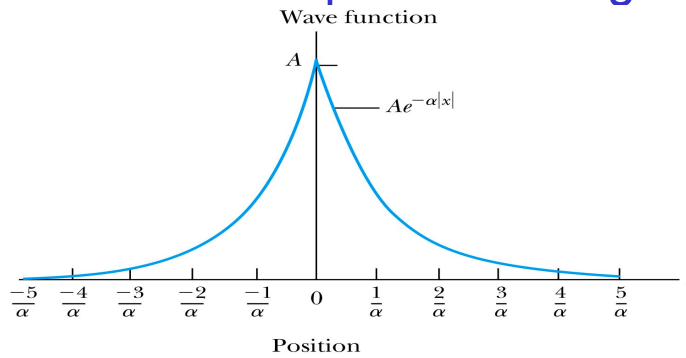
$$P = \int_{x_1}^{x_2} \Psi^* \Psi dx$$

- The wave function must also be normalized so that the probability of the particle being somewhere on the x axis is 1.

$$\int_{-\infty}^{+\infty} \Psi^*(x,t)\Psi(x,t)dx = 1$$

Ex 6.4: Normalization

Consider a wave packet formed by using the wave function that $Ae^{-\alpha|x|}$, where A is a constant to be determined by normalization. Normalize this wave function and find the probabilities of the particle being between 0 and $1/\alpha$, and between $1/\alpha$ and $2/\alpha$.



$$\Psi = Ae^{-\alpha|x|}$$

Probability density

$$\begin{aligned} \int_{-\infty}^{+\infty} \Psi^* \Psi dx &= \int_{-\infty}^{+\infty} (Ae^{-\alpha|x|})^* (Ae^{-\alpha|x|}) dx = \int_{-\infty}^{+\infty} (A^* e^{-\alpha|x|}) (Ae^{-\alpha|x|}) dx = \\ &= \int_{-\infty}^{+\infty} A^2 e^{-2\alpha|x|} dx = 2 \int_0^{+\infty} A^2 e^{-2\alpha|x|} dx = \left. \frac{2A^2}{-2\alpha} e^{-2\alpha|x|} \right|_0^{+\infty} = 0 + \frac{A^2}{\alpha} = 1 \end{aligned}$$

$$A = \sqrt{\alpha}$$

Normalized Wave Function

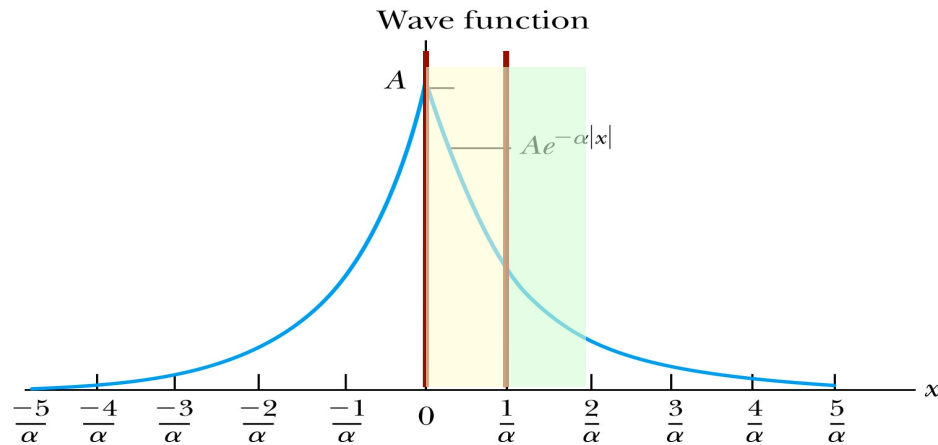
$$\Psi = \sqrt{\alpha} e^{-\alpha|x|}$$

Ex 6.4: Normalization, cont'd

Using the wave function, we can compute the probability for a particle to be with 0 to $1/\alpha$ and $1/\alpha$ to $2/\alpha$.

$$\Psi = \sqrt{\alpha} e^{-\alpha|x|}$$

For 0 to $1/\alpha$:



$$P = \int_0^{1/\alpha} \Psi^* \Psi dx = \int_0^{1/\alpha} \alpha e^{-2\alpha x} dx = \frac{\alpha}{-2\alpha} e^{-2\alpha x} \Big|_0^{1/\alpha} = -\frac{1}{2} (e^{-2} - 1) \approx 0.432$$

For $1/\alpha$ to $2/\alpha$:

$$P = \int_{1/\alpha}^{2/\alpha} \Psi^* \Psi dx = \int_{1/\alpha}^{2/\alpha} \alpha e^{-2\alpha x} dx = \frac{\alpha}{-2\alpha} e^{-2\alpha x} \Big|_{1/\alpha}^{2/\alpha} = -\frac{1}{2} (e^{-4} - e^{-2}) \approx 0.059$$

How about $2/\alpha$ to ∞ ?