PHYS 3313 – Section 001
Lecture #19

Wednesday, Nov. 14, 2012
Dr. Jaehoon Yu

- Historical Overview
- Maxwell Velocity Distribution
- Equipartition Theorem
- Classical and Quantum Statistics
- Fermi-Dirac Statistics
- Quantum Theory of Conductivity
- Bose-Einstein Statistics
- Liquid Helium

Announcements

- Reminder Homework #7
  - CH7 end of chapter problems: 7, 8, 9, 12, 17 and 29
  - Due on Monday, Nov. 19, in class
- Reading assignments
  - Entire CH8 (in particular CH8.1), CH9.4 and CH9.7
- Class is cancelled Wednesday, Nov. 21
- Please be sure to do class evaluations!
- Colloquium Wednesday
  - At 4pm, Wednesday, Nov. 14, in SH101
  - Dr. Masaya Takahashi of UT South Western Medical

Physics Department
The University of Texas at Arlington
COLLOQUIUM

MR Imaging at high magnetic field
-A challenge to new biological information from science to clinical application-

Dr. Masaya Takahashi

University of Texas Southwestern Medical Center/
Advanced Imaging Research Center/Radiology,

4:00 pm Wednesday November 14, 2012 room 101 SH

Abstract:

Recent technical progress in noninvasive imaging techniques - notably magnetic resonance imaging (MRI) - in terms of improvement in achievable signal-to-noise ratio and spatial/temporal resolution, has been overcoming several fundamental difficulties. Subsequently, there is an increase in demand to have a better diagnostic tool with which to determine the mechanism, location and stage of the diseases. An important challenge is the development of more powerful, multi-variate methods for characterization of anatomical and functional changes, predicting individual outcome and responsiveness to particular therapies on the basis of clinical and laboratory characteristics. More investigators have been applying higher magnetic field strengths (3 Tesla or higher) in research and clinical settings. Higher magnetic field strength is expected to afford higher spatial resolution and/or a decrease in the length of total scan time due to its higher signal intensity. In the first half of this lecture, we will review the advance MRI and contrast agents that are state-of-the-art at high magnetic field strength in which we hope one can take a hint in their expertise. We have been dedicated to the development of new acquisition and processing methods by means of MRI during the past years, permitting quantitative characterization of the pathophysiological change. Amide proton transfer (APT) imaging is one of the chemical exchange saturation transfer (CEST) imaging methods that are the most practical molecular MR imaging. With this method the exchange between protons of free tissue water and the amide groups (-NH) of endogenous mobile proteins and peptides is imaged. In the second half of this lecture, we will illustrate the CEST/APT imaging and its contrast agent.

Refreshments will be served at 3:30p.m in the Physics Lounge

Why is statistical physics necessary?

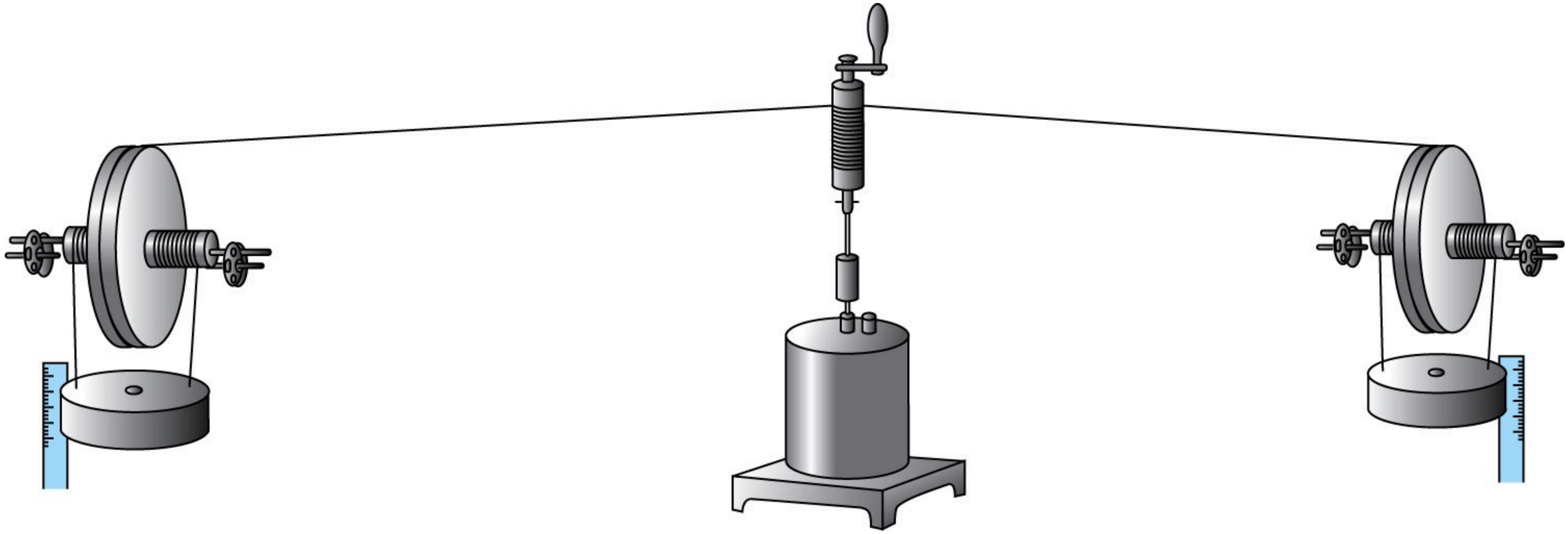
- Does physics perceive inherent uncertainty and indeterminism since everything is probabilistic?
- Statistical physics is necessary since
  - As simple problems as computing probability of coin toss is complex, so it is useful to reduce it to statistical terms
  - When the number of particles gets large, it is rather impractical to describe the motion of individual particle than describing the motion of a group of particles
  - Uncertainties are inherent as Heisenberg's uncertainty principle showed and are of relatively large scale in atomic and subatomic level
- Statistical physics necessary for atomic physics and the description of solid states which consists of many atoms

Historical Overview

- **Statistics and probability:** New mathematical methods developed to understand the Newtonian physics through the eighteenth and nineteenth centuries.
- **Lagrange** around 1790 and **Hamilton** around 1840: They added significantly to the computational power of Newtonian mechanics.
- **Pierre-Simon de Laplace** (1749-1827)
  - Had a view that it is possible to have a perfect knowledge of the universe
  - Can predict the future and the past to the beginning of the universe
  - He told Napoleon that the hypothesis of God is not necessary
  - But he made major contributions to the theory of probability
- **Benjamin Thompson (Count Rumford):** Put forward the idea of heat as merely the motion of individual particles in a substance but not well accepted
- **James Prescott Joule:** Demonstrated experimentally the mechanical equivalence of heat and energy

Joule's experiment

- Showed deterministically the equivalence of heat and energy
- Dropping weight into the water and measuring the change of temperature of the water



Historical Overview

- **James Clark Maxwell**
  - Brought the mathematical theories of probability and statistics to bear on the physical thermodynamics problems
  - Showed that distributions of an ideal gas can be used to derive the observed macroscopic phenomena
  - His electromagnetic theory succeeded to the statistical view of thermodynamics
- **Einstein:** Published a theory of Brownian motion, a theory that supported the view that atoms are real
- **Bohr:** Developed atomic and quantum theory

Maxwell Velocity Distribution

- Laplace claimed that it is possible to know everything about an ideal gas by knowing the position and velocity precisely
- There are six parameters—the position (*x*, *y*, *z*) and the velocity (v_x, v_y, v_z)—per molecule to know the position and instantaneous velocity of an ideal gas.
- These parameters make up 6D **phase space**
- The velocity components of the molecules are more important than positions, because the energy of a gas should depend only on the velocities.
- Define a **velocity distribution function** = the probability of finding a particle with velocity between $\vec{v}$ and $\vec{v}+d^3\vec{v}$ where $d^3\vec{v} = dv_x dv_x dv_x$

Maxwell Velocity Distribution

- Maxwell proved that the probability distribution function is proportional to $\exp\left(-\tfrac{1}{2}mv^2/kT\right)$

  Therefore $f\left(\vec{v}\right)d^3\vec{v} = C\exp\left(-\tfrac{1}{2}\beta mv^2\right)d^3\vec{v}$.

  where C is a proportionality constant and $\beta \equiv (kT)^{-1}$.
- Because $v^2 = v_x^2 + v_y^2 + v_z^2$,

$$f\left(\vec{v}\right)d^3\vec{v} = C\exp\left[-\tfrac{1}{2}\beta m\left(v_x^2 + v_y^2 + v_z^2\right)\right]d^3\vec{v}$$

- Rewrite this as the product of three factors (i.e. probability density).

$$g\left(v_x\right)dv_x = C'\exp\left(-\tfrac{1}{2}\beta mv_x^2\right)dv_x$$
$$g\left(v_y\right)dv_y = C'\exp\left(-\tfrac{1}{2}\beta mv_y^2\right)dv_y$$

$$g\left(v_z\right)dv_z = C'\exp\left(-\tfrac{1}{2}\beta mv_z^2\right)dv_z$$

$$f\left(\vec{v}\right)d^3\vec{v} = Cg\left(v_x\right)g\left(v_y\right)g\left(v_z\right)dv_xdv_ydv_z$$

The solution

- Since the probability is 1 when integrated over entire space, we obtain

$$\int_{-\infty}^{+\infty} g(v_x)\,dv_x = C'\left(\frac{2\pi}{\beta m}\right)^{1/2} = 1 \xrightarrow{\text{Solve for } C'} C' = \left(\frac{\beta m}{2\pi}\right)^{1/2}$$

- Thus $g(v_x)\,dv_x = \sqrt{\frac{\beta m}{2\pi}}\exp\left(-\frac{1}{2}\beta m v_x^2\right)dv_x$

- The average velocity in x direction is

$$\bar{v}_x = \int_{-\infty}^{+\infty} v_x g(v_x)\,dv_x = C'\int_{-\infty}^{+\infty} v_x \exp\left(-\frac{1}{2}\beta m v_x^2\right)dv_x = 0$$

- The average of the square of the velocity in x direction is

$$\overline{v_x^2} = C'\int_{-\infty}^{+\infty} v_x^2 \exp\left(-\frac{1}{2}\beta m v_x^2\right)dv_x = 2C'\int_{0}^{+\infty} v_x^2 \exp\left(-\frac{1}{2}\beta m v_x^2\right)dv_x$$

$$= \sqrt{\frac{\beta m}{2\pi}}\,\frac{\sqrt{\pi}}{2}\left(\frac{2}{\beta m}\right)^{3/2} = \frac{1}{\beta m} = \frac{kT}{m}$$

- Where T is the absolute temperature (temp in C+273), m is the molecular mass and k is the Boltzman constant $k = 1.38 \times 10^{-23}\, J/K$

Maxwell Velocity Distribution

- The results for the *x*, *y*, and *z* velocity components are identical.
- The mean translational kinetic energy of a molecule:

$$\overline{K} = \overline{\frac{1}{2}mv^2} = \frac{1}{2}m\left(\overline{v_x^2}+\overline{v_y^2}+\overline{v_z^2}\right) = \frac{1}{2}m\left(\frac{3kT}{m}\right) = \frac{3}{2}kT$$

- Purely statistical considerations is good evidence of the validity of this statistical approach to thermodynamics.
- Note no dependence of the formula to the mass!!

Ex 9.1: Molecule Kinetic Energy

Compute the mean translational KE of (a) a single ideal gas molecule in eV and (b) a mol of ideal gas in J at room temperature 20°C.

$$(a)\overline{K}=\frac{3}{2}kT=\frac{3}{2}\left(1.38\times10^{-23}\right)\cdot\left(273+20\right)=6.07\times10^{-21}\left(J\right)=$$

$$=0.038\left(eV\right)\approx\frac{1}{25}\left(eV\right)$$

$$(b)\overline{K}=\left(\frac{3}{2}kT\right)N_A=\left[\frac{3}{2}\left(1.38\times10^{-23}\right)\cdot\left(273+20\right)\right]\cdot6.02\times10^{23}=$$

$$=6.07\times10^{-21}\cdot6.02\times10^{23}\left(J\right)=3650\left(J\right)$$

What is the mean translational KE of 1kg of steam at 1atm at 100°C, assuming an ideal gas? Water molecule is 18g/mol.

Equipartition Theorem

- The formula for average kinetic energy 3kT/2 works for monoatomic molecule what is it for diatomic molecule?
- Consider oxygen molecule as two oxygen atoms connected by a massless rod ➔ This will have both translational and rotational energy
- How much rotational energy is there and how is it related to temperature?
- **Equipartition Theorem**:
  - *In equilibrium a mean energy of ½ kT per molecule is associated with each independent quadratic term in the molecule's energy.*
  - Each independent phase space coordinate: ***degree of freedom***
  - Essentially the mean energy of a molecule is *½ kT *NDoF*

Equipartition Theorem

- In a monatomic ideal gas, each molecule has

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right)$$

- There are three degrees of freedom.
- Mean kinetic energy is $3\left(\tfrac{1}{2}kT\right) = \tfrac{3}{2}kT$
- In a gas of *N* helium molecules, the total internal energy is

$$U = N\bar{E} = \tfrac{3}{2}NkT$$

- The heat capacity at constant volume is $C_V = \dfrac{\partial U}{\partial T} = \tfrac{3}{2}Nk$
- For the heat capacity for 1 mole,

$$c_V = \tfrac{3}{2}N_A k = \tfrac{3}{2}R = 12.5\,\mathrm{J/K}$$

- using the ideal gas constant *R* = 8.31 J/K.

Table of Measured Gas Heat Capacities

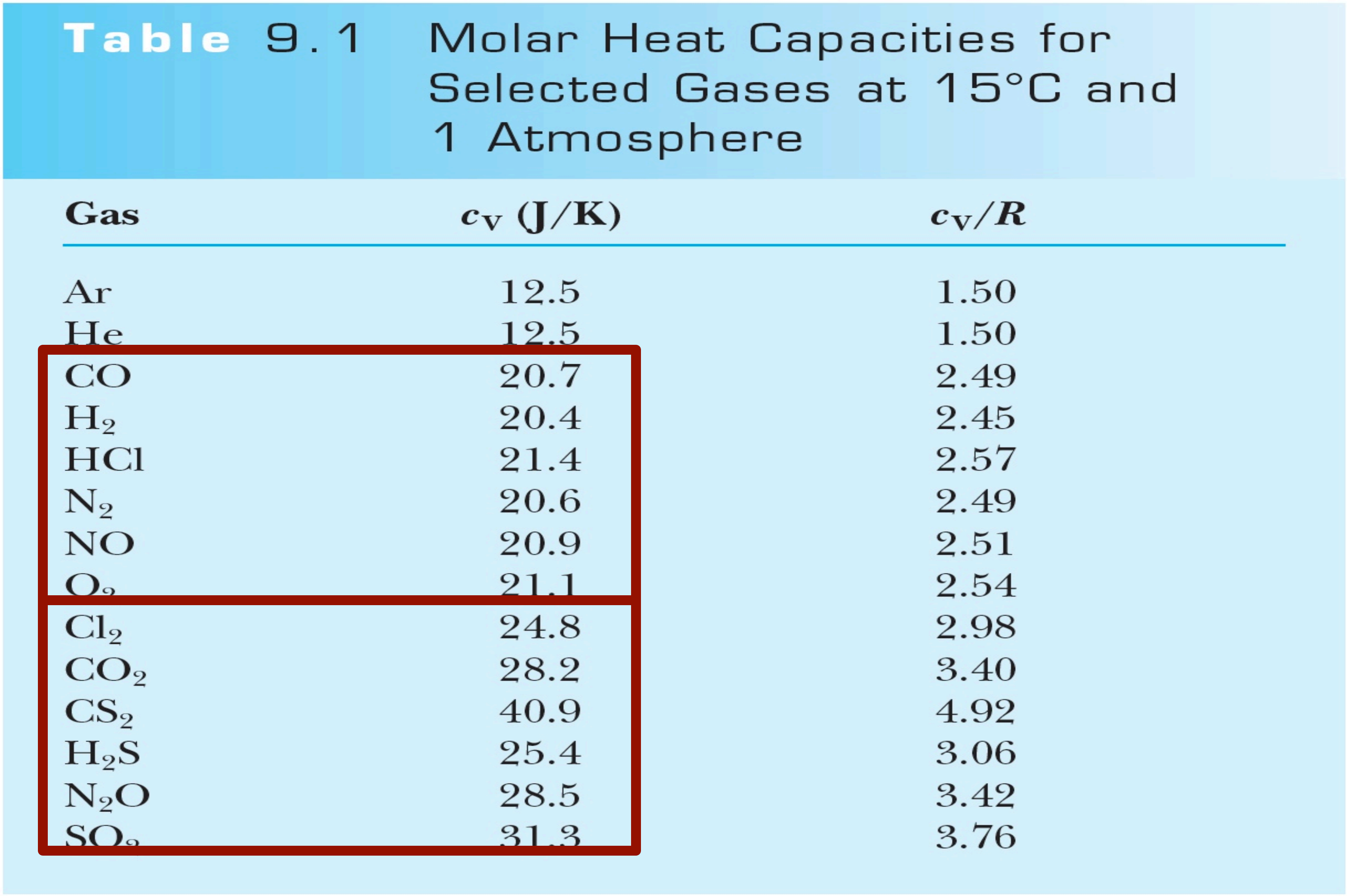
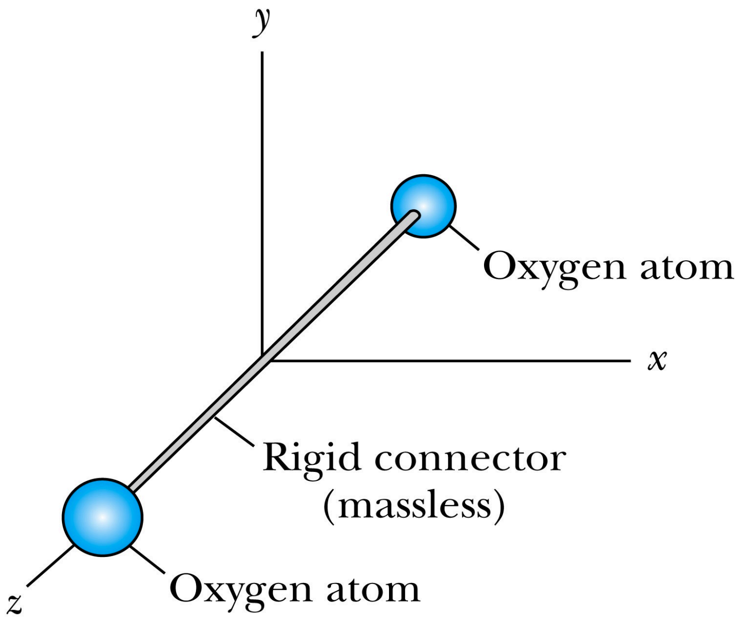


Table 9.1 Molar Heat Capacities for Selected Gases at 15°C and 1 Atmosphere

Gas	c_V (J/K)	c_V/R
Ar	12.5	1.50
He	12.5	1.50
CO	20.7	2.49
H_2	20.4	2.45
HCl	21.4	2.57
N_2	20.6	2.49
NO	20.9	2.51
O_2	21.1	2.54
Cl_2	24.8	2.98
CO_2	28.2	3.40
CS_2	40.9	4.92
H_2S	25.4	3.06
N_2O	28.5	3.42
SO_2	31.3	3.76

The Rigid Rotator Model

- For diatomic gases, consider the rigid rotator model.



- The molecule rotates about either the *x* or *y* axis.
- The corresponding rotational energies are $\frac{1}{2}I_x\omega_x^2$ and $\frac{1}{2}I_y\omega_y^2$
- There are five degrees of freedom (three translational and two rotational)➔ resulting in mean energy of 5kT/2 per molecule according to equi-partition principle (C_V=5R/2)

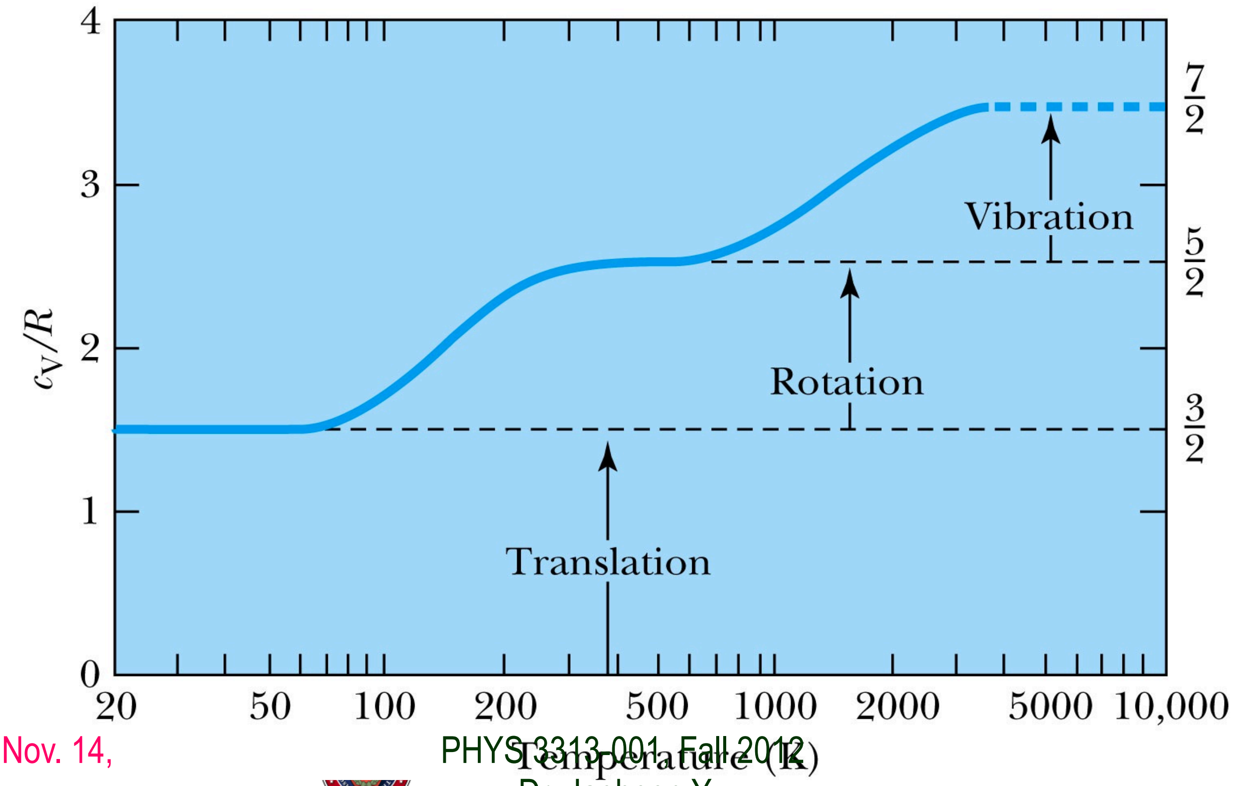
Equipartition Theorem

- From previous chapter, the mass of an atom is confined to a nucleus that magnitude is smaller than the whole atom.
  - I_z is smaller than I_x and I_y.
  - Only rotations about x and y are allowed.
- In some circumstances it is better to think of atoms connected to each other by a massless spring.
- The vibrational kinetic energy is $\frac{1}{2}m\left(dr/dt\right)^2$
- There are seven degrees of freedom (three translational, two rotational, and two vibrational). ➔ 7kT/2 per molecule
- While it works pretty well, the simple assumptions made for equi-partition principle, such as massless connecting rod, is not quite sufficient for detailed molecular behaviors

Molar Heat Capacity

- The heat capacities of diatomic gases are temperature dependent, indicating that the different degrees of freedom are "turned on" at different temperatures.

Example of H_2



Classical and Quantum Statistics

- In gas, particles are so far apart, they do not interact substantially ➔ even if they collide, they can be considered as elastic and do not affect the mean values
- If molecules, atoms, or subatomic particles are in the liquid or solid state, the **Pauli exclusion principle*** prevents two particles with identical quantum states from sharing the same space ➔ limits available energy states in quantum systems
  - Recall there is no restriction on particle energies in classical physics.
- This affects the overall distribution of energies

*Pauli Exclusion Principle: No two electrons in an atom may have the same set of quantum numbers (n,l,m_l,m_s).

Classical Distributions

- Rewrite Maxwell speed distribution in terms of energy.

$$F(v)dv = 4\pi C \exp\left(-\beta m v^2/2\right) v^2 dv = F(E)dE$$

  - Probability for finding a particle between speed v and v+dv
- For a monatomic gas the energy is all translational kinetic energy.

$$E = \tfrac{1}{2} m v^2$$

$$dE = mv\, dv$$

$$dv = \frac{dE}{mv} = \frac{dE}{m\sqrt{2E/m}} = \frac{dE}{\sqrt{2mE}}$$

- where

$$F(E) = \frac{8\pi C}{\sqrt{2} m^{3/2}} \exp(-\beta E) E^{1/2}$$

Classical Distributions

- Boltzmann showed that the statistical factor exp(−βE) is a characteristic of any classical system.
  - regardless of how quantities other than molecular speeds may affect the energy of a given state
- **Maxwell-Boltzmann factor** for classical system:

$$F_{MB} = A\exp(-\beta E)$$

- The energy distribution for classical system:

$$n(E) = g(E)F_{MB}$$

- $n(E)\,dE$: the number of particles with energies between E and $E + dE$
- $g(E)$: the **density of states**, is the number of states available per unit energy range
- F_{MB}: the relative probability that an energy state is occupied at a given temperature

Quantum Distributions

- Identical particles cannot be distinguished if their wave functions overlap significantly
  - Characteristic of indistinguishability that makes quantum statistics different from classical statistics.
- Consider two distinguishable particles in two different energy state with the same probability (0.5 each)
- The possible configurations are

E1	**E2**
A, B	
A	B
B	A
	A, B

- Since the four states are equally likely, the probability of each state is one-fourth (0.25).

Quantum Distributions

- If the two particles are indistinguishable:

State 1	State 2
XX	
X	X
	XX

- There are only three possible configurations
- Thus the probability of each is one-third (~0.33).
- Because some particles do not obey the Pauli exclusion principle, two kinds of quantum distributions are needed.
- **Fermions**: Particles with half-spins (1/2) that **obey** the Pauli principle.
  - Examples? Electron, proton, neutron, any atoms or molecules with odd number of fermions
- **Bosons**: Particles with zero or integer spins that do **NOT obey** the Pauli principle.
  - Examples? Photon, force mediators, pions, any atoms or molecules with even number of fermions

Quantum Distributions

- **Fermi-Dirac** distribution: $n(E) = g(E)F_{FD}$

  where $$F_{FD} = \frac{1}{B_{FD}\exp(\beta E) + 1}$$

- **Bose-Einstein** distribution: $n(E) = g(E)F_{BE}$

  where $$F_{BE} = \frac{1}{B_{BE}\exp(\beta E) - 1}$$

- B_i (i = FD or BE) is a normalization factor.
- Both distributions reduce to the classical Maxwell-Boltzmann distribution when $B_i \exp(\beta E)$ is much greater than 1.
  - the Maxwell-Boltzmann factor $A \exp(-\beta E)$ is much less than 1.
  - In other words, the probability that a particular energy state will be occupied is much less than 1!

Summary of Classical and Quantum Distributions

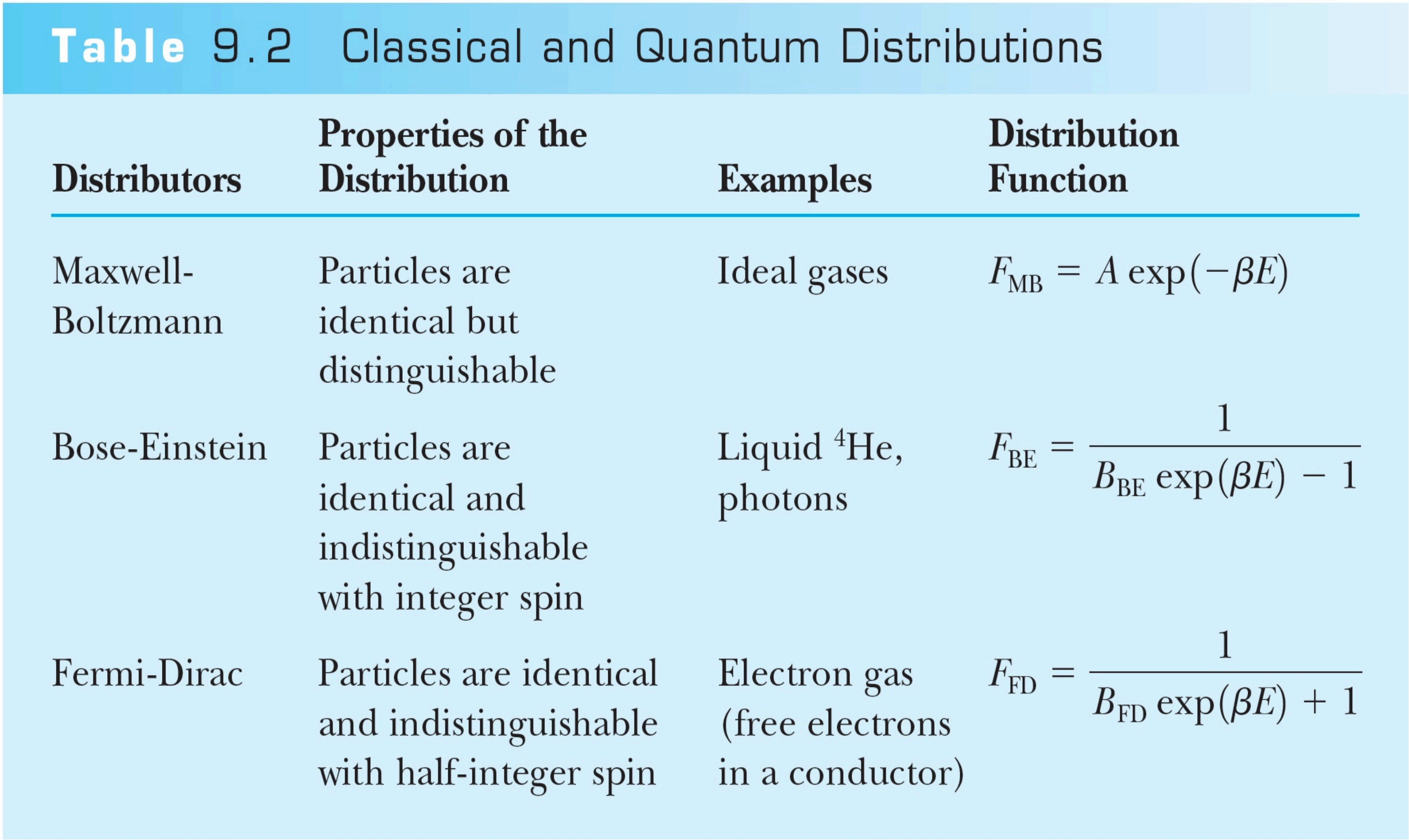
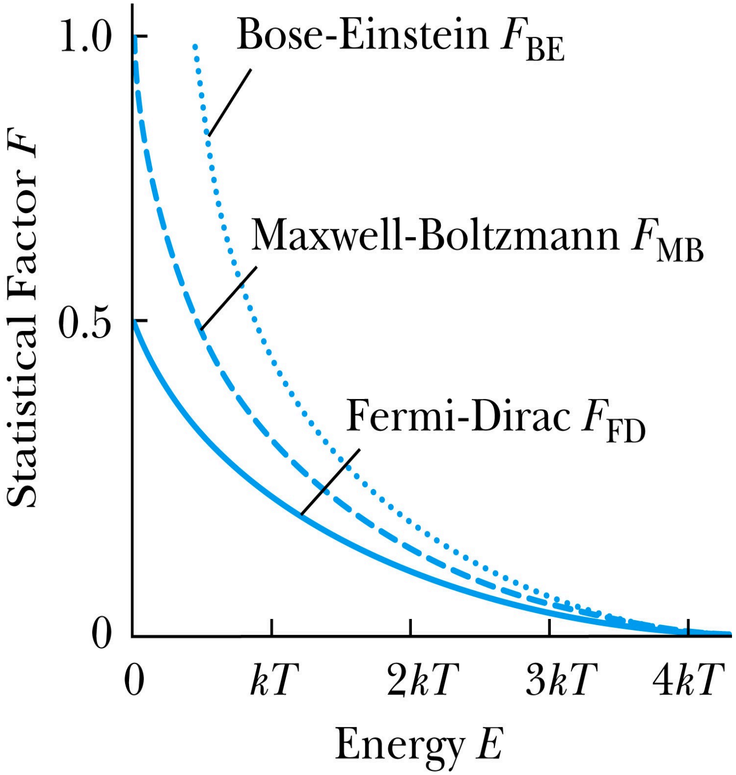


Table 9.2 Classical and Quantum Distributions

Distributors	Properties of the Distribution	Examples	Distribution Function
Maxwell-Boltzmann	Particles are identical but distinguishable	Ideal gases	$F_{\mathrm{MB}} = A\exp(-\beta E)$
Bose-Einstein	Particles are identical and indistinguishable with integer spin	Liquid ^{4}He, photons	$F_{\mathrm{BE}} = \dfrac{1}{B_{\mathrm{BE}}\exp(\beta E) - 1}$
Fermi-Dirac	Particles are identical and indistinguishable with half-integer spin	Electron gas (free electrons in a conductor)	$F_{\mathrm{FD}} = \dfrac{1}{B_{\mathrm{FD}}\exp(\beta E) + 1}$

Quantum Distributions

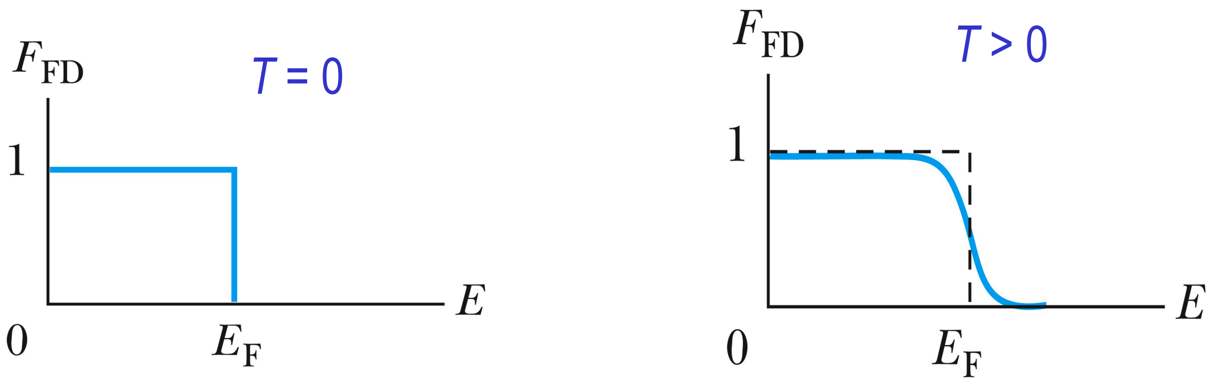
- The normalization constants for the distributions depend on the physical system being considered.
- Because bosons **do not obey the Pauli exclusion principle**, more bosons can fill lower energy states.
- Three graphs coincide at high energies – the classical limit.
- Maxwell-Boltzmann statistics may be used in the classical limit.



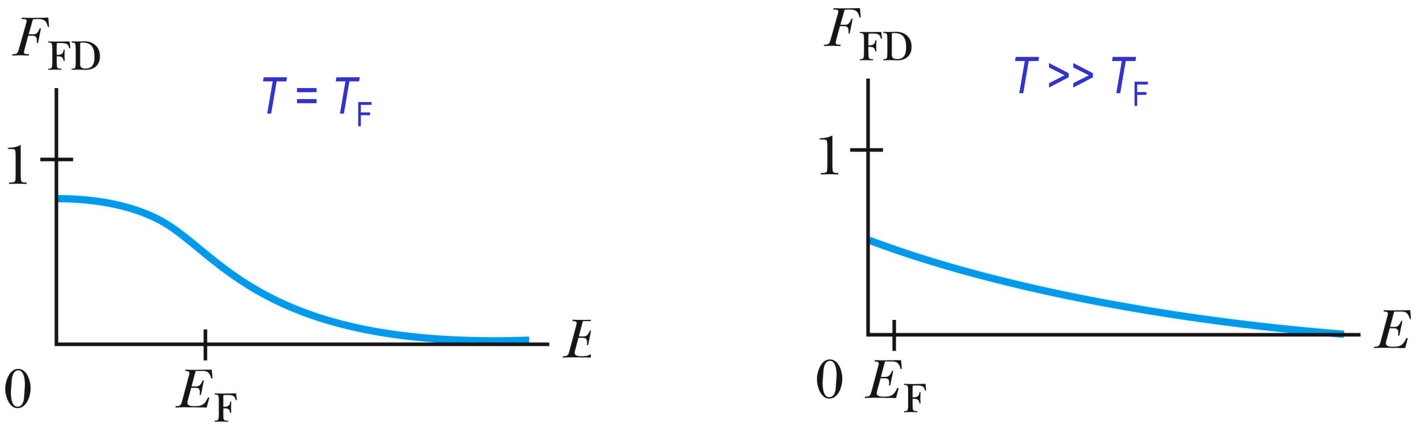
Fermi-Dirac Statistics

- This is most useful for electrical conduction
- The normalization factor B_{FD} $B_{FD} = \exp\left(-\beta E_F\right)$
  - *Where E_F* is called the **Fermi energy**.
- The Fermi-Dirac Factor becomes $F_{FD} = \dfrac{1}{\exp\left[\beta\left(E - E_F\right)\right] + 1}$
- When $E = E_F$, the exponential term is 1. ➔ $F_{FD} = 1/2$
- In the limit as $T \rightarrow 0$, $F_{FD} = \begin{cases} 1 \text{ for } E < E_F \\ 0 \text{ for } E > E_F \end{cases}$
- At $T = 0$, fermions occupy the lowest energy levels available to them
  - Since they cannot all fill the same energy due to Pauli Exclusion principle, they will fill the energy states up to Fermi Energy
- Near $T = 0$, there is little chance that thermal agitation will kick a fermion to an energy greater than E_F.

Fermi-Dirac Statistics



- As the temperature increases from $T = 0$, the Fermi-Dirac factor "smears out", and more fermions jump to higher energy level above Fermi energy
- We can define **Fermi temperature**, defined as $T_F \equiv E_F / k$



- When $T >> T_F$, F_{FD} approaches a simple decaying exponential

Classical Theory of Electrical Conduction

- Paul Drude (1900) showed that the current in a conductor should be linearly proportional to the applied electric field that is consistent with Ohm' s law.
- Prediction of the electrical conductivity $\sigma = \frac{ne^2\tau}{m}$
- **Mean free path** is $\tau = l/\bar{v}$
- True electrical conductivity: $$\sigma = \frac{ne^2 l}{m\bar{v}}$$
- According to the Drude model, the conductivity should be proportional to $T^{-1/2}$.
- But for most conductors is very nearly proportional to T^{-1}
- The heat capacity of the electron gas is R.
- This is not consistent with experimental results.

Free Electron Number Density

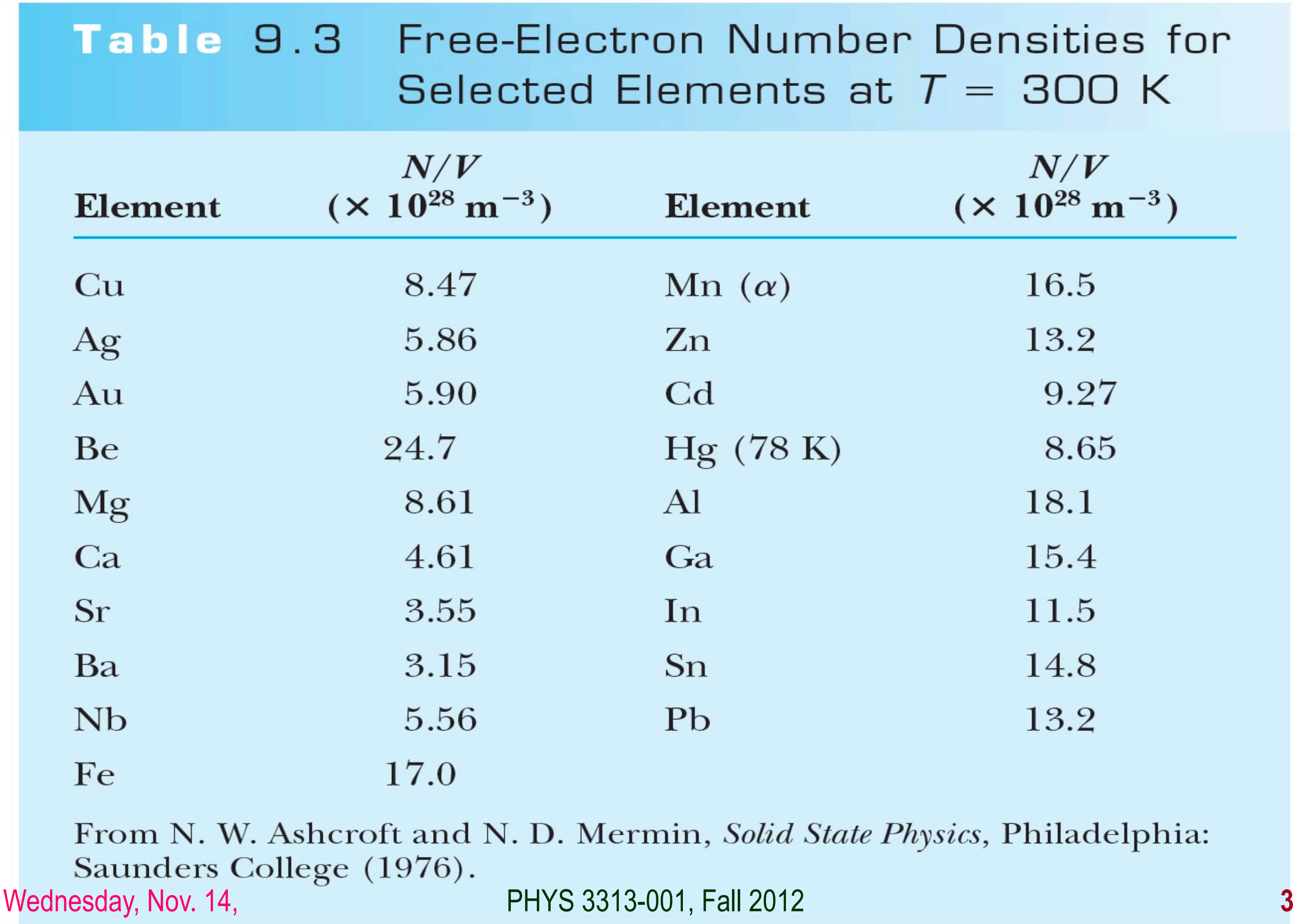


Table 9.3 Free-Electron Number Densities for Selected Elements at $T = 300$ K

Element	N/V ($\times 10^{28}$ m^{-3})	Element	N/V ($\times 10^{28}$ m^{-3})
Cu	8.47	Mn (α)	16.5
Ag	5.86	Zn	13.2
Au	5.90	Cd	9.27
Be	24.7	Hg (78 K)	8.65
Mg	8.61	Al	18.1
Ca	4.61	Ga	15.4
Sr	3.55	In	11.5
Ba	3.15	Sn	14.8
Nb	5.56	Pb	13.2
Fe	17.0		

From N. W. Ashcroft and N. D. Mermin, *Solid State Physics*, Philadelphia: Saunders College (1976).

Table 9-3 p319

Quantum Theory of Electrical Conduction

- Arnold Sommerfield used correct distribution $n(E)$ at room temperature and found a value for α of $\pi^2 / 4$.
- With the value T_F = 80,000 K for copper, we obtain $c_V \approx 0.02R$, which is consistent with the experimental value! Quantum theory has proved to be a success.
- Replace mean speed $\bar{v}$ in the previous page by **Fermi speed** u_F defined from $E_F = \frac{1}{2} m u_F^2$.
- Conducting electrons are loosely bound to their atoms
  - these electrons must be at the high energy level
  - at room temperature the highest energy level is close to the Fermi energy
- We should use $u_F = \sqrt{\frac{2E_F}{m}} \approx 1.6 \times 10^6 \, m/s$

Fermi energies, temperatures and velocities

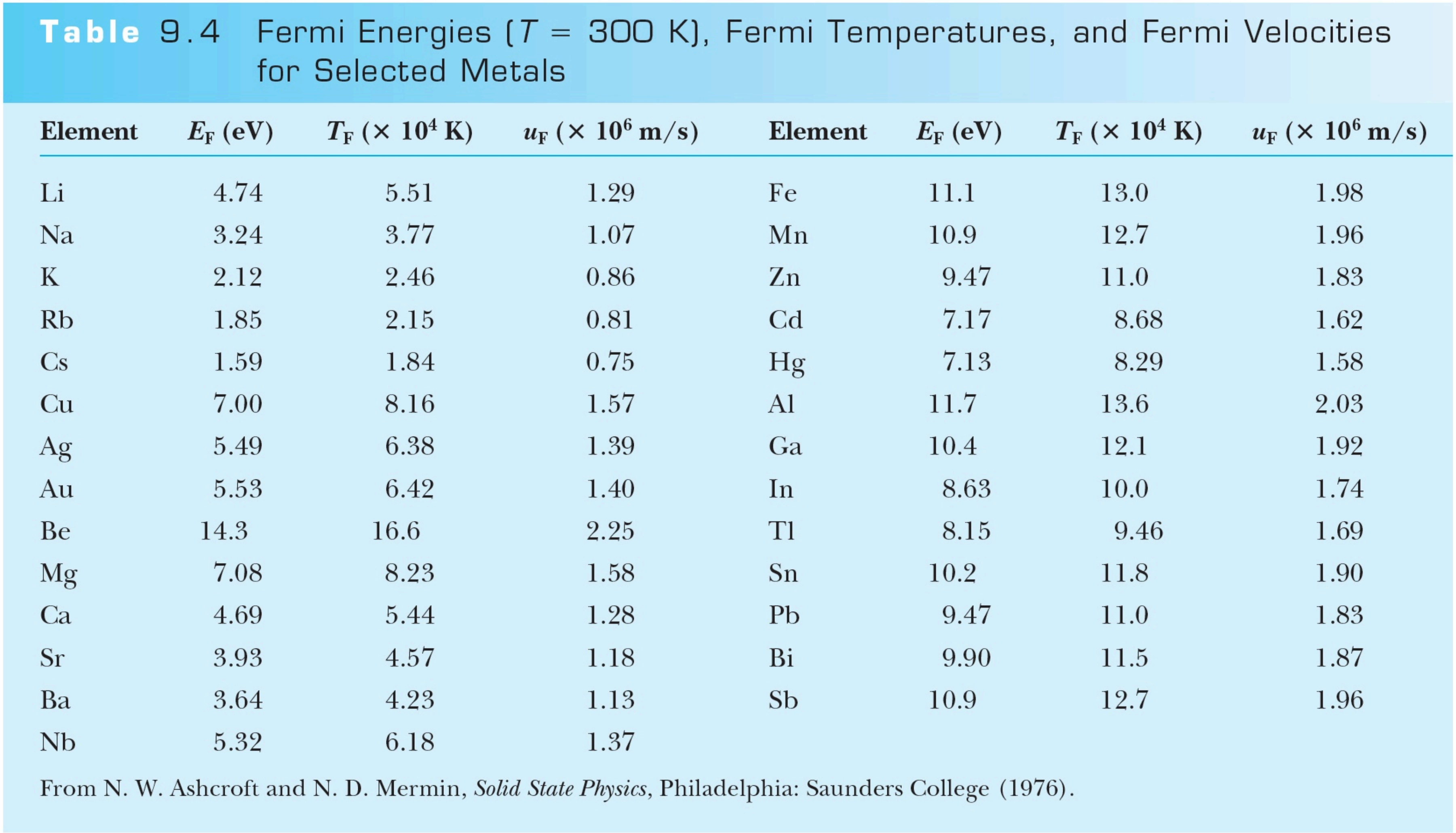


Table 9.4 Fermi Energies (T = 300 K), Fermi Temperatures, and Fermi Velocities for Selected Metals

Element	E_F (eV)	T_F (× 10^4 K)	u_F (× 10^6 m/s)	Element	E_F (eV)	T_F (× 10^4 K)	u_F (× 10^6 m/s)
Li	4.74	5.51	1.29	Fe	11.1	13.0	1.98
Na	3.24	3.77	1.07	Mn	10.9	12.7	1.96
K	2.12	2.46	0.86	Zn	9.47	11.0	1.83
Rb	1.85	2.15	0.81	Cd	7.17	8.68	1.62
Cs	1.59	1.84	0.75	Hg	7.13	8.29	1.58
Cu	7.00	8.16	1.57	Al	11.7	13.6	2.03
Ag	5.49	6.38	1.39	Ga	10.4	12.1	1.92
Au	5.53	6.42	1.40	In	8.63	10.0	1.74
Be	14.3	16.6	2.25	Tl	8.15	9.46	1.69
Mg	7.08	8.23	1.58	Sn	10.2	11.8	1.90
Ca	4.69	5.44	1.28	Pb	9.47	11.0	1.83
Sr	3.93	4.57	1.18	Bi	9.90	11.5	1.87
Ba	3.64	4.23	1.13	Sb	10.9	12.7	1.96
Nb	5.32	6.18	1.37				

From N. W. Ashcroft and N. D. Mermin, *Solid State Physics*, Philadelphia: Saunders College (1976).

Table 9-4 p319

Quantum Theory of Electrical Conduction

- Drude thought that the mean free path could be no more than several tenths of a nanometer, but it was longer than his estimation.
- Einstein calculated the value of ℓ to be on the order of 40 nm in copper at room temperature.
- The conductivity is $\sigma = \dfrac{ne^2 l}{mu_F} \approx 6 \times 10^7 \Omega^{-1} \cdot m^{-1}$
- Sequence of proportions

$$\sigma \propto l \propto r^{-2} = \left(n_x^2 + n_y^2 + n_z^2\right) \propto U^{-1} \propto T^{-1}$$

Bose-Einstein Statistics

Blackbody Radiation

- Intensity of the emitted radiation is $\mathcal{I}(\lambda,T) = \dfrac{2\pi c^2 h}{\lambda^5}\dfrac{1}{e^{hc/\lambda kT} - 1}$

- Use the Bose-Einstein distribution because photons are bosons with spin 1.
- For a free particle in terms of momentum:

$$p = \sqrt{p_x{}^2 + p_y{}^2 + p_z{}^2} = \frac{h}{2L}\sqrt{n_1{}^2 + n_2{}^2 + n_3{}^2}$$

- The energy of a photon is *pc*, so

$$E = \frac{hc}{2L}\sqrt{n_1{}^2 + n_2{}^2 + n_3{}^2}$$

Bose-Einstein Statistics

- The number of allowed energy states within "radius" *r* is

$$N_r = (2)\left(\tfrac{1}{8}\right)\left(\tfrac{4}{3}\pi r^3\right)$$

Where 1/8 comes from the restriction to positive values of n_i and 2 comes from the fact that there are two possible photon polarizations.

- Energy is proportional to *r*, $E = \dfrac{hc}{2L} r$
- The density of states *g*(*E*) is $g(E) = \dfrac{dN_r}{dE} = \dfrac{8\pi L^3}{h^3 c^3} E^2$
- The Bose-Einstein factor: $n(E) = g(E) F_{\mathrm{BE}}$

$$= \frac{8\pi L^3}{h^3 c^3} E^2 \frac{1}{e^{E/kT} - 1}$$

Bose-Einstein Statistics

- Convert from a number distribution to an energy density distribution *u*(*E*).
  - Multiply by a factor E/L^3

$$u(E) = \frac{En(E)}{L^3} = \frac{8\pi}{h^3c^3}E^3\frac{1}{e^{E/kT}-1}$$

  - For all photons in the range *E* to *E* + *dE*

$$u(E)\,dE = \frac{8\pi}{h^3c^3}\frac{E^3\,dE}{e^{E/kT}-1}$$

- Using $E = hc/\lambda$ and $|dE| = (hc/\lambda^2)\,d\lambda$

$$u(\lambda,T)\,d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$$

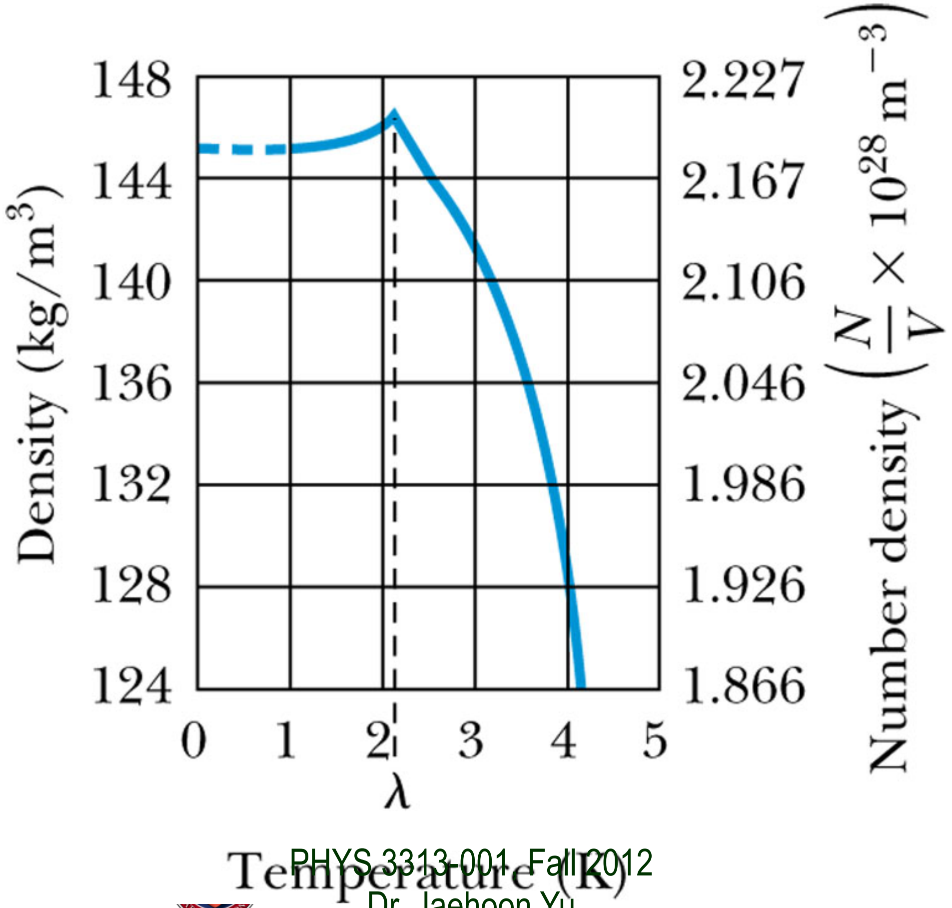
- In the SI system, multiplying by *c*/4 is required.

$$\mathscr{I}(\lambda,T) = \frac{2\pi c^2h}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$$

Liquid Helium

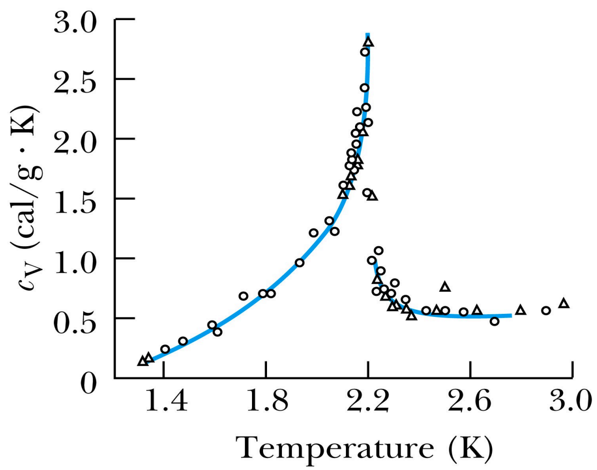
- Has the lowest boiling point of any element (4.2 K at 1 atmosphere pressure) and has no solid phase at normal pressure

The density of liquid helium as a function of temperature:



Liquid Helium

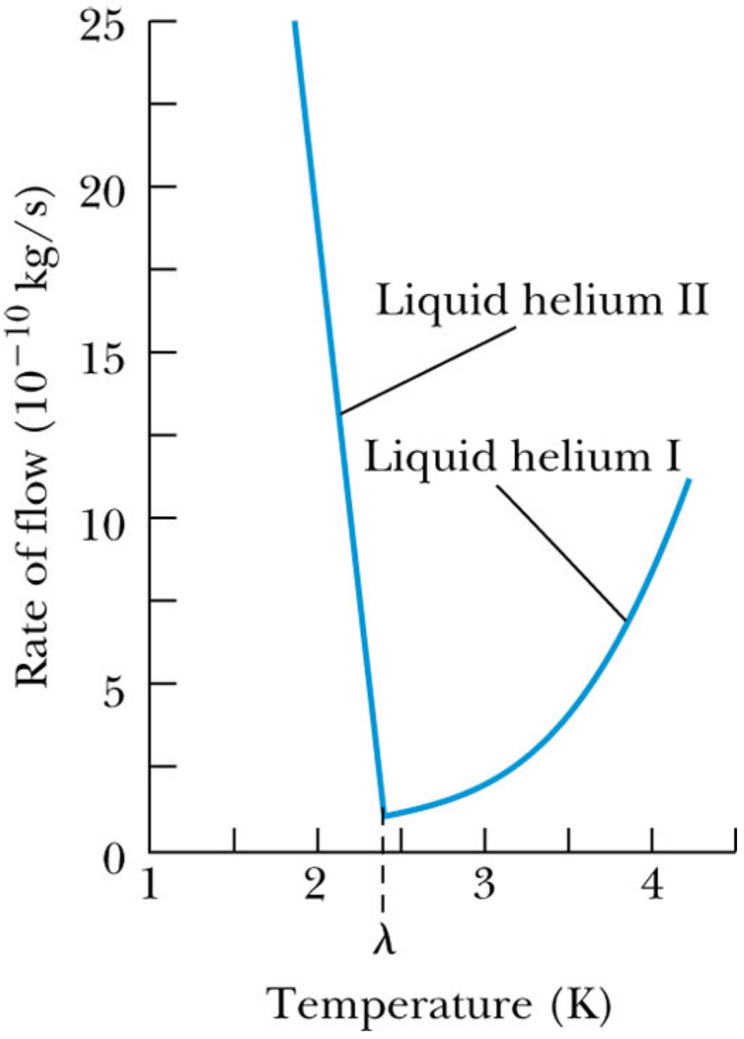
The specific heat of liquid helium as a function of temperature:



- The temperature at about 2.17 K is referred to as the *critical temperature* (T_c), *transition temperature*, or *lambda point*.
- As the temperature is reduced from 4.2 K toward the lambda point, the liquid boils vigorously. At 2.17 K the boiling suddenly stops.
- What happens at 2.17 K is a transition from the **normal phase** to the **superfluid phase**.

Liquid Helium

- The rate of flow increases dramatically as the temperature is reduced because the superfluid has a low viscosity.
- **Creeping film** – formed when the viscosity is very low



Liquid Helium

- Fritz London claimed (1938) that liquid helium below the lambda point is part superfluid and part normal.
  - ❑ As the temperature approaches absolute zero, the superfluid approaches 100% superfluid.
- The fraction of helium atoms in the superfluid state:

$$F = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

- Superfluid liquid helium is referred to as a **Bose-Einstein condensation**.
  - not subject to the Pauli exclusion principle
  - all particles are in the same quantum state

Liquid Helium

- Such a condensation process is not possible with fermions because fermions must "stack up" into their energy states, no more than two per energy state.
- ^{4}He isotope is a fermion and superfluid mechanism is radically different than the Bose-Einstein condensation.
- Use the fermions' density of states function and substituting for the constant E_F yields

$$g_{\mathrm{FD}}(E) = \frac{\pi}{2}\left(\frac{h^2}{8mL^2}\right)^{-3/2} E^{1/2}$$

- Bosons do not obey the Pauli principle, therefore the density of states should be less by a factor of 2.

$$g_{\mathrm{BE}}(E) = \frac{2\pi V}{h^3}(2m)^{3/2} E^{1/2}$$

Liquid Helium

- *m* is the mass of a helium atom.
- The number distribution *n*(*E*) is now

$$n(E) = g_{\mathrm{BE}}(E)F_{\mathrm{BE}}$$

$$= \frac{2\pi V}{h^3}(2m)^{3/2}\ E^{1/2}\frac{1}{B_2 e^{E/kT}-1}$$

- In a collection of *N* helium atoms the normalization condition is

$$N = \int_0^\infty n(E)\ dE$$

$$= \frac{2\pi V}{h^3}(2m)^{3/2}\int_0^\infty \frac{E^{1/2}}{B_2 e^{E/kT}-1}dE$$

- Substituting $u = E / kT$,

$$N = \frac{2\pi V}{h^3}(2mkT)^{3/2}\int_0^\infty \frac{u^{1/2}}{B_2 e^{u}-1}du$$

Liquid Helium

- Use minimum value of $B_{BE} = 1$; this result corresponds to the maximum value of N.

$$N \le \frac{2\pi V}{h^3}(2mkT)^{3/2}(2.315)$$

- Rearrange this, $$T \ge \frac{h^2}{2mk}\left[\frac{N}{2\pi V(2.315)}\right]^{2/3}$$

  The result is $T \ge 3.06$ K.

- The value 3.06 K is an estimate of T_c.

Bose-Einstein Condensation in Gases

- By the strong Coulomb interactions among gas particles it was difficult to obtain the low temperatures and high densities needed to produce the condensate. Finally success was achieved in 1995.
- First, they used laser cooling to cool their gas of ^{87}Rb atoms to about 1 mK. Then they used a magnetic trap to cool the gas to about 20 nK. In their magnetic trap they drove away atoms with higher speeds and further from the center. What remained was an extremely cold, dense cloud at about 170 nK.