PHYS 1443 – Section 004
Lecture #16

Thursday, Oct. 16, 2014
Dr. Jaehoon Yu

- Center of Mass
- Center of mass of a rigid body
- Motion of a Group of Objects
- Fundamentals of Rotational Motion
- Rotational Kinematics

Announcements

- Mid-term comprehensive exam
  - In class 9:30 – 10:50am, next Tuesday, Oct. 21
  - Covers CH 1.1 through what we finish today (CH10.2) plus the math refresher
  - Mixture of multiple choice and free response problems
  - Bring your calculator but DO NOT input formula into it!
    - Your phones or portable computers are NOT allowed as a replacement!
  - You can prepare a one 8.5x11.5 sheet (front and back) of **handwritten** formulae and values of constants for the exam
    - None of the parts of the solutions of any problems
    - No derived formulae, derivations of equations or word definitions!
  - Do NOT Miss the exam!

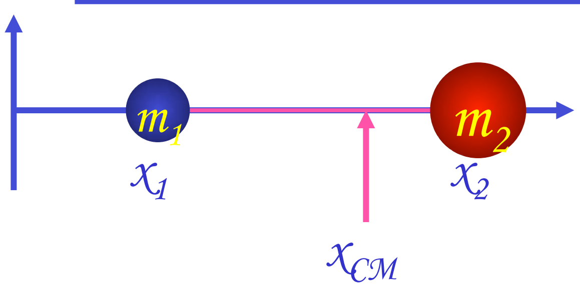
Center of Mass

We've been solving physical problems treating objects as sizeless points with masses, but in realistic situations objects have shapes with masses distributed throughout the body.

Center of mass of a system is the average position of the system's mass and represents the motion of the system as if all the mass is on the point.

What does above statement tell you concerning the forces being exerted on the system?

The total external force exerted on the system of total mass M causes the center of mass to move at an acceleration given by $\vec{a} = \sum \vec{F} / M$ *as if all the mass of the system is concentrated on the center of mass.*



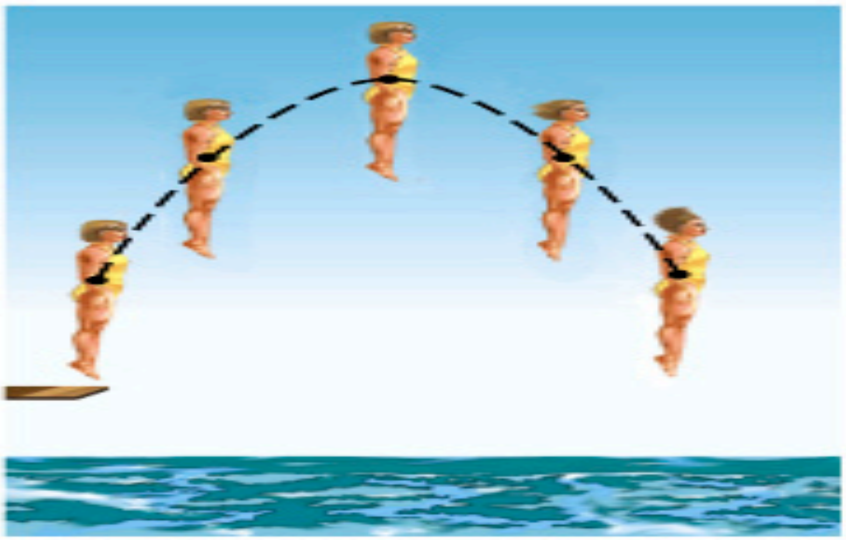
Consider a massless rod with two balls attached at either end.

The position of the center of mass of this system is the mass averaged position of the system

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

CM is closer to the heavier object

Motion of a Diver and the Center of Mass



(a)

A diver performs a simple dive. The motion of the center of mass follows a parabola since it is a projectile motion.

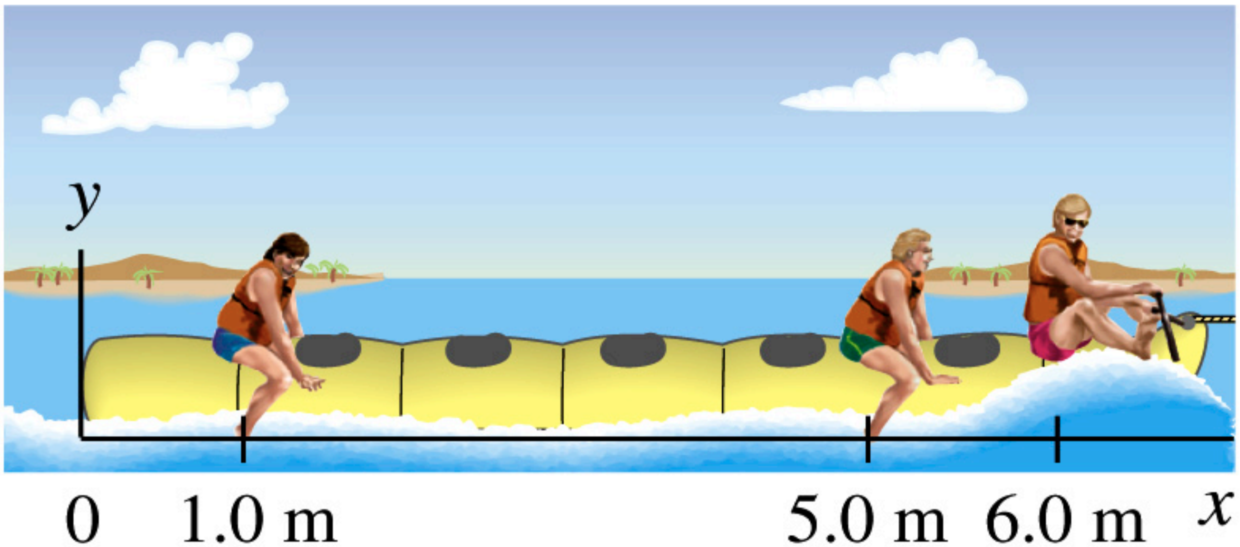
(b)

A diver performs a complicated dive. The motion of the center of mass still follows the same parabola since it still is a projectile motion.

The motion of the center of mass of the diver is always the same.

Example for CM

Thee people of roughly equivalent mass M on a lightweight (air-filled) banana boat sit along the x axis at positions x_1=1.0m, x_2=5.0m, and x_3=6.0m. Find the position of CM.



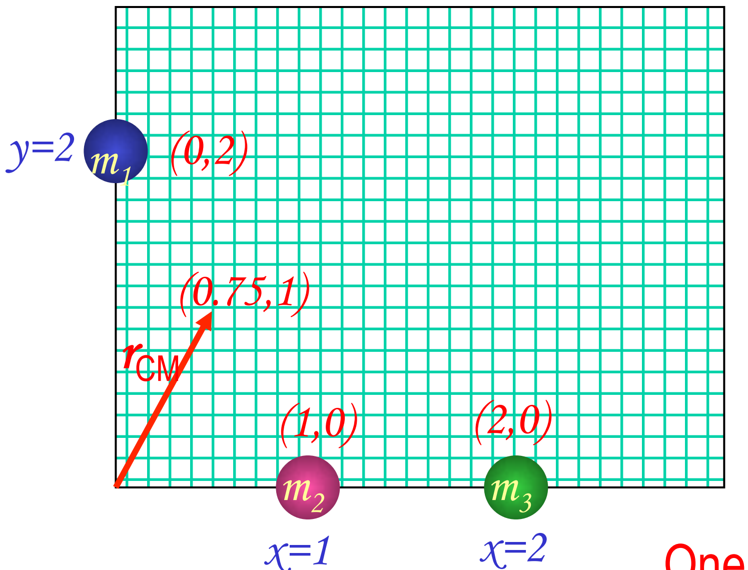
Using the formula for CM

$$x_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

$$= \frac{M \cdot 1.0 + M \cdot 5.0 + M \cdot 6.0}{M + M + M} = \frac{12.0M}{3M} = 4.0(m)$$

Example for Center of Mass in 2-D

A system consists of three particles as shown in the figure. Find the position of the center of mass of this system.



Using the formula for CM for each position vector component

$$x_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i} \qquad y_{CM} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

One obtains $\vec{r}_{CM} = x_{CM}\vec{i} + y_{CM}\vec{j} = \frac{(m_2 + 2m_3)\vec{i} + 2m_1\vec{j}}{m_1 + m_2 + m_3}$

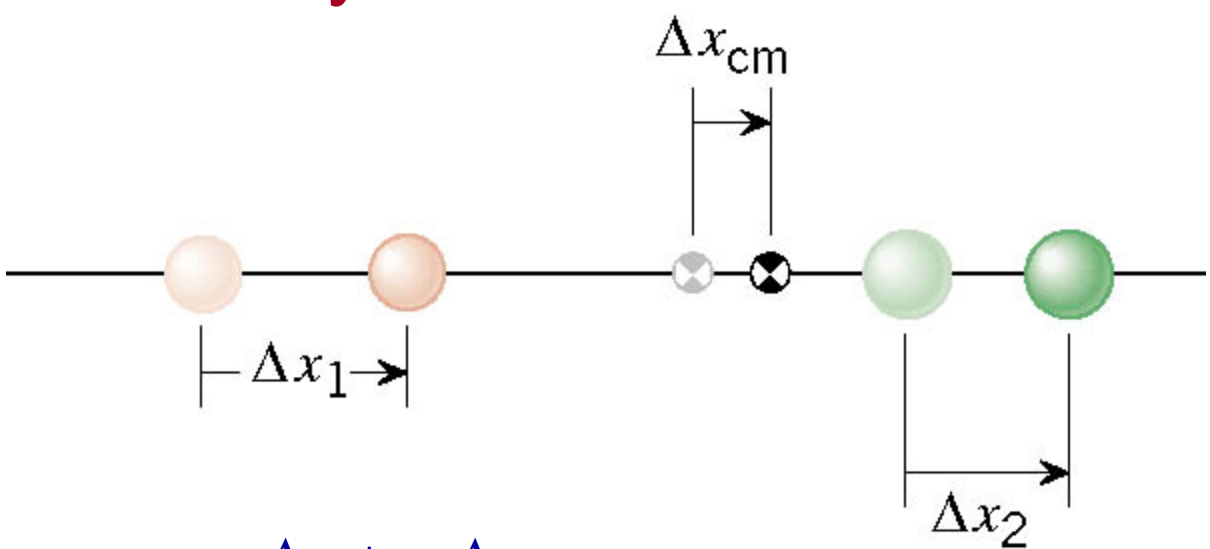
$$x_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{m_2 + 2m_3}{m_1 + m_2 + m_3}$$

$$y_{CM} = \frac{\sum_i m_i y_i}{\sum_i m_i} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{2m_1}{m_1 + m_2 + m_3}$$

If $m_1 = 2kg; m_2 = m_3 = 1kg$

$$\vec{r}_{CM} = \frac{3\vec{i} + 4\vec{j}}{4} = 0.75\vec{i} + \vec{j}$$

Velocity of the Center of Mass



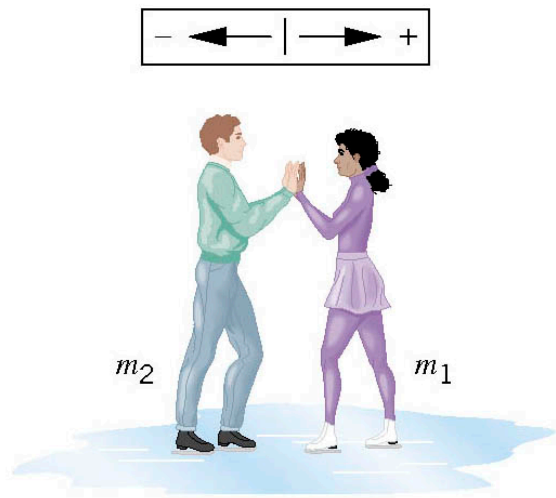
$$\Delta x_{cm} = \frac{m_1 \Delta x_1 + m_2 \Delta x_2}{m_1 + m_2}$$

$$v_{cm} = \frac{\Delta x_{cm}}{\Delta t} = \frac{m_1 \Delta x_1/\Delta t + m_2 \Delta x_2/\Delta t}{m_1 + m_2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

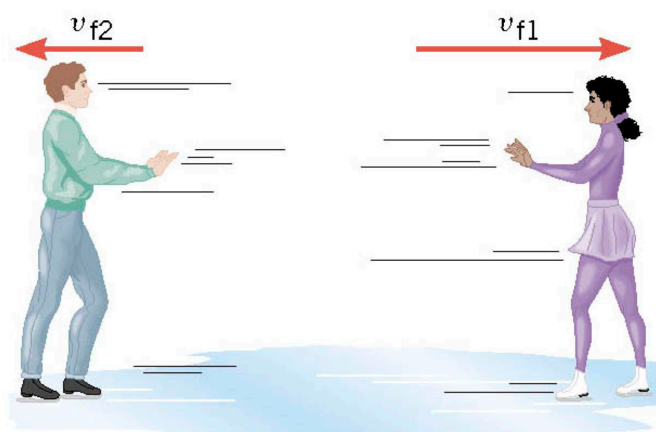
In an isolated system, the total linear momentum does not change, therefore the velocity of the center of mass does not change.

Another Look at the Ice Skater Problem

Starting from rest, two skaters push off against each other on ice where friction is negligible. One is a 54-kg woman and one is a 88-kg man. The woman moves away with a velocity of +2.5 m/s. Man's velocity?



(a) Before



(b) After

$$v_{10} = 0\,m/s \qquad v_{20} = 0\,m/s$$

$$v_{cm0} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = 0$$

$$v_{1f} = +2.5\,m/s \qquad v_{2f} = -1.5\,m/s$$

$$v_{cmf} = \frac{m_1 v_{1f} + m_2 v_{2f}}{m_1 + m_2}$$

$$= \frac{54 \cdot (+2.5) + 88 \cdot (-1.5)}{54 + 88} = \frac{3}{142} = 0.02 \approx 0\,m/s$$

Center of Mass of a Rigid Object

The formula for CM can be extended to a system of many particles or a Rigid Object

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

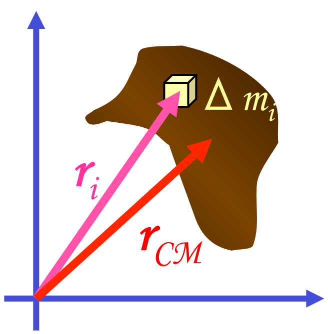
$$y_{CM} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

$$z_{CM} = \frac{\sum_i m_i z_i}{\sum_i m_i}$$

The position vector of the center of mass of a many particle system is

$$\vec{r}_{CM} = x_{CM}\vec{i} + y_{CM}\vec{j} + z_{CM}\vec{k} = \frac{\sum_i m_i x_i \vec{i} + \sum_i m_i y_i \vec{j} + \sum_i m_i z_i \vec{k}}{\sum_i m_i}$$

$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{M}$$



A rigid body – an object with shape and size with mass spread throughout the body, ordinary objects – can be considered as a group of particles with mass m_i densely spread throughout the given shape of the object

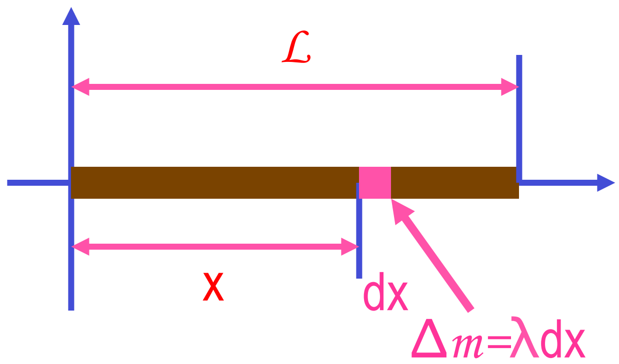
$$x_{CM} \approx \frac{\sum_i \Delta m_i x_i}{M}$$

$$x_{CM} = \lim_{\Delta m_i \to 0} \frac{\sum_i \Delta m_i x_i}{M} = \frac{1}{M}\int x\,dm$$

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\,dm$$

Example: CM of a thin rod

Show that the center of mass of a rod of mass $\mathcal{M}$ and length $\mathcal{L}$ lies in midway between its ends, assuming the rod has a uniform mass per unit length.



The formula for CM of a continuous object is

$$\boxed{x_{CM} = \frac{1}{M}\int_{x=0}^{x=L} x\,dm}$$

Since the density of the rod (λ) is constant; $\lambda = M/L$

The mass of a small segment $dm = \lambda dx$

Therefore
$$x_{CM} = \frac{1}{M}\int_{x=0}^{x=L}\lambda x\,dx = \frac{1}{M}\left[\frac{1}{2}\lambda x^2\right]_{x=0}^{x=L} = \frac{1}{M}\left(\frac{1}{2}\lambda L^2\right) = \frac{1}{M}\left(\frac{1}{2}ML\right) = \frac{L}{2}$$

Find the CM when the density of the rod non-uniform but varies linearly as a function of x, $\lambda=\alpha x$

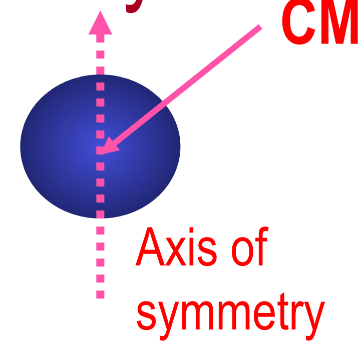
$$M = \int_{x=0}^{x=L}\lambda dx = \int_{x=0}^{x=L}\alpha x\,dx = \left[\frac{1}{2}\alpha x^2\right]_{x=0}^{x=L} = \frac{1}{2}\alpha L^2$$

$$x_{CM} = \frac{1}{M}\int_{x=0}^{x=L}\lambda x\,dx = \frac{1}{M}\int_{x=0}^{x=L}\alpha x^2\,dx = \frac{1}{M}\left[\frac{1}{3}\alpha x^3\right]_{x=0}^{x=L}$$

$$x_{CM} = \frac{1}{M}\left(\frac{1}{3}\alpha L^3\right) = \frac{1}{M}\left(\frac{2}{3}ML\right) = \frac{2L}{3}$$

Center of Mass and Center of Gravity

The center of mass of any symmetric object lies on the axis of symmetry and on any plane of symmetry, if the object's mass is evenly distributed throughout the body.

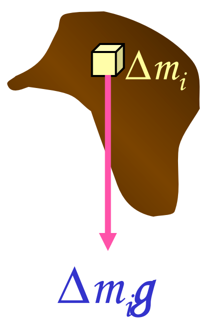


How do you think you can determine the CM of the objects that are not symmetric?

One can use gravity to locate CM.

1. Hang the object by one point and draw a vertical line following a plum-bob.
2. Hang the object by another point and do the same.
3. The point where the two lines meet is the CM.

Center of Gravity



Since a rigid object can be considered as a **collection of small masses**, one can see the total gravitational force exerted on the object as

$$\vec{F}_g = \sum_i \vec{F}_i = \sum_i \Delta m_i \vec{g} = M\vec{g}$$

What does this equation tell you?

The net effect of these small gravitational forces is equivalent to a single force acting on a point (Center of Gravity) with mass M.

The CoG is the point in an object as if all the gravitational force is acting on!

Motion of a Group of Particles

We've learned that the CM of a system can represent the motion of a system. Therefore, for an isolated system of many particles in which the total mass M is preserved, the velocity, total momentum, acceleration of the system are

Velocity of the system

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{d}{dt}\left(\frac{1}{M}\sum_i m_i\vec{r}_i\right) = \frac{1}{M}\sum_i \frac{m_i d\vec{r}_i}{dt} = \frac{\sum_i m_i\vec{v}_i}{M}$$

Total Momentum of the system

$$\vec{p}_{CM} = M\vec{v}_{CM} = M\frac{\sum_i m_i\vec{v}_i}{M} = \sum_i m_i\vec{v}_i = \sum_i \vec{p}_i = \vec{p}_{tot}$$

Acceleration of the system

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{d}{dt}\left(\frac{1}{M}\sum_i m_i\vec{v}_i\right) = \frac{1}{M}\left(\sum_i m_i\frac{d\vec{v}_i}{dt}\right) = \frac{\sum_i m_i\vec{a}_i}{M}$$

The external force acting on the system

$$\sum \vec{F}_{ext} = M\vec{a} = \left(\sum_i m_i\right)\vec{a} = \frac{d\vec{p}_{tot}}{dt}$$

What about the internal forces?

If net external force is 0

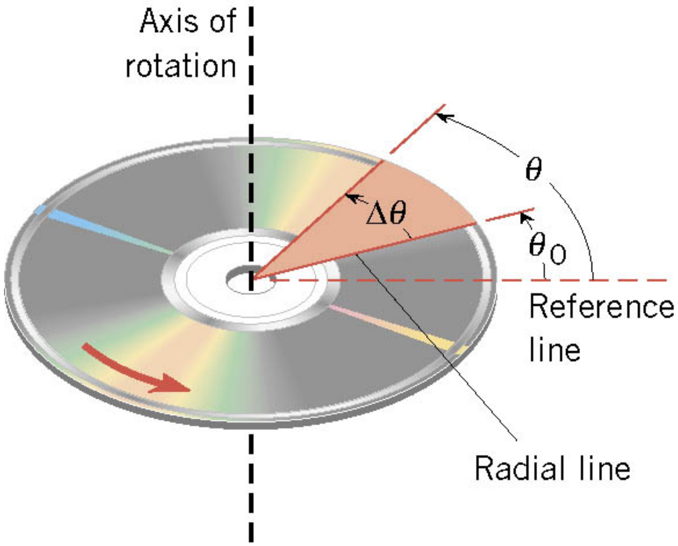
$$\sum \vec{F}_{ext} = 0 = \frac{d\vec{p}_{tot}}{dt}$$

$$\boxed{\vec{p}_{tot} = const}$$

System's momentum is conserved.

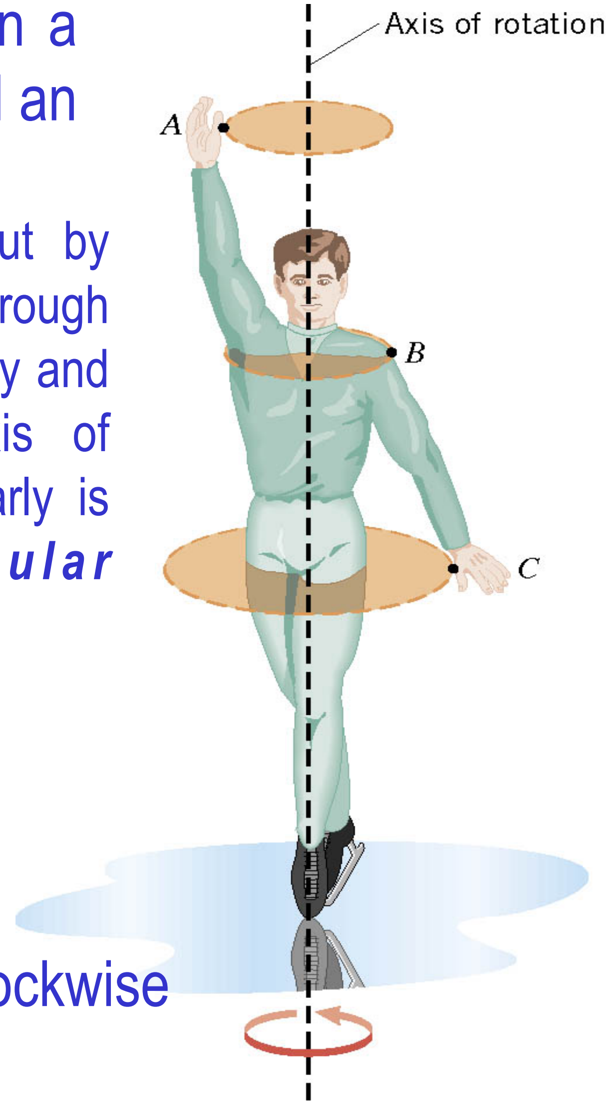
Rotational Motion and Angular Displacement

In the simplest kind of rotation, points on a rigid object move on circular paths around an ***axis of rotation.***



The angle swept out by the line passing through any point on the body and intersecting the axis of rotation perpendicularly is called the ***angular displacement.***

$$\Delta\theta = \theta - \theta_o$$



It's a vector!! So there must be a direction…

How do we define directions?

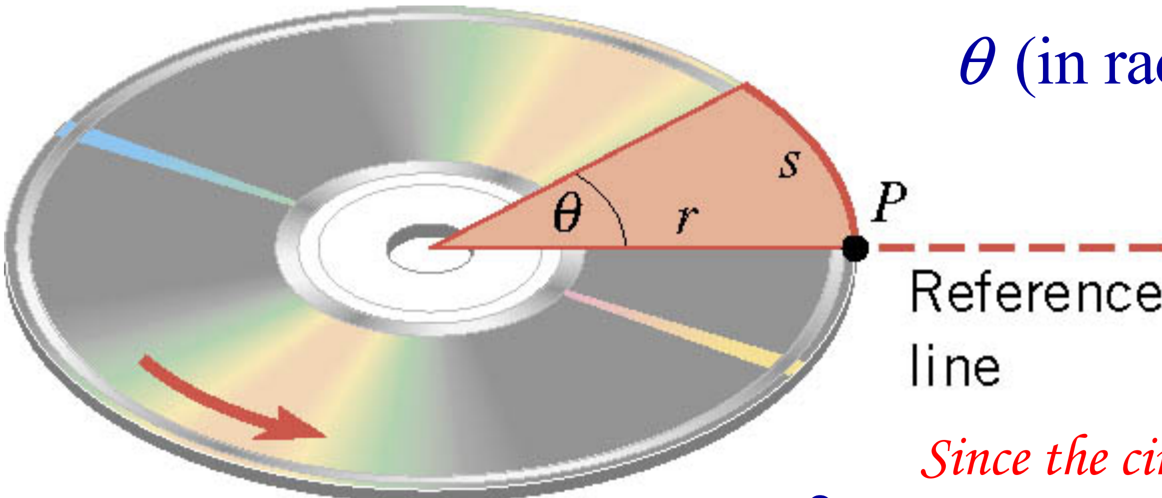
+:if counter-clockwise

-:if clockwise

The direction vector points gets determined based on the right-hand rule.

These are just conventions!!

SI Unit of the Angular Displacement



$$\theta \text{ (in radians)} = \frac{\text{Arc length}}{\text{Radius}} = \frac{s}{r}$$

Dimension? None

For one full revolution:

Since the circumference of a circle is $2\pi r$

$$\theta = \frac{2\pi r}{r} = 2\pi \text{ rad} \quad \Rightarrow \quad 2\pi \text{ rad} = 360^{o}$$

One radian is an angle subtended by an arc of the same length as the radius!

Unit of the Angular Displacement

How many degrees are in one radian?

1 radian is

$$1\text{ rad} = \frac{360^\circ}{2\pi rad}\cdot 1rad = \frac{180^\circ}{\pi} \cong \frac{180^\circ}{3.14} \cong 57.3^\circ$$

How radians is one degree?

And one degrees is

$$1^\circ = \frac{2\pi}{360^\circ}\cdot 1^\circ = \frac{\pi}{180^\circ}\cdot 1^\circ \cong \frac{3.14}{180^\circ}\cdot 1^\circ \cong 0.0175rad$$

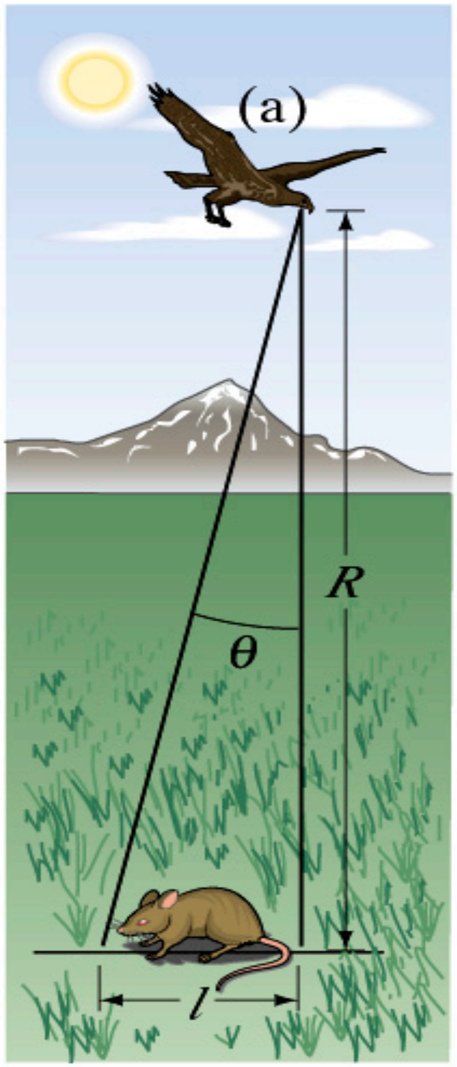
How many radians are in 10.5 revolutions?

$$10.5rev = 10.5rev\cdot 2\pi\frac{rad}{rev} = 21\pi\left(rad\right)$$

Very important: In solving angular problems, all units, degrees or revolutions, must be converted to radians.

Example

A particular bird's eyes can just distinguish objects that subtend an angle no smaller than about $3\text{x}10^{-4}$ rad. (a) How many degrees is this? (b) How small an object can the bird just distinguish when flying at a height of 100m?



(a) One radian is $360^o/2\pi$. Thus

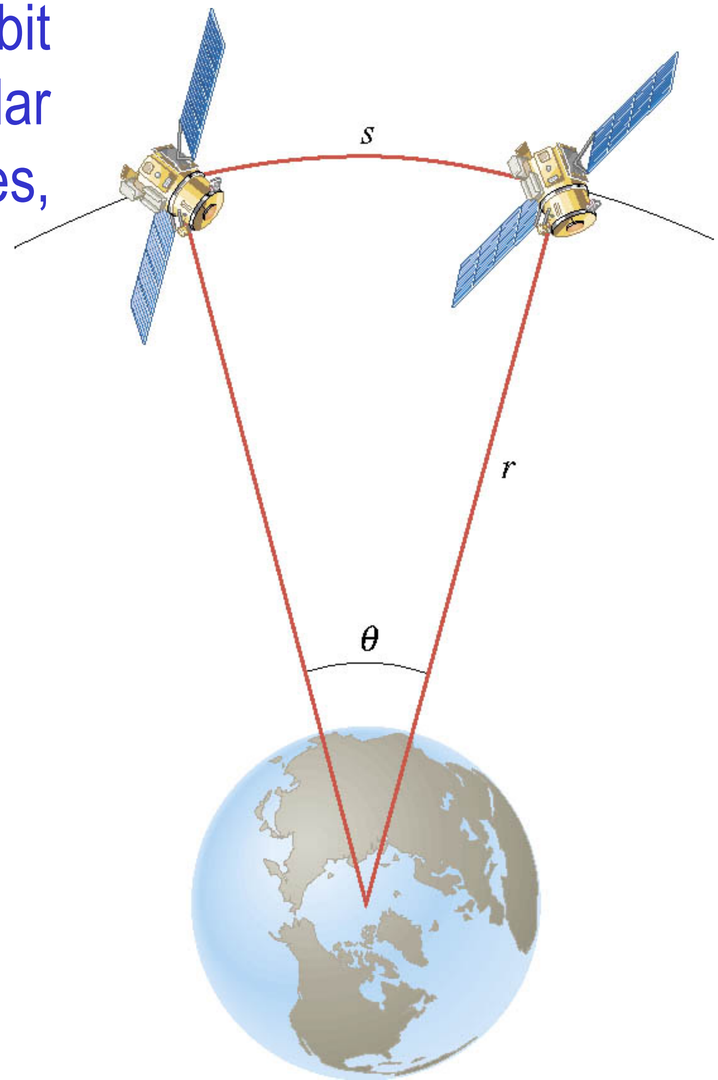
$$3\times10^{-4}\,rad = \left(3\times10^{-4}\,rad\right)\times\left(360^o/2\pi\,rad\right) = 0.017^o$$

(b) Since $l=r\theta$ and for small angle arc length is approximately the same as the chord length.

$$l = r\theta = 100m\times3\times10^{-4}\,rad = 3\times10^{-2}\,m = 3cm$$

Ex. Adjacent Synchronous Satellites

Synchronous satellites are put into an orbit whose radius is 4.23×10^{7}m. If the angular separation of the two satellites is 2.00 degrees, find the arc length that separates them.



What do we need to find out? The Arc length!!!

$$\theta\ \text{(in radians)} = \frac{\text{Arc length}}{\text{Radius}} = \frac{s}{r}$$

Convert degrees to radians

$$2.00\,\text{deg}\left(\frac{2\pi\ \text{rad}}{360\,\text{deg}}\right) = 0.0349\ \text{rad}$$

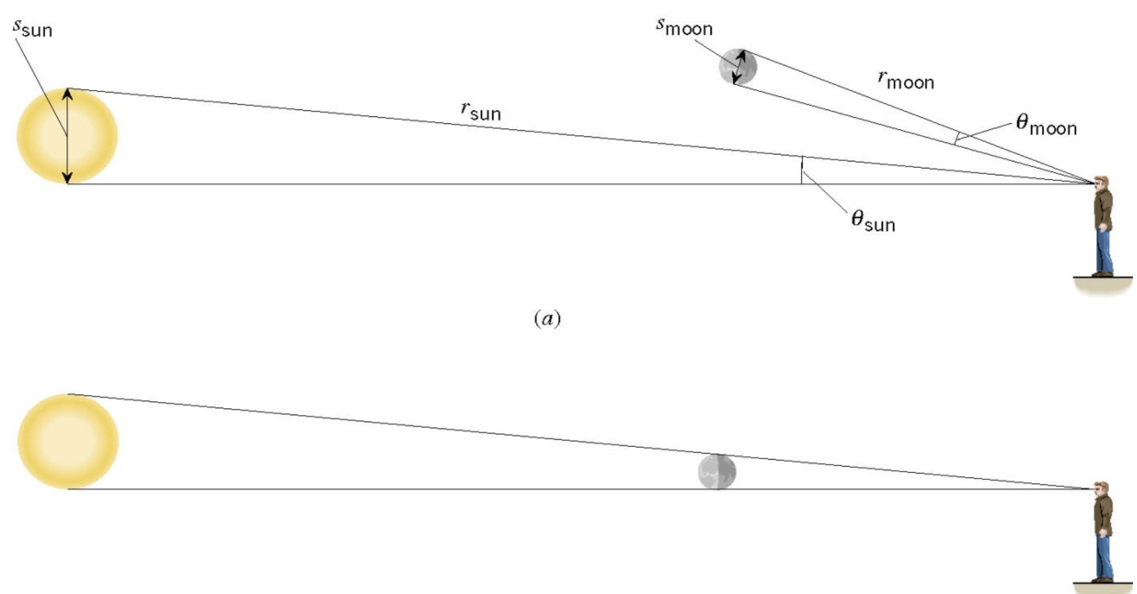
$$s = r\theta = \left(4.23\times10^{7}\,\text{m}\right)\left(0.0349\ \text{rad}\right)$$

$$= 1.48\times10^{6}\,\text{m}\ \ (920\ \text{miles})$$

Ex. A Total Eclipse of the Sun

The diameter of the sun is about 400 times greater than that of the moon. By coincidence, the sun is also about 400 times farther from the earth than is the moon. For an observer on the earth, compare the angle subtended by the moon to the angle subtended by the sun and explain why this result leads to a total solar eclipse.

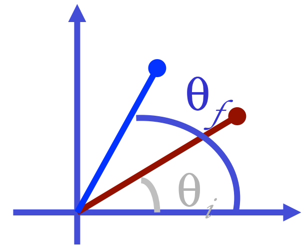
$$\theta \text{ (in radians)} = \frac{\text{Arc length}}{\text{Radius}} = \frac{s}{r}$$



I can even cover the entire sun with my thumb!! Why?

Because the distance (r) from my eyes to my thumb is far shorter than that to the sun.

Angular Displacement, Velocity, and Acceleration



Angular displacement is defined as

$$\Delta\theta = \theta_f - \theta_i$$

How about the average angular velocity, the rate of change of angular displacement?

$$\overline{\omega} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

Unit? rad/s Dimension? $[T^{-1}]$

By the same token, the average angular acceleration, rate of change of the angular velocity, is defined as…

$$\overline{\alpha} \equiv \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

Unit? rad/s^2 Dimension? $[T^{-2}]$

When rotating about a fixed axis, every particle on a rigid object rotates through the same angle and has the same angular speed and angular acceleration.

Problem Solving Strategy

- Visualize the problem by drawing a picture.
- Write down the values that are given for any of the five kinematic variables and convert them to SI units.
  - Remember that the unit of the angle must be radians!!
- Verify that the information contains values for at least three of the five kinematic variables. Select the appropriate equation.
- When the motion is divided into segments, remember that the final angular velocity of one segment is the initial velocity for the next.
- Keep in mind that there may be two possible answers to a kinematics problem.

Ex. Rotational Kinematics

A wheel rotates with a constant angular acceleration of 3.50 rad/s^2. If the angular speed of the wheel is 2.00 rad/s at t_i=0, a) through what angle does the wheel rotate in 2.00s?

Using the angular displacement formula in the previous slide, one gets

$$\theta_f - \theta_i = \omega t + \frac{1}{2}\alpha t^2$$

$$= 2.00 \times 2.00 + \frac{1}{2} 3.50 \times (2.00)^2 = 11.0 rad$$

$$= \frac{11.0}{2\pi} rev. = 1.75 rev.$$

Example for Rotational Kinematics cnt'd

What is the angular speed at t=2.00s?

Using the angular speed and acceleration relationship

$$\omega_f = \omega_i + \alpha t = 2.00 + 3.50 \times 2.00 = 9.00 rad/s$$

Find the angle through which the wheel rotates between t=2.00s and t=3.00s.

Using the angular kinematic formula

$$\theta_f - \theta_i = \omega t + \frac{1}{2}\alpha t^2$$

At t=2.00s

$$\theta_{t=2} = 2.00 \times 2.00 + \frac{1}{2} 3.50 \times 2.00 = 11.0 rad$$

At t=3.00s

$$\theta_{t=3} = 2.00 \times 3.00 + \frac{1}{2} 3.50 \times (3.00)^2 = 21.8 rad$$

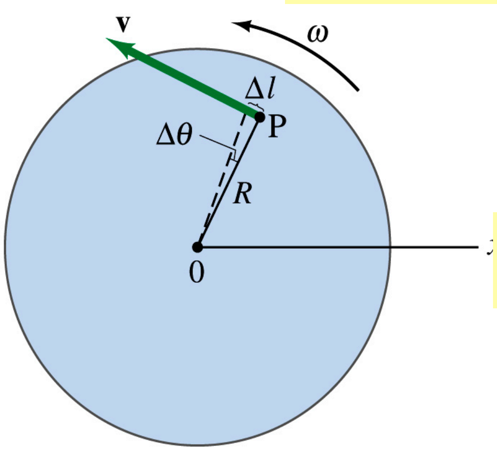
Angular displacement

$$\Delta\theta = \theta_3 - \theta_2 = 10.8 rad = \frac{10.8}{2\pi} rev. = 1.72 rev.$$

Relationship Between Angular and Linear Quantities

What do we know about a rigid object that rotates about a fixed axis of rotation?

Every particle (or masslet) in the object moves in a circle centered at the same axis of rotation.



When a point rotates, it has both the linear and angular components in its motion.

What is the linear component of the motion you see?

Linear velocity along the tangential direction.

The direction of ω follows the right-hand rule.

How do we related this linear component of the motion with angular component?

The arc-length is $\boxed{l = r\theta}$ So the tangential speed v is

$$v = \frac{dl}{dt} = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} = r\omega$$

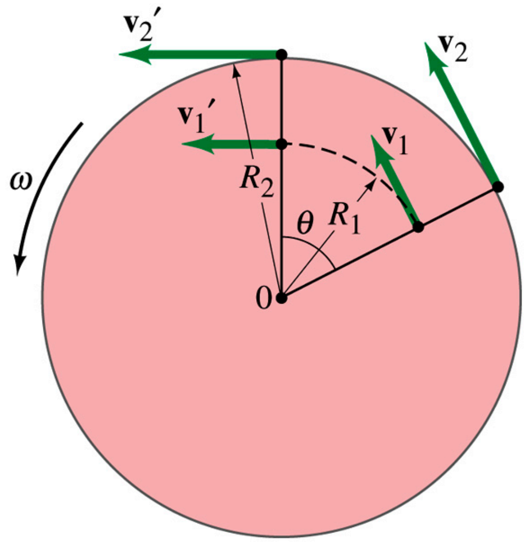
What does this relationship tell you about the tangential speed of the points in the object and their angular speed?:

Although every particle in the object has the same angular speed, its tangential speed differs and is proportional to its distance from the axis of rotation.

The farther away the particle is from the center of rotation, the higher the tangential speed.

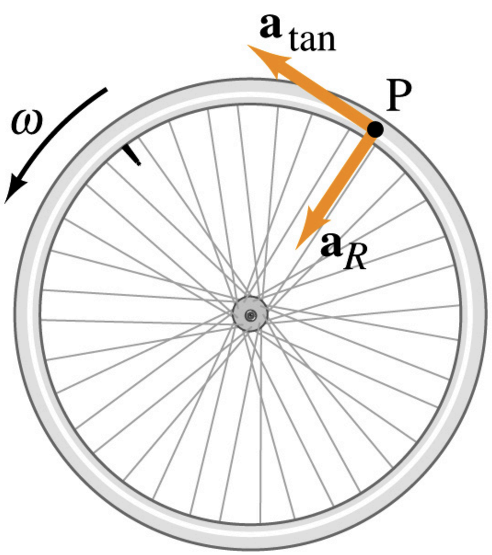
Is the lion faster than the horse?

A rotating carousel has one child sitting on a horse near the outer edge and another child on a lion halfway out from the center. (a) Which child has the greater linear speed? (b) Which child has the greater angular speed?



(a) Linear speed is the distance traveled divided by the time interval. So the child sitting at the outer edge travels more distance within the given time than the child sitting closer to the center. Thus, the horse is faster than the lion.

(b) Angular speed is the angle traveled divided by the time interval. The angle both the children travel in the given time interval is the same. Thus, both the horse and the lion have the same angular speed.



How about the acceleration?

How many different linear acceleration components do you see in a circular motion and what are they? Two

Tangential, a_t, and the radial acceleration, a_r.

Since the tangential speed v is $v = r\omega$

The magnitude of tangential acceleration a_t is $a_t = \frac{dv}{dt} = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} = r\alpha$

What does this relationship tell you?

Although every particle in the object has the same angular acceleration, its tangential acceleration differs proportional to its distance from the axis of rotation.

The radial or centripetal acceleration a_r is $a_r = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$

What does this tell you?

The father away the particle is from the rotation axis, the more radial acceleration it receives. In other words, it receives more centripetal force.

Total linear acceleration is $a = \sqrt{a_t^2 + a_r^2} = \sqrt{(r\alpha)^2 + (r\omega^2)^2} = r\sqrt{\alpha^2 + \omega^4}$

Example

(a) What is the linear speed of a child seated 1.2m from the center of a steadily rotating merry-go-around that makes one complete revolution in 4.0s? (b) What is her total linear acceleration?

First, figure out what the angular speed of the merry-go-around is.

$$\varpi = \frac{1rev}{4.0s} = \frac{2\pi}{4.0s} = 1.6rad/s$$

Using the formula for linear speed

$$v = r\omega = 1.2m \times 1.6rad/s = 1.9m/s$$

Since the angular speed is constant, there is no angular acceleration.

Tangential acceleration is

$$a_t = r\alpha = 1.2m \times 0rad/s^2 = 0m/s^2$$

Radial acceleration is

$$a_r = r\varpi^2 = 1.2m \times (1.6rad/s)^2 = 3.1m/s^2$$

Thus the total acceleration is

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{0 + (3.1)^2} = 3.1m/s^2$$

Example for Rotational Motion

Audio information on compact discs are transmitted digitally through the readout system consisting of laser and lenses. The digital information on the disc are stored by the pits and flat areas on the track. Since the speed of readout system is constant, it reads out the same number of pits and flats in the same time interval. In other words, the linear speed is the same no matter which track is played. a) Assuming the linear speed is 1.3 m/s, find the angular speed of the disc in revolutions per minute when the inner most (r=23mm) and outer most tracks (r=58mm) are read.

Using the relationship between angular and tangential speed $v = r\omega$

$$r = 23mm \qquad \omega = \frac{v}{r} = \frac{1.3m/s}{23mm} = \frac{1.3}{23\times10^{-3}} = 56.5rad/s$$

$$= 9.00rev/s = 5.4\times10^{2} rev/\min$$

$$r = 58mm \qquad \omega = \frac{1.3m/s}{58mm} = \frac{1.3}{58\times10^{-3}} = 22.4rad/s$$

$$= 2.1\times10^{2} rev/\min$$

b) The maximum playing time of a standard music CD is 74 minutes and 33 seconds. How many revolutions does the disk make during that time?

$$\overline{\omega} = \frac{(\omega_i + \omega_f)}{2} = \frac{(540+210)rev/\min}{2} = 375 rev/\min$$

$$\theta_f = \theta_i + \overline{\omega} t = 0 + \frac{375}{60} rev/s \times 4473 s = 2.8\times 10^4 rev$$

c) What is the total length of the track past through the readout mechanism?

$$l = v_t \Delta t = 1.3 m/s \times 4473 s = 5.8\times 10^3 m$$

d) What is the angular acceleration of the CD over the 4473s time interval, assuming constant a?

$$\alpha = \frac{(\omega_f - \omega_i)}{\Delta t} = \frac{(22.4 - 56.5) rad/s}{4473 s} = 7.6\times 10^{-3} rad/s^2$$