PHYS 1443 – Section 004
Lecture #17

Thursday, Oct. 23, 2014
Dr. Jaehoon Yu

- Torque & Vector product
- Moment of Inertia
- Calculation of Moment of Inertia
- Parallel Axis Theorem
- Torque and Angular Acceleration
- Rolling Motion and Rotational Kinetic Energy

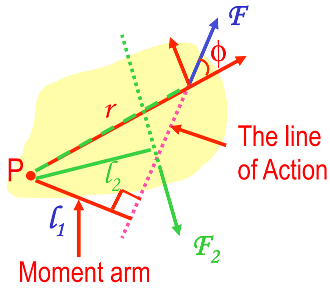
Today's homework is homework #9, due 11pm, Thursday, Oct. 30!!

Announcements

- Mid-term grade discussion
  - Coming Tuesday, Oct. 28
  - Will have 40min of class followed by a 45min grade discussion
  - In my office, CPB342

Torque

Torque is the tendency of a force to rotate an object about an axis. Torque, τ, is a vector quantity.



Consider an object pivoting about the point P by the force F being exerted at a distance r from P.

The line that extends out of the tail of the force vector is called the line of action.

The perpendicular distance from the pivoting point P to the line of action is called the moment arm.

Magnitude of torque is defined as the product of the force exerted on the object to rotate it and the moment arm.

$$|\vec{\tau}| \equiv (\text{Magnitude of the Force}) \times (\text{Lever Arm}) = (F)(r\sin\phi) = Fl_1$$

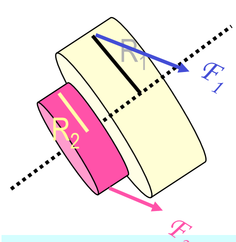
When there are more than one force being exerted on certain points of the object, one can sum up the torque generated by each force vectorially. The convention for sign of the torque is **positive if rotation is in counter-clockwise** and **negative if clockwise**.

$$\sum \tau = \tau_1 + \tau_2 = F_1 l_1 - F_2 l_2$$

Unit? $N \cdot m$

Ex. Torque

A one piece cylinder is shaped as in the figure with core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the picture. A rope wrapped around the drum whose radius is R_1 exerts force F_1 to the right on the cylinder, and another force exerts F_2 on the core whose radius is R_2 downward on the cylinder. A) What is the net torque acting on the cylinder about the rotation axis?



The torque due to F_1 $\tau_1 = -R_1 F_1$ and due to F_2 $\tau_2 = R_2 F_2$

So the total torque acting on the system by the forces is $\sum \tau = \tau_1 + \tau_2 = -R_1 F_1 + R_2 F_2$

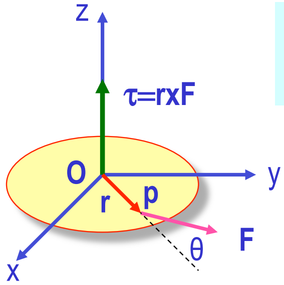
Suppose F_1=5.0 N, R_1=1.0 m, F_2= 15.0 N, and R_2=0.50 m. What is the net torque about the rotation axis and which way does the cylinder rotate from the rest?

Using the above result

$$\sum \tau = -R_1 F_1 + R_2 F_2 = -5.0 \times 1.0 + 15.0 \times 0.50 = 2.5\ N \cdot m$$

The cylinder rotates in counter-clockwise.

Torque and Vector Product



Let's consider a disk fixed onto the origin O and the force $\mathcal{F}$ exerts on the point p. What happens?

The disk will start rotating counter clockwise about the Z axis

The magnitude of the torque given to the disk by the force $\mathcal{F}$ is

$$\tau = Fr\sin\theta$$

But torque is a vector quantity, what is the direction?
How is torque expressed mathematically?

$$\vec{\tau} \equiv \vec{r} \times \vec{F}$$

What is the direction? The direction of the torque follows the right-hand rule!!

The above operation is called the Vector product or Cross product

$$\vec{C} \equiv \vec{A} \times \vec{B}$$

$$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$$

What is the result of a vector product?

Another vector

What is another vector operation we've learned?

Scalar product $C \equiv \vec{A} \cdot \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$

Result? A scalar

Properties of Vector Product

Vector Product is Non-commutative What does this mean?

If the order of operation changes the result changes $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$

Following the right-hand rule, the direction changes

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

Vector Product of two parallel vectors is 0.

$$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta = \left|\vec{A}\right|\left|\vec{B}\right|\sin 0 = 0$$

Thus, $\vec{A} \times \vec{A} = 0$

If two vectors are perpendicular to each other

$$\left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta = \left|\vec{A}\right|\left|\vec{B}\right|\sin 90^\circ = \left|\vec{A}\right|\left|\vec{B}\right| = AB$$

Vector product follows distribution law

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

The derivative of a Vector product with respect to a scalar variable is

$$\frac{d\left(\vec{A} \times \vec{B}\right)}{dt} = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

More Properties of Vector Product

The relationship between unit vectors, $\vec{i}$, $\vec{j}$ and $\vec{k}$

$$\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0$$
$$\vec{i}\times\vec{j}=-\vec{j}\times\vec{i}=\vec{k}$$
$$\vec{j}\times\vec{k}=-\vec{k}\times\vec{j}=\vec{i}$$
$$\vec{k}\times\vec{i}=-\vec{i}\times\vec{k}=\vec{j}$$

Vector product of two vectors can be expressed in the following determinant form

$$\vec{A}\times\vec{B}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\vec{i}\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}-\vec{j}\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}+\vec{k}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}$$

$$=\left(A_yB_z-A_zB_y\right)\vec{i}-\left(A_xB_z-A_zB_x\right)\vec{j}+\left(A_xB_y-A_yB_x\right)\vec{k}$$

Moment of Inertia

Rotational Inertia: The measure of resistance of an object to changes in its rotational motion. Equivalent to mass in linear motion.

For a group of particles $I\equiv\sum_i m_i r_i^2$ For a rigid body $I\equiv\int r^2 dm$

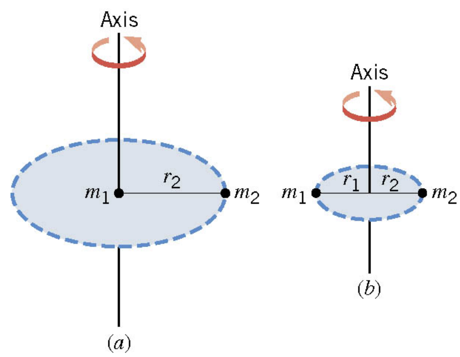
What are the dimension and unit of Moment of Inertia? $\left[ML^2\right]$ $kg\cdot m^2$

Determining Moment of Inertia is extremely important for computing equilibrium of a rigid body, such as a building.

Dependent on the axis of rotation!!!

Ex. The Moment of Inertia Depends on Where the Axis Is.

Two particles each have mass m_1 and m_2 and are fixed at the ends of a thin rigid rod. The length of the rod is *L*. Find the moment of inertia when this object rotates relative to an axis that is perpendicular to the rod at (a) one end and (b) the center.



(a) $I = \sum\left(mr^2\right) = m_1 r_1^2 + m_2 r_2^2$

$m_1 = m_2 = m \qquad r_1 = 0 \quad r_2 = L$

$I = m(0)^2 + m(L)^2 = mL^2$

(b) $I = \sum\left(mr^2\right) = m_1 r_1^2 + m_2 r_2^2$

$m_1 = m_2 = m \qquad r_1 = L/2 \quad r_2 = L/2$

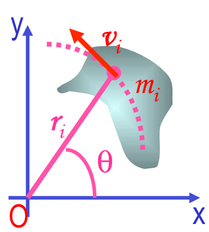
$I = m(L/2)^2 + m(L/2)^2 = \frac{1}{2}mL^2$

Which case is easier to spin?

Case (b)

Why? Because the moment of inertia is smaller

Rotational Kinetic Energy



What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, m_i, moving at a tangential speed, v_i, is $K_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i r_i^2 \omega^2$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

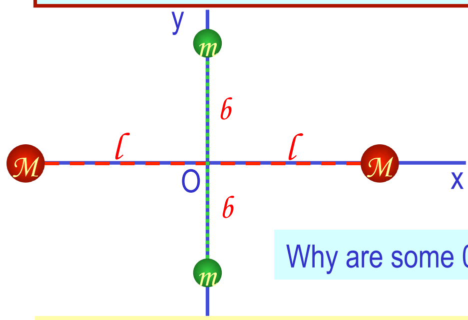
$$K_R = \sum_i K_i = \frac{1}{2}\sum_i m_i r_i^2 \omega^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

Since moment of Inertia, I, is defined as $I = \sum_i m_i r_i^2$

The above expression is simplified as $K_R = \frac{1}{2}I\omega^2$

Example for Moment of Inertia

In a system of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at angular speed ω.



Since the rotation is about y axis, the moment of inertia about y axis, I_y, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m \cdot 0^2 + m \cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2\right)\omega^2 = Ml^2\omega^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2\left(Ml^2 + mb^2\right) \qquad K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2 + 2mb^2\right)\omega^2 = \left(Ml^2 + mb^2\right)\omega^2$$

Calculation of Moments of Inertia

Moments of inertia for large rigid objects can be computed, if we assume the object consists of small volume elements with mass, Δm_i.

The moment of inertia for the large rigid object is

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$

It is sometimes easier to compute moments of inertia in terms of volume of the elements rather than their mass

How can we do this?

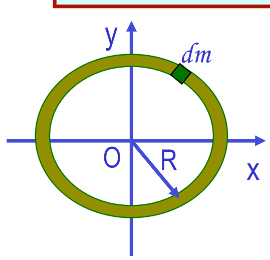
Using the volume density, ρ, replace dm in the above equation with dV.

$$\rho = \frac{dm}{dV} \Rightarrow dm = \rho dV$$

The moments of inertia becomes

$$I = \int \rho r^2 \, dV$$

Example: Find the moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.



The moment of inertia is

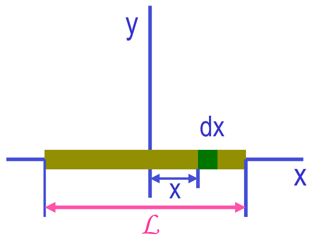
$$I = \int r^2 \, dm = R^2 \int dm = MR^2$$

What do you notice from this result?

The moment of inertia for this object is the same as that of a point of mass M at the distance R.

Ex. Rigid Body Moment of Inertia

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing through its center of mass.



The line density of the rod is $\lambda = \frac{M}{L}$

so the mass of a masslet is $dm = \lambda dx = \frac{M}{L}dx$

The moment of inertia is

$$I = \int r^2\,dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L}dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

What is the moment of inertia when the rotational axis is at one end of the rod.

$$I = \int r^2\,dm = \int_0^L \frac{x^2 M}{L}dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_0^L$$

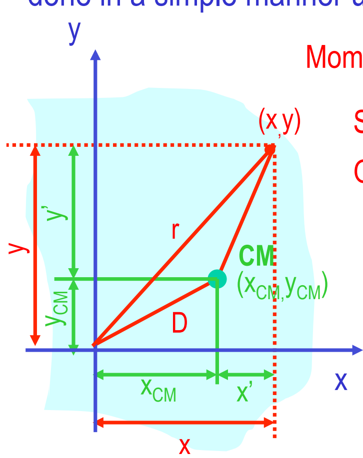
$$= \frac{M}{3L}\left[(L)^3 - 0\right] = \frac{M}{3L}\left(L^3\right) = \frac{ML^2}{3}$$

Will this be the same as the above. Why or why not?

Since the moment of inertia is resistance to motion, it makes perfect sense for it to be harder to move when it is rotating about the axis at one end.

Parallel Axis Theorem

Moments of inertia for a highly symmetric object is easy to compute if the rotational axis is the same as the axis of symmetry. However if the axis of rotation does not coincide with axis of symmetry, the calculation can still be done in a simple manner using the **parallel-axis theorem**. $I = I_{CM} + MD^2$



Moment of inertia is defined as $I = \int r^2\,dm = \int \left(x^2 + y^2\right)dm$ (1)

Since x and y are $x = x_{CM} + x'$ $y = y_{CM} + y'$

One can substitute x and y in Eq. 1 to obtain

$$I = \int\left[\left(x_{CM} + x'\right)^2 + \left(y_{CM} + y'\right)^2\right]dm$$

$$= \left(x_{CM}^2 + y_{CM}^2\right)\int dm + 2x_{CM}\int x'dm + 2y_{CM}\int y'dm + \int\left(x'^2 + y'^2\right)dm$$

Since x' and y' are the distance from CM, by definition $\int x'dm = 0$ $\int y'dm = 0$

Therefore, the parallel-axis theorem

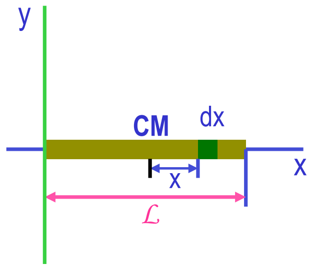
$$I = \left(x_{CM}^2 + y_{CM}^2\right)\int dm + \int\left(x'^2 + y'^2\right)dm = MD^2 + I_{CM}$$

What does this theorem tell you?

Moment of inertia of any object about any arbitrary axis are the same as the sum of moment of inertia for **a rotation about the CM** and **that of the CM about the rotation axis**.

Example for Parallel Axis Theorem

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis that goes through one end of the rod, using parallel-axis theorem.



The line density of the rod is $\lambda = \frac{M}{L}$

so the mass of a masslet is $dm = \lambda dx = \frac{M}{L} dx$

The moment of inertia about the CM

$$I_{CM} = \int r^2\, dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

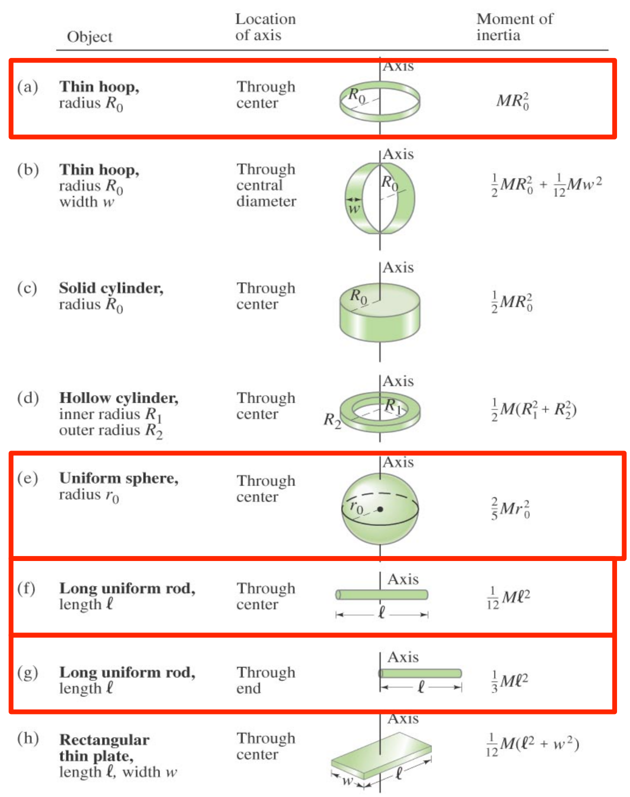
Using the parallel axis theorem

$$I = I_{CM} + D^2 M = \frac{ML^2}{12} + \left(\frac{L}{2}\right)^2 M = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

The result is the same as using the definition of moment of inertia.

Parallel-axis theorem is useful to compute moment of inertia of a rotation of a rigid object with complicated shape about an arbitrary axis

Check out Table 10 – 2 for moment of inertia for various shaped objects



	Object	Location of axis	Moment of inertia
(a)	**Thin hoop,** radius R_0	Through center	MR_0^2
(b)	**Thin hoop,** radius R_0 width w	Through central diameter	$\frac{1}{2}MR_0^2 + \frac{1}{12}Mw^2$
(c)	**Solid cylinder,** radius R_0	Through center	$\frac{1}{2}MR_0^2$
(d)	**Hollow cylinder,** inner radius R_1 outer radius R_2	Through center	$\frac{1}{2}M(R_1^2 + R_2^2)$
(e)	**Uniform sphere,** radius r_0	Through center	$\frac{2}{5}Mr_0^2$
(f)	**Long uniform rod,** length ℓ	Through center	$\frac{1}{12}M\ell^2$
(g)	**Long uniform rod,** length ℓ	Through end	$\frac{1}{3}M\ell^2$
(h)	**Rectangular thin plate,** length ℓ, width w	Through center	$\frac{1}{12}M(\ell^2 + w^2)$