PHYS 1444 – Section 002
Lecture #24

Monday, Dec. 2, 2019

Dr. Jaehoon Yu

- Chapter 31: Maxwell’s equations
  - Expansion of Ampere’s Law
  - Gauss’ Law for Magnetism
  - Production of EM Waves
  - Light as EM Waves

Announcements

- Reading assignments: CH31.6 – 10
- Submit planetarium extra credit!
- Final comprehensive exam
  - In class 1:00 – 2:20pm, THIS Wednesday, Dec. 4
  - Covers: CH21.1 – CH31.6 plus math refresher in Appendices
  - Bring your calculator but DO NOT input formula into it!
    - Cell phones or any types of computers cannot replace a calculator!
  - BYOF: You may bring a one 8.5x11.5 sheet (front and back) of **handwritten** formulae and values of constants for the quiz
  - No derivations, word definitions, set ups or solutions of any problems!
  - NO MAXWELL's Equations!!
  - No additional formulae or values of constants will be provided!
- Quiz 4 results
  - Class average: 33.6/80
    - Equivalent to 42/100
    - Previous quizzes: 56.4, 46.3 and 66.8
  - Top score: 78/80

Maxwell's Equations

- The development of EM theory by Oersted, Ampere and others was not done in terms of EM fields
  - The idea of fields was introduced somewhat by Faraday
- Scottish physicist James C. Maxwell unified all the phenomena of electricity and magnetism in one theory with only four equations (Maxwell's Equations) using the concept of fields
  - This theory provided the prediction of EM waves
  - As important as Newton's law since it provides dynamics of electromagnetism
  - This theory is also in agreement with Einstein's special relativity
- The biggest achievement of 19th century electromagnetic theory is the prediction and experimental verifications that the electromagnetic waves can travel through the empty space
  - What do you think this accomplishment did?
    - Open a new world of communication
    - It also yielded the prediction that the light is an EM wave
- Since all of Electromagnetism is contained in the four Maxwell's equations, this is considered as one of the greatest achievements of human intellect

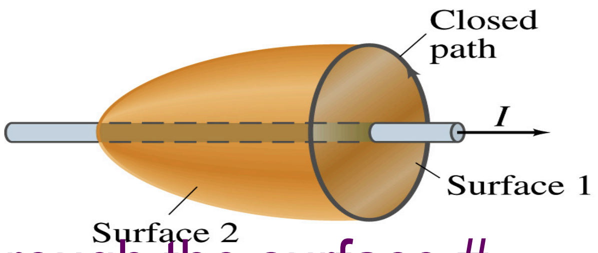
Ampere's Law

- Do you remember the mathematical expression of Oersted discovery of a magnetic field produced by an electric current, given by Ampere?

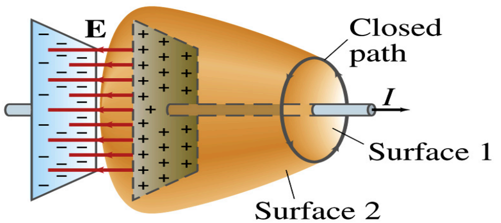
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$

- We've learned that a varying magnetic field produces an electric field
- Then can the reverse phenomena, that a changing electric field producing a magnetic field, possible?
  - If this is the case, it would demonstrate a beautiful symmetry in nature between electricity and magnetism

Expanding Ampere's Law



- Let's consider a wire carrying current I
  - The current that is enclosed in the loop passes through the surface # 1 in the figure
  - We could imagine a different surface # 2 that shares the same enclosed path but cuts through the wire in a different location. What is the current that passes through this surface?
    - Still I.
  - So the Ampere's law still works



- We could then consider a capacitor being charged up or being discharged.
  - The current I enclosed in the loop passes through the surface #1
  - However the surface #2 that shares the same closed loop do not have any current passing through it.
    - There is magnetic field present since there is current ➔ In other words there is a changing electric field in between the plates
    - Maxwell resolved this by adding an additional term to Ampere's law involving the changing electric field

Modifying Ampere's Law

- To determine what the extra term should be, we first have to figure out what the electric field between the two plates is
  - The charge Q on the capacitor with capacitance C is Q=CV
    - Where V is the potential difference between the plates
  - Since V=Ed
    - Where E is the uniform field between the plates, and d is the separation of the plates
  - And for parallel plate capacitor $C=\varepsilon_0 A/d$
  - We obtain

$$Q = CV = \left(\varepsilon_0 \frac{A}{d}\right) Ed = \varepsilon_0 AE$$

Modifying Ampere's Law

- If the charge on the plate changes with time, we can write

$$\frac{dQ}{dt} = \varepsilon_0 A \frac{dE}{dt}$$

- Using the relationship between the current and charge we obtain

$$I = \frac{dQ}{dt} = \varepsilon_0 A \frac{dE}{dt} = \varepsilon_0 \frac{d\left(AE\right)}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt}$$

  - Where Φ_E=EA is the electric flux through the surface between the plates

- So in order to make Ampere's law work for the surface 2 in the figure, we must write it in the following form

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}}$$

Extra term by Maxwell

- This equation represents the general form of Ampere's law
  - This means that a magnetic field can be caused not only by an ordinary electric current but also by a changing electric flux

Example 31 – 1

Charging capacitor. A 30-pF air-gap capacitor has circular plates of area A=100cm^2. It is charged by a 70-V battery through a 2.0-Ω resistor. At the instance the battery is connected, the electric field between the plates is changing most rapidly. At this instance, calculate (a) the current into the plates, and (b) the rate of change of electric field between the plates. (c) Determine the magnetic field induced between the plates. Assume **E** is uniform between the plates at any instant and is zero at all points beyond the edges of the plates.

Since this is an RC circuit, the charge on the plates is: $Q = CV_0\left(1 - e^{-t/RC}\right)$

For the initial current (t=0), we differentiate the charge with respect to time.

$$I_0 = \left.\frac{dQ}{dt}\right|_{t=0} = \left.\frac{CV_0}{RC} e^{-t/RC}\right|_{t=0} = \frac{V_0}{R} = \frac{70V}{2.0\Omega\cdot} = 35A$$

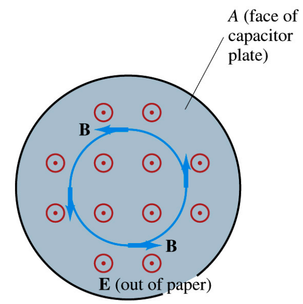
The electric field is $E = \dfrac{\sigma}{\varepsilon_0} = \dfrac{Q/A}{\varepsilon_0}$

Change of the electric field is $\dfrac{dE}{dt} = \dfrac{dQ/dt}{A\varepsilon_0} = \dfrac{35A}{\left(8.85\times10^{-12}\ C^2/N\cdot m^2\right)\cdot\left(1.0\times10^{-2}\ m^2\right)} = 4.0\times10^{14}\ V/m\cdot s$

Example 31 – 1

(c) Determine the magnetic field induced between the plates. Assume **E** is uniform between the plates at any instant and is zero at all points beyond the edges of the plates.

The magnetic field lines generated by changing electric field is perpendicular to E and is circular due to symmetry



Whose law can we use to determine B?

Extended Ampere's Law w/ I_{encl}=0!

$$\oint \vec{B}\cdot d\vec{l} = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$

We choose a circular path of radius r, centered at the center of the plane, following the B.

For $r<r_{plate}$, the electric flux is $\Phi_E = EA = E\pi r^2$ since E is uniform throughout the plate

So from Ampere's law, we obtain $$B\cdot(2\pi r) = \mu_0\varepsilon_0 \frac{d\left(E\pi r^2\right)}{dt} = \mu_0\varepsilon_0\pi r^2\frac{dE}{dt}$$

Solving for B

$$\boxed{B = \mu_0\varepsilon_0 \frac{r}{2}\frac{dE}{dt}}$$

For $r<r_{plate}$

Since we assume E=0 for $r>r_{plate}$, the electric flux beyond the plate is fully contained inside the surface. $\Phi_E = EA = E\pi r_{plate}^2$

So from Ampere's law, we obtain $$B\cdot(2\pi r) = \mu_0\varepsilon_0 \frac{d\left(E\pi r_{plate}^2\right)}{dt} = \mu_0\varepsilon_0\pi r_{plate}^2\frac{dE}{dt}$$

Solving for B

$$\boxed{B = \frac{\mu_0\varepsilon_0 r_{plate}^2}{2r}\frac{dE}{dt}}$$

For $r>r_{plate}$

Displacement Current

- Maxwell interpreted the second term in the generalized Ampere’s law equivalent to an electric current
  - He called this term as **the displacement current, I_D**
  - While the other term is called as **the conduction current, I**
- Ampere’s law then can be written as

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(I_{encl} + I_D \right)}$$

  - Where

$$I_D = \varepsilon_0 \frac{d\Phi_E}{dt}$$

  - While it is in effect equivalent to an electric current, a flow of electric charge, this actually does not have anything to do with the flow itself

Gauss' Law for Magnetism

- If there is a symmetry between electricity and magnetism, there must be an equivalent law in magnetism as the Gauss' Law in electricity
- For a magnetic field **B**, the magnetic flux Φ_B through the surface is defined as

$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$

  - Where the integration is over the area of either an open or a closed surface
- The magnetic flux through a closed surface which completely encloses a volume is

$$\Phi_B = \oint \vec{B} \cdot d\vec{A}$$

- What was the Gauss' law in the electric case?
  - The electric flux through a closed surface is equal to the total net charge Q enclosed by the surface divided by ε_0.

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0}$$

Gauss' Law for electricity

- Similarly, we can write Gauss' law for magnetism as

$$\oint \vec{B} \cdot d\vec{A} = 0$$

Gauss' Law for magnetism

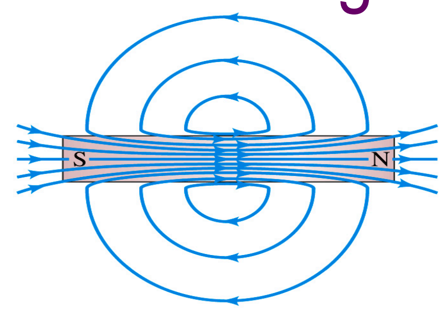
- Why is result of the integral zero?
  - There is no isolated magnetic poles, the magnetic equivalent of single electric charges

Gauss' Law for Magnetism

- What does the Gauss' law in magnetism mean physically?

$$\boxed{\oint \vec{B} \cdot d\vec{A} = 0}$$

  - There are as many magnetic flux lines that enter the enclosed volume as leave it
  - If magnetic monopole does not exist, there is no starting or stopping point of the flux lines
    - Electricity do have the source and the sink



  - Magnetic field lines must be continuous
  - Even for a bar magnet, the field lines exist both insides and outside of the magnet

Maxwell's Equations

- In the absence of dielectric or magnetic materials, the four equations developed by Maxwell are:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0}$$

Gauss' Law for electricity

A generalized form of Coulomb's law relating electric field to its sources, the electric charge

$$\oint \vec{B} \cdot d\vec{A} = 0$$

Gauss' Law for magnetism

A magnetic equivalent of Coulomb's law relating magnetic field to its sources. This says there are no magnetic monopoles.

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

Faraday's Law

An electric field is produced by a changing magnetic field

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

Ampére's Law

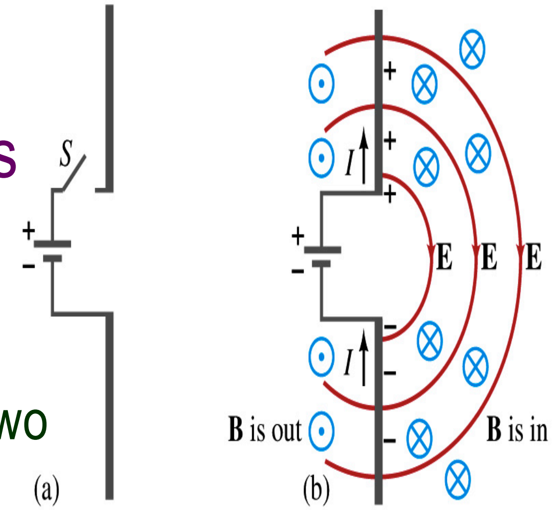
A magnetic field is produced by an electric current or by a changing electric field

Maxwell's Amazing Leap of Faith

- According to Maxwell, a magnetic field will be produced even in an empty space if there is a changing electric field
  - He then took this concept one step further & concluded that
    - If a changing magnetic field produces an electric field, the electric field is also changing in time.
    - This changing electric field in turn produces a magnetic field that changes.
    - This changing magnetic field then in turn produces an electric field that changes.
    - This process continues.
  - With a manipulation of the equations, Maxwell found that the net result of this interacting changing fields is a wave of electric and magnetic fields that can actually propagate (travel) through the space

Production of EM Waves

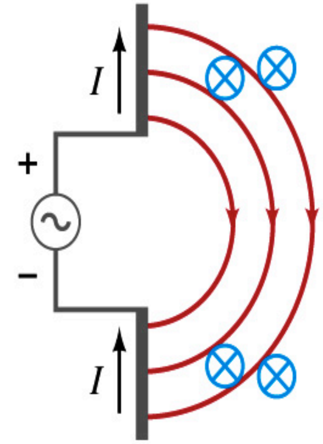
- Consider two conducting rods that will serve as an antenna are connected to a DC power source
  - What do you think will happen when the switch is closed?
    - The rod connected to the positive terminal is charged positively and the other negatively
    - Then the electric field will be generated between the two rods
    - Since there is current that flows through, the rods generates a magnetic field around them



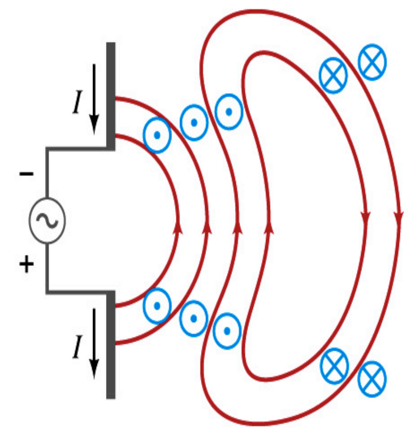
- How far would the electric and magnetic fields extend?
  - In static case, the field extends indefinitely
  - When the switch is closed, the fields are formed nearby the rods quickly but
  - The stored energy in the fields won't propagate w/ infinite speed

Production of EM Waves

- What happens if the antenna is connected to an AC power source?
  - When the connection was initially made, the rods are charging up quickly w/ the current flowing in one direction as shown in the figure
    - The field lines form as in the dc case
    - The field lines propagate away from the antenna



  - Then the direction of the voltage reverses
    - The new field lines with the opposite direction forms
    - While the original field lines still propagates away from the rod reaching out far
      - Since the original field propagates through an empty space, the field lines must form a closed loop (no charge exist)
    - Since changing electric and magnetic fields produce changing magnetic and electric fields, the fields moving outward is self supporting and do not need antenna with flowing charge



  - The fields far from the antenna is called the **radiation field**
  - Both electric and magnetic fields form closed loops perpendicular to each other

Properties of Radiation Fields – I

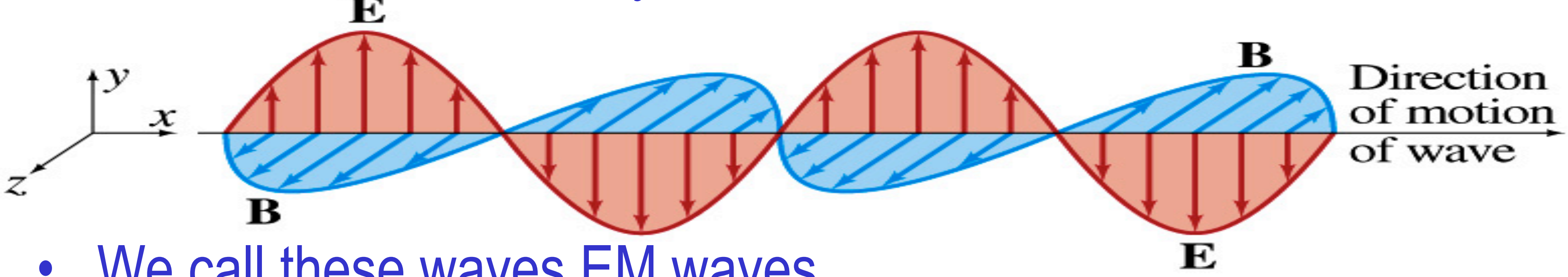
- The fields travel on the other side of the antenna as well
- The field strength is the greatest in the direction perpendicular to the oscillating charge while it is 0 along the direction of the current
- The magnitude of E and B in the radiation field decrease with distance as 1/r
- The energy carried by the EM wave is proportional to the square of the amplitude, E^2 or B^2
  - So the intensity of wave decreases as $1/r^2$

Properties of Radiation Fields – II

- The electric and magnetic fields at any point are perpendicular to each other and to the direction of motion
- The fields alternate in direction
  - The field strengths vary from maximum in one direction, to 0 and to max in the opposite direction
- The electric and magnetic fields are in phase
- Very far from the antenna, the field lines are quite flat over a reasonably large area
  - Called plane waves

EM Waves

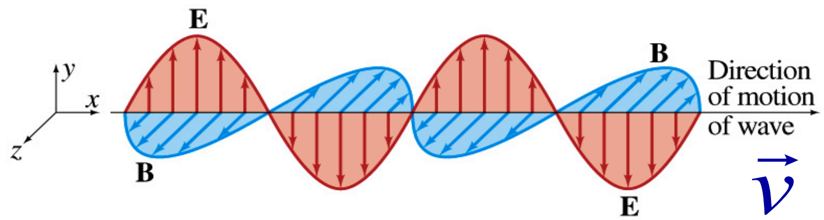
- If the voltage of the source varies sinusoidal, the field strengths of the radiation field vary sinusoidal



- We call these waves EM waves
- They are transverse waves
- EM waves are always waves of fields
  - Since these are fields, they can propagate through an empty space
- In general **accelerating electric charges give rise to electromagnetic waves**
- This prediction from Maxwell's equations was experimentally proven by Heinrich Hertz through the discovery of radio waves

EM Waves and Their Speeds

- Let's consider a region of free space. What's a free space?
  - An area of space where there is no charges or conduction currents
  - In other words, far from emf sources so that the wave fronts are essentially flat or not distorted over a reasonable area
  - What are these flat waves called?
    - **Plane waves**
    - At any instance **E** and **B** are uniform over a large plane perpendicular to the direction of propagation
  - So we can also assume that the wave is traveling in the x-direction w/ velocity, **v**=v**i**, and that **E** is parallel to y axis and **B** is parallel to z axis



Maxwell's Equations w/ Q=I=0

- In this region of free space, Q=0 and I=0, thus the four Maxwell's equations become

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0} \quad \xrightarrow{Q_{encl}=0} \quad \oint \vec{E}\cdot d\vec{A} = 0$$

$$\oint \vec{B}\cdot d\vec{A} = 0 \quad \xrightarrow{\text{No Changes}} \quad \oint \vec{B}\cdot d\vec{A} = 0$$

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} \quad \xrightarrow{\text{No Changes}} \quad \oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl} + \mu_0\varepsilon_0\frac{d\Phi_E}{dt} \quad \xrightarrow{I_{encl}=0} \quad \oint \vec{B}\cdot d\vec{l} = \mu_0\varepsilon_0\frac{d\Phi_E}{dt}$$

One can observe the symmetry between electricity and magnetism.

The last equation is the most important one for EM waves and their propagation!!

EM Waves from Maxwell's Equations

- If the wave is sinusoidal w/ wavelength λ and frequency *f*, such traveling wave can be written as

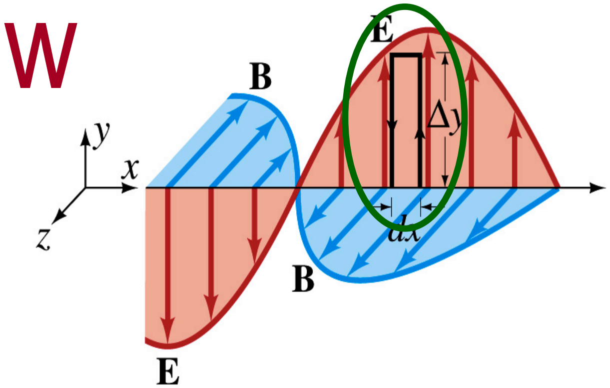
$$E = E_y = E_0 \sin\left(kx - \omega t\right)$$
$$B = B_z = B_0 \sin\left(kx - \omega t\right)$$

  – Where

$$k = \frac{2\pi}{\lambda} \qquad \varpi = 2\pi f \quad \textbf{Thus} \Rightarrow \quad f\lambda = \frac{\omega}{k} = v$$

  – What is v?
    - It is the speed of the traveling wave
  – What are E_0 and B_0?
    - The amplitudes of the EM wave. Maximum values of **E** and **B** field strengths.

From Faraday's Law



- Let's apply Faraday's law

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

  – to the rectangular loop of height Δy and width dx

- $\vec{E} \cdot d\vec{l}$ along the top and bottom of the loop is 0. Why?
  – Since **E** is perpendicular to d*l*.
  – So the result of the integral through the loop counterclockwise becomes $\oint \vec{E} \cdot d\vec{l} = \vec{E} \cdot d\vec{x} + \left(\vec{E} + d\vec{E}\right) \cdot \Delta\vec{y} + \vec{E} \cdot d\vec{x} + \vec{E} \cdot \Delta\vec{y} =$

$$= 0 + \left(E + dE\right)\Delta y - 0 - E\Delta y = dE\Delta y$$

  – For the right-hand side of Faraday's law, the magnetic flux through the loop changes as

$$-\frac{d\Phi_B}{dt} = \frac{dB}{dt}dx\Delta y$$

Thus

$$dE\Delta y = -\frac{dB}{dt}dx\Delta y$$

$$\frac{dE}{dx} = -\frac{dB}{dt}$$

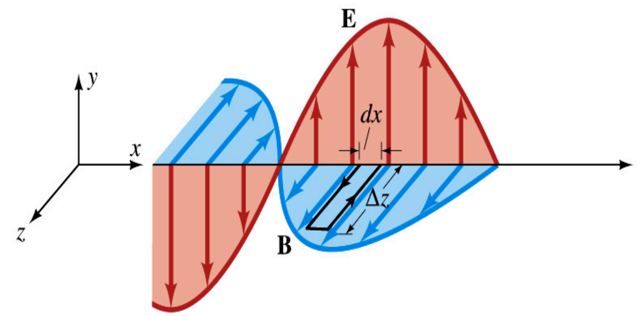
Since E and B depend on x and t

$$\boxed{\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}}$$

From Modified Ampére's Law

- Let's apply Maxwell's 4th equation

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$



  – to the rectangular loop of length Δz and width dx
- $\vec{B} \cdot d\vec{l}$ along the x-axis of the loop is 0
  – Since **B** is perpendicular to d*l*
  – So the result of the integral through the loop counterclockwise becomes

$$\oint \vec{B} \cdot d\vec{l} = B\Delta z - \left(B + dB\right)\Delta z = -dB\Delta z$$

  – For the right-hand side of the equation is

$$\mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \varepsilon_0 \frac{dE}{dt} dx\Delta z$$

Thus

$$-dB\Delta z = \mu_0 \varepsilon_0 \frac{dE}{dt} dx\Delta z$$

  – $$\frac{dB}{dx} = -\mu_0 \varepsilon_0 \frac{dE}{dt}$$

Since E and B depend on x and t

$$\boxed{\frac{\partial B}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}}$$

Relationship between **E**, **B** and **v**

- Let's now use the relationship from Faraday's law $\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$
- Taking the derivatives of E and B as given their traveling wave form, we obtain

$$\frac{\partial E}{\partial x} = \frac{\partial}{\partial x}\left(E_0 \sin\left(kx - \omega t\right)\right) = kE_0 \cos\left(kx - \omega t\right)$$

$$\frac{\partial B}{\partial t} = \frac{\partial}{\partial t}\left(B_0 \sin\left(kx - \omega t\right)\right) = -\omega B_0 \cos\left(kx - \omega t\right)$$

Since $\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$ **We obtain** $kE_0 \cancel{\cos\left(kx - \omega t\right)} = \omega B_0 \cancel{\cos\left(kx - \omega t\right)}$

Thus $\frac{E_0}{B_0} = \frac{\omega}{k} = \mathrm{v}$

  - Since E and B are in phase, we can write $E/B = \mathrm{v}$
    - This is valid at any point and time in space. What is v?
      - The velocity of the wave

Speed of EM Waves

- Let's now use the relationship from Apmere's law $\frac{\partial B}{\partial x} = -\varepsilon_0 \mu_0 \frac{\partial E}{\partial t}$
- Taking the derivatives of E and B as given their traveling wave form, we obtain

$$\frac{\partial B}{\partial x} = \frac{\partial}{\partial x}\left(B_0 \sin\left(kx - \omega t\right)\right) = kB_0 \cos\left(kx - \omega t\right)$$

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial t}\left(E_0 \sin\left(kx - \omega t\right)\right) = -\omega E_0 \cos\left(kx - \omega t\right)$$

Since $\frac{\partial B}{\partial x} = -\varepsilon_0 \mu_0 \frac{\partial E}{\partial t}$ **We obtain** $kB_0 \cancel{\cos\left(kx - \omega t\right)} = \varepsilon_0 \mu_0 \omega E_0 \cancel{\cos\left(kx - \omega t\right)}$

Thus $\frac{B_0}{E_0} = \frac{\varepsilon_0 \mu_0 \omega}{k} = \varepsilon_0 \mu_0 v$

  - However, from the previous page we obtain $E_0 / B_0 = \mathrm{v} = \frac{1}{\varepsilon_0 \mu_0 \mathrm{v}}$
  - Thus $\mathrm{v}^2 = \frac{1}{\varepsilon_0 \mu_0}$

$$\boxed{\mathrm{v} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = \frac{1}{\sqrt{\left(8.85\times10^{-12}\, C^2/N\cdot m^2\right)\cdot\left(4\pi\times10^{-7}\, T\cdot m/A\right)}} = 3.00\times10^8\, m/s}$$

The speed of EM waves is the same as the speed of light. EM waves behaves like the light.