PHYS 1444 – Section 501
Lecture #21

Wednesday, Apr. 19, 2006
Dr. Jaehoon Yu

- Energy Stored in the Magnetic Field
- LR circuit
- LC Circuit and EM Oscillation
- LRC circuit
- AC Circuit w/ Resistance only
- AC Circuit w/ Inductance only

Today's homework is homework #11, due 7pm, next Thursday!!

Announcements

- Quiz Monday, Apr. 24 early in class
  – Covers: CH 29 to CH30
- Reading assignments:
  – CH29-8

Energy Stored in a Magnetic Field

- When an inductor of inductance $\mathcal{L}$ is carrying current I which is changing at a rate dI/dt, energy is supplied to the inductor at a rate
  - $P = I\varepsilon = IL\frac{dI}{dt}$
- What is the work needed to increase the current in an inductor from 0 to I?
  - The work, dW, done in time dt is $dW = Pdt = LIdI$
  - Thus the total work needed to bring the current from 0 to I in an inductor is

$$W = \int dW = \int_0^I LIdI = L\left[\frac{1}{2}I^2\right]_0^I = \frac{1}{2}LI^2$$

Energy Stored in a Magnetic Field

- The work done to the system is the same as the energy stored in the inductor when it is carrying current I
  - $$U = \frac{1}{2}LI^2$$

    Energy Stored in a magnetic field inside an inductor

  - This is compared to the energy stored in a capacitor, C, when the potential difference across it is V $U = \frac{1}{2}CV^2$
  - Just like the energy stored in a capacitor is considered to reside in the electric field between its plates
  - The energy in an inductor can be considered to be stored in its magnetic field

Stored Energy in terms of B

- So how is the stored energy written in terms of magnetic field B?
  - Inductance of an ideal solenoid without a fringe effect $L = \mu_0 N^2 A / l$
  - The magnetic field in a solenoid is $B = \mu_0 NI / l$
  - Thus the energy stored in an inductor is

$$U = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\mu_0 N^2 A}{l} \left(\frac{Bl}{\mu_0 N} \right)^2 = \frac{1}{2} \frac{B^2}{\mu_0} Al$$

What is this?

Volume V

$$U = \frac{1}{2} \frac{B^2}{\mu_0} Al$$

E

  - Thus the energy density is

$$u = \frac{U}{V} = \frac{U}{Al} = \frac{1}{2} \frac{B^2}{\mu_0}$$

$$u = \frac{1}{2} \frac{B^2}{\mu_0}$$

E density

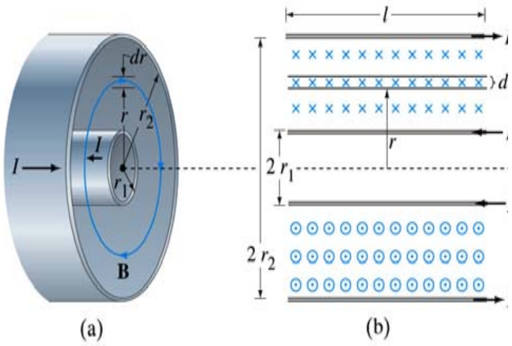
  - This formula is valid to any region of space
  - If a ferromagnetic material is present, μ_0 becomes μ.

What volume does Al represent?

The volume inside a solenoid!!

Example 30 – 5

Energy stored in a coaxial cable. (a) How much energy is being stored per unit length in a coaxial cable whose conductors have radii r_1 and r_2 and which carry a current I? (b) Where is the energy density highest?



(a) The inductance per unit length for a coaxial cable is $\frac{L}{l} = \frac{\mu_0}{2\pi} \ln \frac{r_2}{r_1}$

Thus the energy stored per unit length is $$\frac{U}{l} = \frac{1}{2}\frac{LI^2}{l} = \frac{\mu_0 I^2}{4\pi} \ln \frac{r_2}{r_1}$$

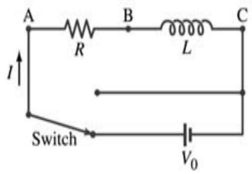
(b) Since the magnetic field is $B = \frac{\mu_0 I}{2\pi r}$

And the energy density is $u = \frac{1}{2}\frac{B^2}{\mu_0}$

The energy density is highest where B is highest. B is highest close to $r=r_1$, near the surface of the inner conductor.

LR Circuits

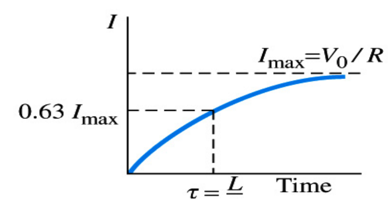
- What happens when an emf is applied to an inductor?
  - An inductor has some resistance, however negligible



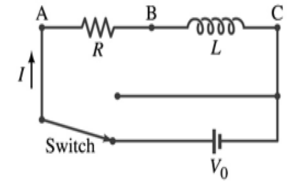
    - So an inductor can be drawn as a circuit of separate resistance and coil. What is the name this kind of circuit?

LR Circuit

  - What happens at the instance the switch is thrown to apply emf to the circuit?
    - The current starts to flow, gradually increasing from 0
    - This change is opposed by the induced emf in the inductor ➔ the emf at point B is higher than point C
    - However there is a voltage drop at the resistance which reduces the voltage across inductance
    - Thus the current increases less rapidly
    - The overall behavior of the current is gradual increase, reaching to the maximum current $I_{max}=V_0/R$.

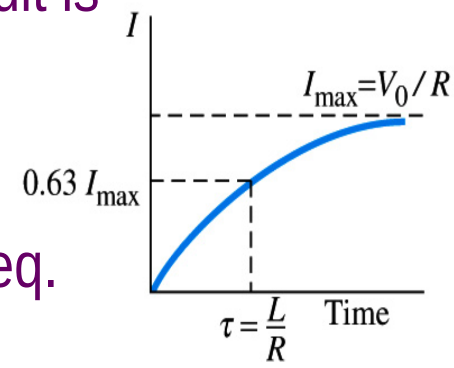


LR Circuits



- This can be shown w/ Kirchhoff rule loop rules
  - The emfs in the circuit are the battery voltage V_0 and the emf $\varepsilon=-\mathcal{L}(d\mathcal{I}/dt)$ in the inductor opposing the current increase
  - The sum of the potential changes through the circuit is

$$V_0 + \varepsilon - IR = V_0 - L\,dI/dt - IR = 0$$

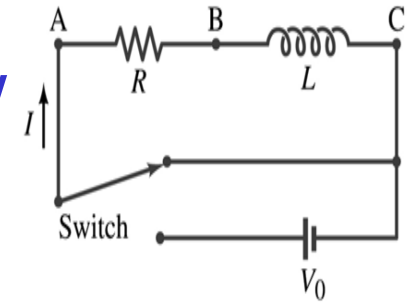


  - Where $\mathcal{I}$ is the current at any instance
  - By rearranging the terms, we obtain a differential eq.
  - $L\,dI/dt + IR = V_0$
  - We can integrate just as in RC circuit $\int_{I=0}^{I} \frac{dI}{V_0 - IR} = \int_{t=0}^{t} \frac{dt}{L}$
  - So the solution is $-\frac{1}{R}\ln\left(\frac{V_0 - IR}{V_0}\right) = \frac{t}{L}$

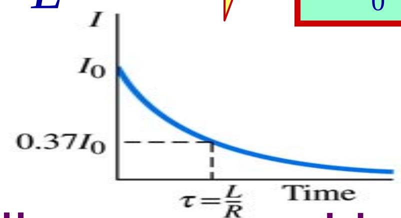
$$I = V_0\left(1-e^{-t/\tau}\right)/R = I_{max}\left(1-e^{-t/\tau}\right)$$

  - Where τ=L/R
    - This is the time constant τ of the LR circuit and is the time required for the current $\mathcal{I}$ to reach 0.63 of the maximum

Discharge of LR Circuits



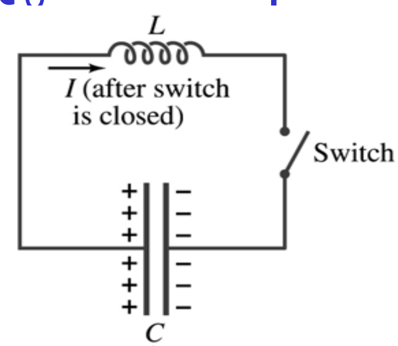
- If the switch is flipped away from the battery
  - The differential equation becomes
  - $L\,dI/dt + IR = 0$
  - So the integration is $\int_{I=0}^{I}\frac{dI}{IR} = \int_{t=0}^{t}\frac{dt}{L}$ $\Rightarrow$ $\ln\frac{I}{I_0} = -\frac{R}{L}t$
  - Which results in the solution
  - $I = I_0 e^{-\frac{R}{L}t} = I_0 e^{-t/\tau}$



  - The current decays exponentially to zero with the time constant τ=L/R
  - So there always is a reaction time when a system with a coil, such as an electromagnet, is turned on or off.
  - The current in LR circuit behaves almost the same as that in RC circuit but the time constant is inversely proportional to R in LR circuit unlike the RC circuit

LC Circuit and EM Oscillations

- What's an LC circuit?
  - A circuit that contains only an inductor and a capacitor
    - How is this possible? There is no source of emf!!
    - Well, you can imagine a circuit with a fully charged capacitor
    - In this circuit, we assume the inductor does not have any resistance
- Let's assume that the capacitor originally has $+Q_0$ on one plate and $-Q_0$ on the other



  - Suppose the switch is closed at t=0
  - The capacitor starts discharging
  - The current flow through the inductor increases
  - Applying Kirchhoff's loop rule, we obtain $-L\,dI/dt + Q/C = 0$
  - Since the current flows out of the plate with positive charge, the charge on the plate reduces, so $I=-dQ/dt$. Thus the differential equation can be rewritten

$$\frac{d^2Q}{dt} + \frac{Q}{LC} = 0$$

LC Circuit and EM Oscillations

- This equation looks the same as that of the harmonic oscillation
  - So the solution for this second order differential equation is
  - $Q = Q_0 \cos(\varpi t + \phi)$ The charge on the capacitor oscillates sinusoidally
  - Inserting the solution back into the differential equation
  - $\frac{d^2Q}{dt} + \frac{Q}{LC} = -\varpi^2 Q_0 \cos(\varpi t + \phi) + Q_0 \cos(\varpi t + \phi)/LC = 0$
  - Solving this equation for ω, we obtain $\varpi = 2\pi f = 1/\sqrt{LC}$
  - The current in the inductor is
  - $I = -dQ/dt = \varpi Q_0 \sin(\varpi t + \phi) = I_0 \sin(\varpi t + \phi)$
  - So the current also is sinusoidal with the maximum value

$$I_0 = \varpi Q_0 = Q_0/\sqrt{LC}$$

Energies in LC Circuit & EM Oscillation

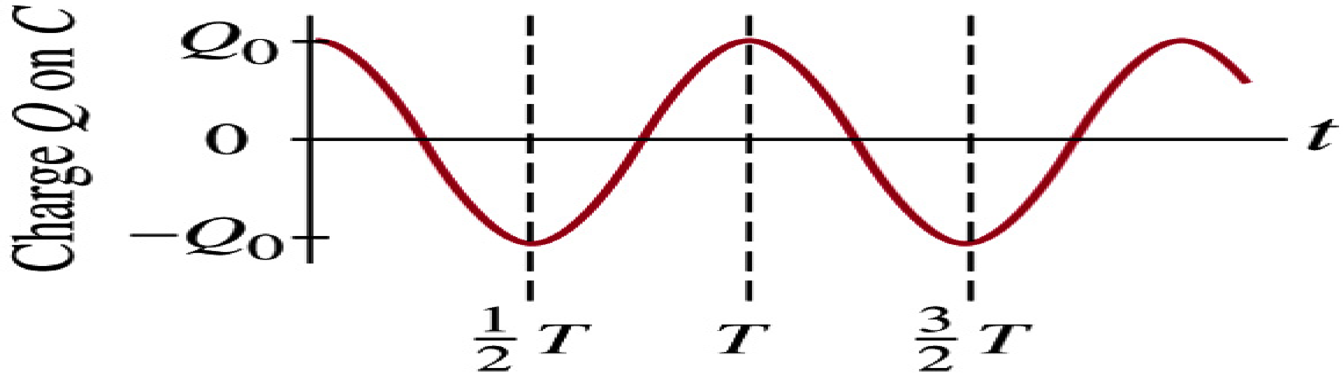
- The energy stored in the electric field of the capacitor at any time t is $U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{Q_0^2}{2C}\cos^2(\varpi t + \phi)$
- The energy stored in the magnetic field in the inductor at the same instant is $U_B = \frac{1}{2}LI^2 = \frac{Q_0^2}{2C}\sin^2(\varpi t + \phi)$
- Thus, the total energy in LC circuit at any instant is

$$U = U_E + U_B = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2 = \frac{Q_0^2}{2C}\left[\cos^2(\varpi t + \phi) + \sin^2(\varpi t + \phi)\right] = \frac{Q_0^2}{2C}$$

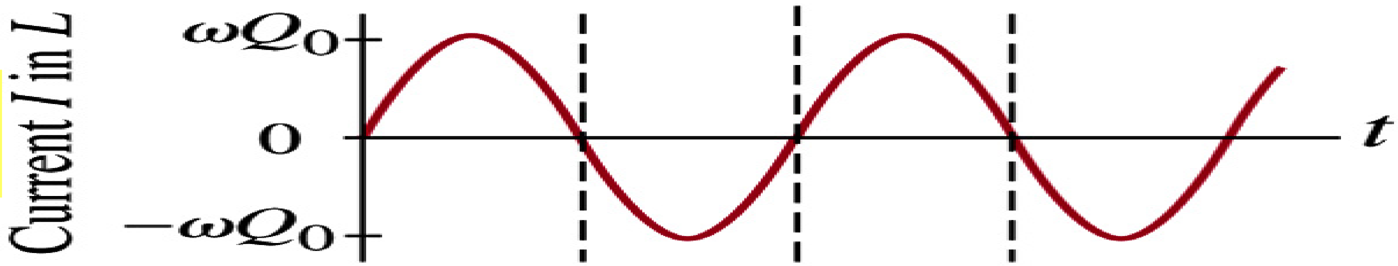
- So the total EM energy is constant and is conserved.
- This LC circuit is an LC oscillator or EM oscillator
  - The charge Q oscillates back and forth, from one plate of the capacitor to the other
  - The current also oscillates back and forth as well

LC Circuit Behaviors

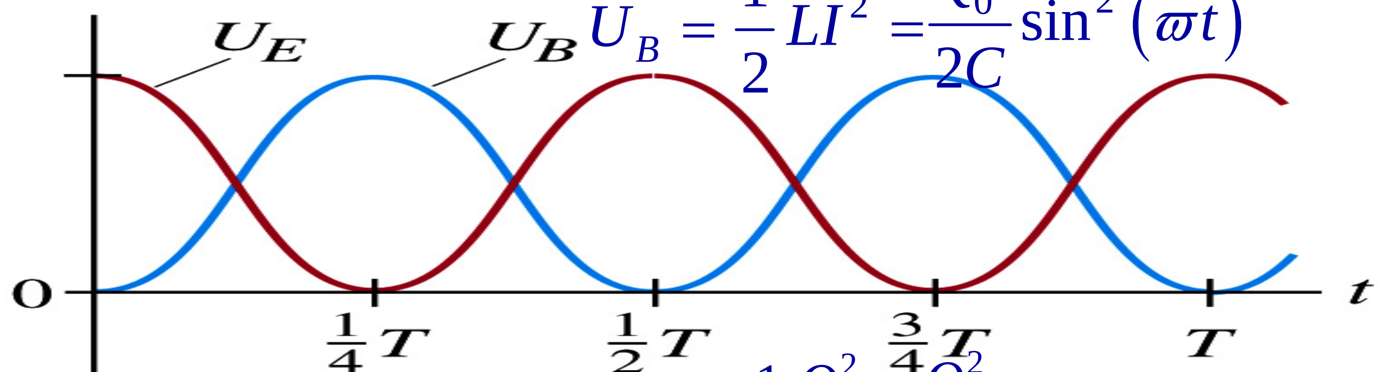
$$Q = Q_0 \cos(\varpi t)$$



$$I = \varpi Q_0 \sin(\varpi t)$$



$$U = \frac{Q_0^2}{2C}$$



$$U_B = \frac{1}{2}LI^2 = \frac{Q_0^2}{2C}\sin^2(\varpi t)$$

$$U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{Q_0^2}{2C}\cos^2(\varpi t)$$

Example 30 – 7

LC Circuit. A 1200-pF capacitor is fully charged by a 500-V dc power supply. It is disconnected from the power supply and is connected, at t=0, to a 75-mH inductor. Determine: (a) The initial charge on the capacitor, (b) the maximum current, (c) the frequency *f* and period T of oscillation; and (d) the total energy oscillating in the system.

(a) The 500-V power supply, charges the capacitor to

$$Q = CV = \left(1200\times10^{-12}F\right)\cdot 500V = 6.0\times10^{-7}C$$

(b) The maximum current is

$$I_{\max} = \varpi Q_0 = \frac{Q_0}{\sqrt{LC}} = \frac{6.0\times10^{-7}C}{\sqrt{75\times10^{-3}H\times1.2\times10^{-9}F}} = 63mA$$

(c) The frequency is

$$f = \frac{\varpi}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{7.5\times10^{-3}H\cdot1.2\times10^{-9}F}} = 1.7\times10^{3}Hz$$

The period is

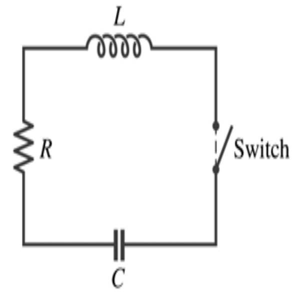
$$T = \frac{1}{f} = 6.0\times10^{-5}S$$

(d) The total energy in the system

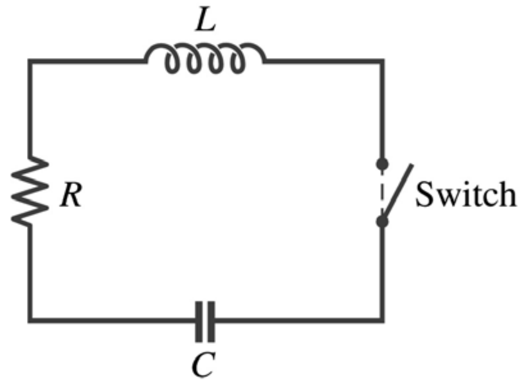
$$U = \frac{Q_0^2}{2C} = \frac{\left(6.0\times10^{-7}C\right)^2}{2\cdot1.2\times10^{-9}F} = 1.5\times10^{-4}J$$

LC Oscillations w/ Resistance (LRC circuit)

- There is no such thing as zero resistance coil so all LC circuits have some resistance
  - So to be more realistic, the effect of the resistance should be taken into account
  - Suppose the capacitor is charged up to Q_0 initially and the switch is closed in the circuit at t=0
  - What do you expect to happen to the energy in the circuit?
    - Well, due to the resistance we expect some energy will be lost through the resister via a thermal conversion
  - What about the oscillation? Will it look the same as the ideal LC circuit we dealt with?
  - No? OK then how would it be different?
  - The oscillation would be damped due to the energy loss.



LC Oscillations w/ Resistance (LRC circuit)



- Now let's do some analysis
- From Kirchhoff's loop rule, we obtain

$$-L\,dI/dt \quad -IR + \frac{Q}{C} = 0$$

- Since I=dQ/dt, the equation becomes

$$-L\frac{d^2Q}{dt} - R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

  - Which is identical to that of a damped oscillator
- The solution of the equation is $Q = Q_0 e^{-\frac{R}{2L}t}\cos(\varpi' t + \phi)$
  - Where the angular frequency is $\varpi' = \sqrt{1/LC - R^2/4L^2}$
    - $R^2<4L/C$: Underdamped
    - $R^2>4L/C$: Overdampled