PHYS 5326 – Lecture #9

Wednesday, Feb. 28, 2007
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1. Quantum Electro-dynamics (QED)
2. Local Gauge Invariance
3. Introduction of Massless Vector Gauge Field

Announcements

- First term exam will be on Wednesday, Mar. 7
- It will cover up to what we finish today
- The due for all homework up to last week's is Monday, Mar. 19

Prologue

- How is a motion described?
  - Motion of a particle or a group of particles can be expressed in terms of the position of the particle at any given time in classical mechanics.
- A state (or a motion) of particle is expressed in terms of wave functions that represent probability of the particle occupying certain position at any given time in Quantum mechanics
  - With the operators provide means for obtaining values for observables, such as momentum, energy, etc
- A state or motion in relativistic quantum field theory is expressed in space and time.
- Equation of motion in any framework starts with Lagrangians.

Non-relativistic Equation of Motion for Spin 0 Particle

Energy-momentum relation in classical mechanics give

$$\frac{\mathbf{p}^2}{2m} + V = E$$

Quantum prescriptions; $\mathbf{p} \to \frac{\hbar}{i}\nabla, \quad E \to i\hbar\frac{\partial}{\partial t}$.

provides the non-relativistic equation of motion for field, ψ, the Schrödinger Equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$|\Psi|^2$ represents the probability of finding the particle of mass m at the position (x,y,z)

Relativistic Equation of Motion for Spin 0 Particle

Relativistic energy-momentum relationship

$$E^2 - \mathbf{p}^2 c^2 = m^2 c^4 \Rightarrow p^\mu p_\mu - m^2 c^2 = 0$$

With four vector notation of quantum prescriptions;

$$p_\mu \to \frac{\hbar}{i}\partial_\mu \quad \text{where} \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}; \quad \left(\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}, \ \partial_1 = \frac{\partial}{\partial x}, \ \partial_2 = \frac{\partial}{\partial y}, \ \partial_3 = \frac{\partial}{\partial z} \right)$$

Relativistic equation of motion for field, ψ, the Klein-Gordon Equation

$$-\hbar^2 \partial_\mu \partial^\mu \Psi - m^2 c^2 \Psi = 0$$

$$-\frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} + \nabla^2 \Psi = \left(\frac{mc}{\hbar}\right)^2 \Psi$$

2nd order in time

Relativistic Equation of Motion (Dirac Equation) for Spin 1/2 Particle

To avoid 2nd order time derivative term, Dirac attempted to factor relativistic energy-momentum relation

$$p^{\mu}p_{\mu}-m^{2}c^{2}=0$$

This works for the case with zero three momentum

$$\left(p^{0}\right)^{2}-m^{2}c^{2}=\left(p^{0}+mc\right)\left(p^{0}-mc\right)=0$$

This results in two first order equations

$$p^{0}+mc=0$$

$$p^{0}-mc=0$$

Dirac Equation Continued…

The previous prescription does not work for the case with non-0 three momentum

$$p^\mu p_\mu - m^2c^2 = \left(\beta^k p_k + mc\right)\left(\gamma^\lambda p_\lambda - mc\right) =$$

$$\beta^k \gamma^\lambda p_k p_\lambda - mc\left(\beta^k - \gamma^k\right) p_k - m^2c^2$$

The terms linear to momentum should disappear, so $\beta^k = \gamma^k$

To make it work, we must find coefficients γ^k to satisfy: $p^\mu p_\mu = \gamma^k \gamma^\lambda p_k p_\lambda$

$$\left(p^0\right)^2 - \left(p^1\right)^2 - \left(p^2\right)^2 - \left(p^3\right)^2$$

$$= \left(\gamma^0\right)^2 \left(p^0\right)^2 + \left(\gamma^1\right)^2 \left(p^1\right)^2 + \left(\gamma^2\right)^2 \left(p^2\right)^2 + \left(\gamma^3\right)^2 \left(p^3\right)^2$$

$$+ \left(\gamma^0\gamma^1 + \gamma^1\gamma^0\right) p_0 p_1 + \left(\gamma^0\gamma^2 + \gamma^2\gamma^0\right) p_0 p_2 + \left(\gamma^0\gamma^3 + \gamma^3\gamma^0\right) p_0 p_3 + \text{Other Cross Terms}$$

The coefficients like $\gamma^0=1$ and $\gamma^1= \gamma^2= \gamma^3=i$ do not work since they do not eliminate the cross terms.

Dirac Equation Continued…

It would work if these coefficients are matrices that satisfy the conditions

$$\left(\gamma^0\right)^2=1,\ \left(\gamma^1\right)^2=\left(\gamma^2\right)^2=\left(\gamma^3\right)^2=-1$$

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=0 \text{ when } \mu\neq\nu$$

Or using Minkowski metric, $g^{\mu\nu}$

$$\{\gamma^\mu,\gamma^\nu\}=\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}$$

where $g^{\mu\nu}=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$

Using gamma matrices with the standard Bjorken and Drell convention

$$\gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{bmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}&\begin{pmatrix}0&0\\0&0\end{pmatrix}\\\begin{pmatrix}0&0\\0&0\end{pmatrix}&\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\end{bmatrix}$$

$$\gamma^i=\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}$$

Where σ^i are Pauli spin matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \sigma_3=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Dirac Equation Continued…

Using Pauli matrix as components in coefficient matrices whose smallest size is 4x4, the energy-momentum relation can now be factored

$$p^{\mu}p_{\mu}-m^2c^2=\left(\gamma^k p_k+mc\right)\left(\gamma^{\lambda}p_{\lambda}-mc\right)=0$$

w/ a solution $\gamma^{\lambda}p_{\lambda}-mc=0$

By applying quantum prescription of momentum $p_{\mu}\rightarrow i\hbar\partial_{\mu}$

Acting the 1-D solution on a wave function, ψ, we obtain Dirac equation

$$i\hbar\gamma^k\partial_{\mu}\psi-mc\psi=0$$

where Dirac spinor, ψ

$$\psi=\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix}$$

Euler-Lagrange Equation

For a conservative force, the force can be expressed as the gradient of the corresponding scalar potential, U

$$\vec{F} = -\nabla U$$

Therefore the Newton's law can be written $m\frac{d\vec{v}}{dt} = -\nabla U$.

Starting from Lagrangian $L = T - U = \frac{1}{2}mv^2 - U$

The 1-D Euler-Lagrange fundamental equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

In 1D Cartesian Coordinate system

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{dT}{dv_x} = mv_x$$

$$\frac{\partial L}{\partial q_1} = -\frac{\partial U}{\partial x}$$

Euler-Lagrange equation in QFT

- Unlike particles, field occupies regions of space. Therefore in field theory, the motion is expressed in terms of space and time.
- Euler-Larange equation for relativistic fields is, therefore,

$$\partial_\mu \left(\frac{\partial L}{\partial \left(\partial_\mu \phi_i \right)} \right) = \frac{\partial L}{\partial \phi_i}$$

Note the four vector form

Klein-Gordon Largangian for scalar (S=0) Field

For a single, scalar field ϕ, the Lagrangian is

$$\mathcal{L}=\frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right)-\frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^2$$

Since $\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_i\right)}=\partial^\mu\phi_i$ and $\frac{\partial\mathcal{L}}{\partial\phi_i}=-\left(\frac{mc}{\hbar}\right)^2\phi_i$

From the Euler-Largange equation, we obtain

$$\partial_\mu\partial^\mu\phi+\left(\frac{mc}{\hbar}\right)^2\phi=0$$

This equation is the Klein-Gordon equation describing a free, scalar particle (spin 0) of mass m.

Dirac Largangian for Spinor (S=1/2) Field

For a spinor field ψ, the Lagrangian

$$\mathcal{L} = i(\hbar c)\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - (mc^2)\overline{\psi}\psi$$

Since $\dfrac{\partial \mathcal{L}}{\partial(\partial_{\mu}\overline{\psi})} = 0$ and $\dfrac{\partial \mathcal{L}}{\partial \overline{\psi}} = i(\hbar c)\gamma^{\mu}\partial_{\mu}\psi - mc^2\psi$

From the Euler-Largange equation for $\overline{\psi}$, we obtain

$$i\gamma^{\mu}\partial_{\mu}\psi - \left(\frac{mc}{\hbar}\right)\psi = 0$$

Dirac equation for a particle of spin ½ and mass m.

How's Euler Lagrangian equation looks like for ψ?

Proca Largangian for Vector (S=1) Field

Suppose we take the Lagrangian for a vector field A^μ

$$\mathcal{L} = -\frac{1}{16\pi}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A^\nu A_\nu$$

$$= -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A^\nu A_\nu$$

Where $\mathcal{F}^{\mu\nu}$ is the field strength tensor in relativistic notation, E and B in Maxwell's equation form an anti-symmetic second-rank tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

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$$= -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A^\nu A_\nu$$

Since $\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} = -\frac{1}{4\pi}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$ and $\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{1}{4\pi}\left(\frac{mc}{\hbar}\right)^2 A^\nu$

From the Euler-Largange equation for A^μ, we obtain

$$\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \left(\frac{mc}{\hbar}\right)^2 A^\nu = \partial_\mu F^{\mu\nu} + \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0$$

Proca equation for a particle of spin 1 and mass *m*.

For *m*=0, this equation is for an electromagnetic field.

Lagrangians

- Lagrangians we discussed are concocted to produce desired field equations
  - $\mathcal{L}$ derived (L=T-V) in classical mechanics
  - $\mathcal{L}$ taken as axiomatic in field theory
- The Lagrangian for a particular system is not unique
  - Can always multiply by a constant
  - Or add a divergence
  - Since these do not affect field equations due to cancellations

Homework

- Prove that Fmn can represent Maxwell's equations, pg. 225 of Griffith's book.
- Derive Eq. 11.17 in Griffith's book
- Due is Wednesday, Mar. 7