# PHYS 1443 – Section 001
# Lecture #18

*Monday, April 18, 2011*

*Dr. Jaehoon Yu*

- Torque & Vector product
- Moment of Inertia
- Calculation of Moment of Inertia
- Parallel Axis Theorem
- Torque and Angular Acceleration
- Rolling Motion and Rotational Kinetic Energy

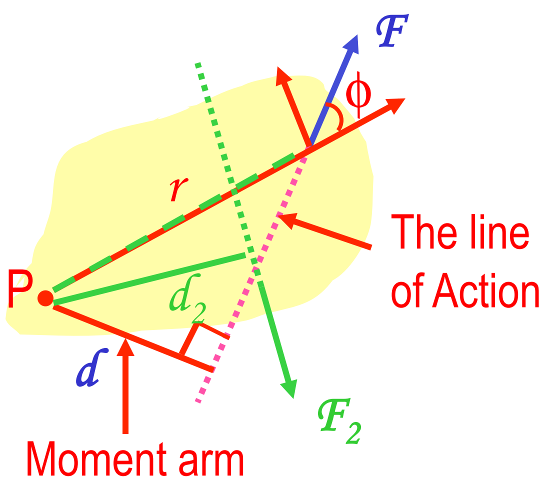
Today's homework is homework #10, due 10pm, Tuesday, Apr. 26!!

# Announcements

- Quiz Wednesday, Apr. 20
  - Beginning of the class
  - Covers from CH9.5 to CH10
- Reading assignment: CH10 – 10.
- Remember to submit your extra credit special project

# Torque

Torque is the tendency of a force to rotate an object about an axis. Torque, $\tau$, is a vector quantity.




Consider an object pivoting about the point P by the force $F$ being exerted at a distance r from P.

The line that extends out of the tail of the force vector is called the line of action.

The perpendicular distance from the pivoting point P to the line of action is called the moment arm.

Magnitude of torque is defined as the product of the force exerted on the object to rotate it and the moment arm.

$$\tau \equiv rF\sin\phi = Fd$$

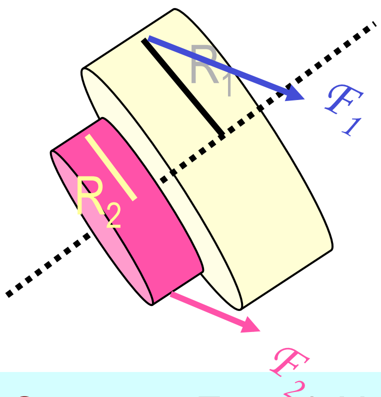
When there are more than one force being exerted on certain points of the object, one can sum up the torque generated by each force vectorially. The convention for sign of the torque is **positive if rotation is in counter-clockwise** and **negative if clockwise**.

$$\sum \tau = \tau_1 + \tau_2$$

$$= F_1 d_1 - F_2 d_2$$

# Ex. 10 – 7: Torque

A one piece cylinder is shaped as in the figure with core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the picture. A rope wrapped around the drum whose radius is $R_1$ exerts force $F_1$ to the right on the cylinder, and another force exerts $F_2$ on the core whose radius is $R_2$ downward on the cylinder. A) What is the net torque acting on the cylinder about the rotation axis?




The torque due to $F_1$ $\tau_1 = -R_1 F_1$ and due to $F_2$ $\tau_2 = R_2 F_2$

So the total torque acting on the system by the forces is

$$\sum \tau = \tau_1 + \tau_2 = -R_1 F_1 + R_2 F_2$$

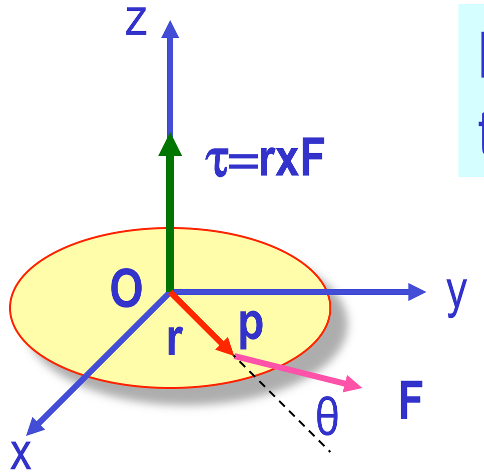
Suppose $F_1$=5.0 N, $R_1$=1.0 m, $F_2$= 15.0 N, and $R_2$=0.50 m. What is the net torque about the rotation axis and which way does the cylinder rotate from the rest?

Using the above result

$$\sum \tau = -R_1 F_1 + R_2 F_2$$
$$= -5.0 \times 1.0 + 15.0 \times 0.50 = 2.5 \ N \cdot m$$

The cylinder rotates in counter-clockwise.

# Torque and Vector Product




Let's consider a disk fixed onto the origin O and the force $\mathcal{F}$ exerts on the point p. What happens?

The disk will start rotating counter clockwise about the Z axis

The magnitude of torque given to the disk by the force $\mathcal{F}$ is

$$\tau = Fr\sin\theta$$

But torque is a vector quantity, what is the direction?
How is torque expressed mathematically?

$$\boxed{\vec{\tau} \equiv \vec{r} \times \vec{F}}$$

What is the direction?

The direction of the torque follows the right-hand rule!!

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The above operation is called the
Vector product or Cross product

$$\vec{C} \equiv \vec{A} \times \vec{B}$$

$$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$$

What is the result of a vector product?

Another vector

What is another vector operation we've learned?

Scalar product

$$C \equiv \vec{A} \cdot \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$$

Result? A scalar

# Properties of Vector Product

**Vector Product is Non-commutative**

What does this mean?

If the order of operation changes the result changes $\vec{A}\times\vec{B}\neq\vec{B}\times\vec{A}$

Following the right-hand rule, the direction changes $\boxed{\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}}$

**Vector Product of two parallel vectors is 0.**

$$\left|\vec{C}\right|=\left|\vec{A}\times\vec{B}\right|=\left|\vec{A}\right|\left|\vec{B}\right|\sin\theta=\left|\vec{A}\right|\left|\vec{B}\right|\sin 0=0$$

Thus, $\boxed{\vec{A}\times\vec{A}=0}$

**If two vectors are perpendicular to each other**

$$\left|\vec{A}\times\vec{B}\right|=\left|\vec{A}\right|\left|\vec{B}\right|\sin\theta=\left|\vec{A}\right|\left|\vec{B}\right|\sin 90^\circ=\left|\vec{A}\right|\left|\vec{B}\right|=AB$$

**Vector product follows distribution law**

$$\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

**The derivative of a Vector product with respect to a scalar variable is**

$$\frac{d\left(\vec{A}\times\vec{B}\right)}{dt}=\frac{d\vec{A}}{dt}\times\vec{B}+\vec{A}\times\frac{d\vec{B}}{dt}$$

# More Properties of Vector Product

The relationship between unit vectors, $\vec{i}$, $\vec{j}$ and $\vec{k}$

$$\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0$$

$$\vec{i}\times\vec{j}=-\vec{j}\times\vec{i}=\vec{k}$$

$$\vec{j}\times\vec{k}=-\vec{k}\times\vec{j}=\vec{i}$$

$$\vec{k}\times\vec{i}=-\vec{i}\times\vec{k}=\vec{j}$$

Vector product of two vectors can be expressed in the following determinant form

$$\vec{A}\times\vec{B}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\vec{i}\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}-\vec{j}\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}+\vec{k}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}$$

$$=\left(A_yB_z-A_zB_y\right)\vec{i}-\left(A_xB_z-A_zB_x\right)\vec{j}+\left(A_xB_y-A_yB_x\right)\vec{k}$$

# Moment of Inertia

Rotational Inertia: Measure of resistance of an object to changes in its rotational motion. Equivalent to mass in linear motion.

For a group of particles $I \equiv \sum_i m_i r_i^2$

For a rigid body $I \equiv \int r^2 dm$

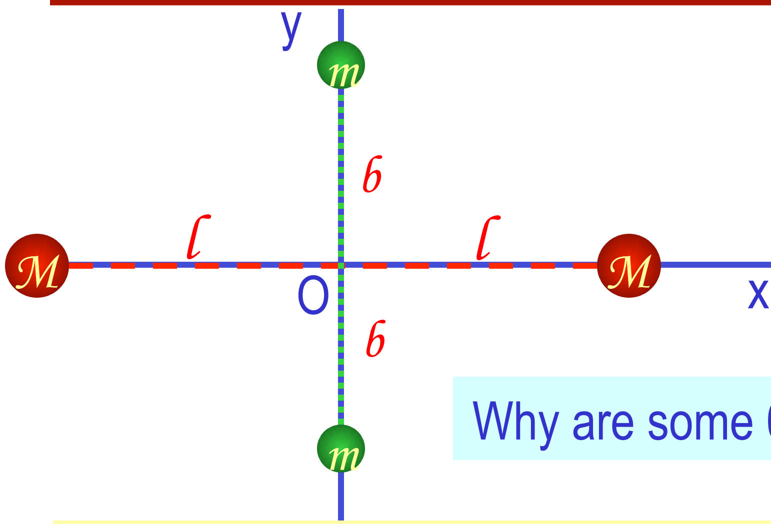
What are the dimension and unit of Moment of Inertia? $\left[ML^2\right]$ $kg \cdot m^2$

Determining Moment of Inertia is extremely important for computing equilibrium of a rigid body, such as a building.

$\boxed{\text{Dependent on the axis of rotation!!!}}$

# Example for Moment of Inertia

In a system of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at angular speed ω.




Since the rotation is about y axis, the moment of inertia about y axis, $I_y$, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m \cdot 0^2 + m \cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2\right)\omega^2 = Ml^2\omega^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2\left(Ml^2 + mb^2\right) \qquad K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2 + 2mb^2\right)\omega^2 = \left(Ml^2 + mb^2\right)\omega^2$$

# Calculation of Moments of Inertia

Moments of inertia for large objects can be computed, if we assume the object consists of small volume elements with mass, $\Delta m_i$.

The moment of inertia for the large rigid object is $I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$

It is sometimes easier to compute moments of inertia in terms of volume of the elements rather than their mass

How can we do this?

Using the volume density, ρ, replace $dm$ in the above equation with dV. $\rho = \frac{dm}{dV} \Rightarrow dm = \rho dV$ The moments of inertia becomes $\boxed{I = \int \rho r^2 \, dV}$

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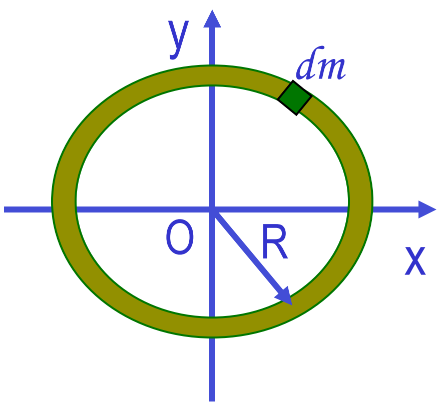
Example: Find the moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.



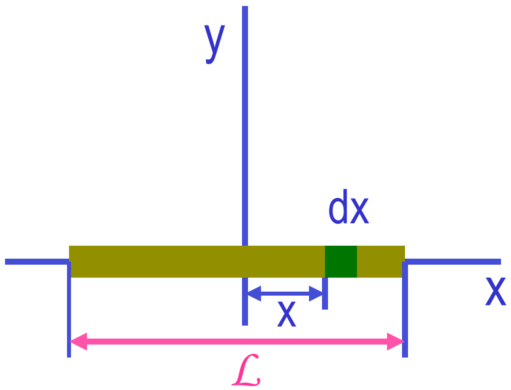

The moment of inertia is $I = \int r^2 \, dm = R^2 \int dm = MR^2$

What do you notice from this result?

The moment of inertia for this object is the same as that of a point of mass M at the distance R.

# Ex.10 – 11 Rigid Body Moment of Inertia

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing through its center of mass.




The line density of the rod is $\lambda = \frac{M}{L}$

so the masslet is $dm = \lambda dx = \frac{M}{L}dx$

The moment of inertia is

$$I = \int r^2\, dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

What is the moment of inertia when the rotational axis is at one end of the rod.

$$I = \int r^2\, dm = \int_0^L \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_0^L$$

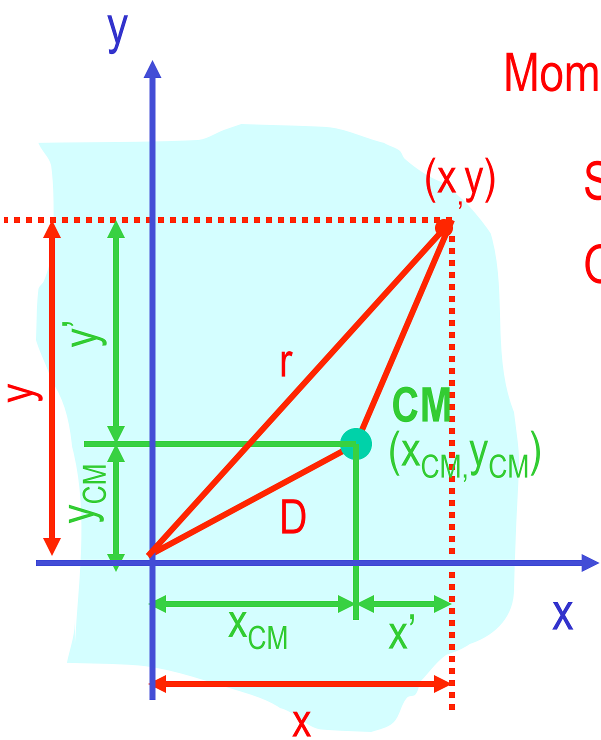
$$= \frac{M}{3L}\left[(L)^3 - 0\right] = \frac{M}{3L}\left(L^3\right) = \frac{ML^2}{3}$$

Will this be the same as the above. Why or why not?

Since the moment of inertia is resistance to motion, it makes perfect sense for it to be harder to move when it is rotating about the axis at one end.

# Parallel Axis Theorem

Moments of inertia for highly symmetric object is easy to compute if the rotational axis is the same as the axis of symmetry. However if the axis of rotation does not coincide with axis of symmetry, the calculation can still be done in a simple manner using **parallel-axis theorem**. $I = I_{CM} + MD^2$




Moment of inertia is defined as $I = \int r^2\,dm = \int \left(x^2 + y^2\right) dm \quad (1)$

Since x and y are $x = x_{CM} + x' \quad y = y_{CM} + y'$

One can substitute x and y in Eq. 1 to obtain

$$I = \int \left[ \left(x_{CM} + x'\right)^2 + \left(y_{CM} + y'\right)^2 \right] dm$$

$$= \left(x_{CM}^2 + y_{CM}^2\right) \int dm + 2x_{CM} \int x'\,dm + 2y_{CM} \int y'\,dm + \int \left(x'^2 + y'^2\right) dm$$

Since the x' and y' are the distance from CM, by definition $\int x'\,dm = 0 \quad \int y'\,dm = 0$

Therefore, the parallel-axis theorem

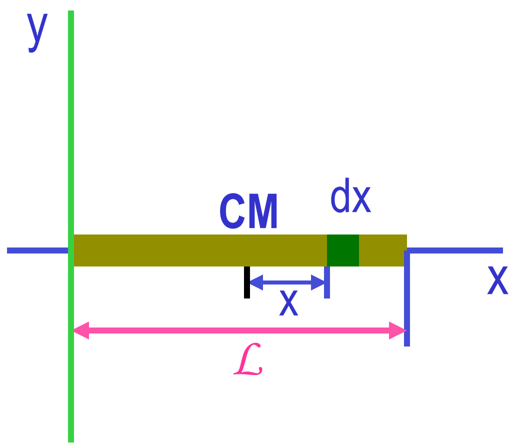
$$I = \left(x_{CM}^2 + y_{CM}^2\right) \int dm + \int \left(x'^2 + y'^2\right) dm = MD^2 + I_{CM}$$

What does this theorem tell you?

Moment of inertia of any object about any arbitrary axis are the same as the sum of moment of inertia for **a rotation about the CM** and **that of the CM about the rotation axis**.

# Example for Parallel Axis Theorem

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis that goes through one end of the rod, using parallel-axis theorem.




The line density of the rod is $\lambda = \frac{M}{L}$

so the masslet is $dm = \lambda dx = \frac{M}{L} dx$

The moment of inertia about the CM

$$I_{CM} = \int r^2\, dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

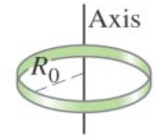
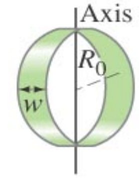
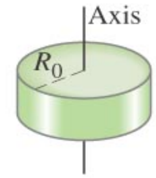
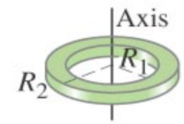
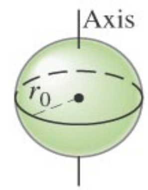
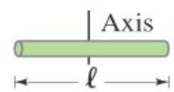
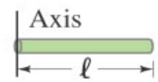
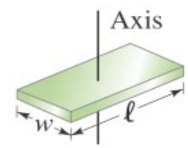
Using the parallel axis theorem

$$I = I_{CM} + D^2 M = \frac{ML^2}{12} + \left(\frac{L}{2}\right)^2 M = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

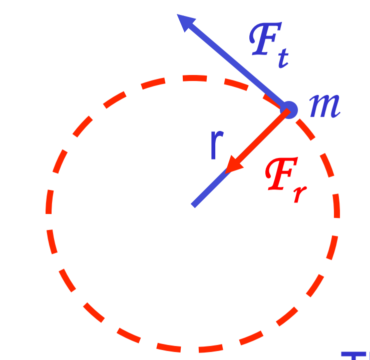
The result is the same as using the definition of moment of inertia.

Parallel-axis theorem is useful to compute moment of inertia of a rotation of a rigid object with complicated shape about an arbitrary axis

# Check out Figure 10 – 20 for moment of inertia for various shaped objects

| | Object | Location of axis | | Moment of inertia |
|---|---|---|---|---|
| (a) | **Thin hoop,** radius $R_0$ | Through center |   | $MR_0^2$ |
| (b) | **Thin hoop,** radius $R_0$ width $w$ | Through central diameter |   | $\frac{1}{2}MR_0^2 + \frac{1}{12}Mw^2$ |
| (c) | **Solid cylinder,** radius $R_0$ | Through center |   | $\frac{1}{2}MR_0^2$ |
| (d) | **Hollow cylinder,** inner radius $R_1$ outer radius $R_2$ | Through center |   | $\frac{1}{2}M(R_1^2 + R_2^2)$ |
| (e) | **Uniform sphere,** radius $r_0$ | Through center |   | $\frac{2}{5}Mr_0^2$ |
| (f) | **Long uniform rod,** length $\ell$ | Through center |   | $\frac{1}{12}M\ell^2$ |
| (g) | **Long uniform rod,** length $\ell$ | Through end |   | $\frac{1}{3}M\ell^2$ |
| (h) | **Rectangular thin plate,** length $\ell$, width $w$ | Through center |   | $\frac{1}{12}M(\ell^2 + w^2)$ |

# Torque & Angular Acceleration




Let's consider a point object with mass $m$ rotating on a circle.

What forces do you see in this motion?

The tangential force $\mathcal{F}_t$ and the radial force $\mathcal{F}_r$

The tangential force $\mathcal{F}_t$ is $\quad F_t = ma_t = mr\alpha$

The torque due to tangential force $\mathcal{F}_t$ is $\quad \tau = F_t r = ma_t r = mr^2\alpha = I\alpha$

What do you see from the above relationship? $\quad \tau = I\alpha$

What does this mean?

Torque acting on a particle is proportional to the angular acceleration.

What law do you see from this relationship?

Analogs to Newton's 2nd law of motion in rotation.

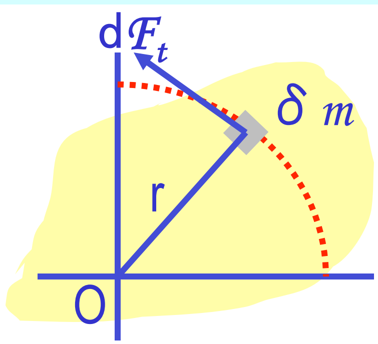
How about a rigid object?

The external tangential force $\delta\mathcal{F}_t$ is $\quad \delta F_t = \delta m a_t = \delta m r\alpha$

The torque due to tangential force $\mathcal{F}_t$ is $\quad \delta\tau = \delta F_t r = \left(r^2\delta m\right)\alpha$

The total torque is

$$\tau = \lim_{\delta\tau\to 0}\sum\delta\tau = \int d\tau = \alpha\lim_{\delta m\to 0}\sum r^2\delta m = \alpha\int r^2\,dm = I\alpha$$




What is the contribution due to radial force and why?

Contribution from radial force is 0, because its line of action passes through the pivoting point, making the moment arm 0.

# Rolling Motion of a Rigid Body

What is a rolling motion?

A more generalized case of a motion where the rotational axis moves together with an object

A rotational motion about a moving axis

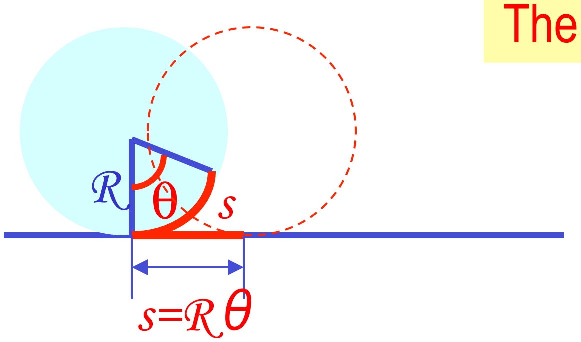
To simplify the discussion, let's make a few assumptions

1. Limit our discussion on very symmetric objects, such as cylinders, spheres, etc
2. The object rolls on a flat surface

Let's consider a cylinder rolling on a flat surface, without slipping.

Under what condition does this "Pure Rolling" happen?

The total linear distance the CM of the cylinder moved is $s = R\theta$




Thus the linear speed of the CM is

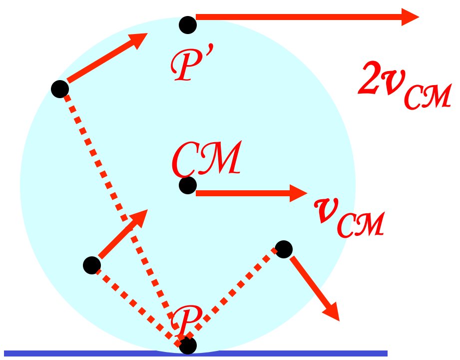
$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

The condition for a "Pure Rolling motion"

# More Rolling Motion of a Rigid Body

The magnitude of the linear acceleration of the CM is

$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha$$



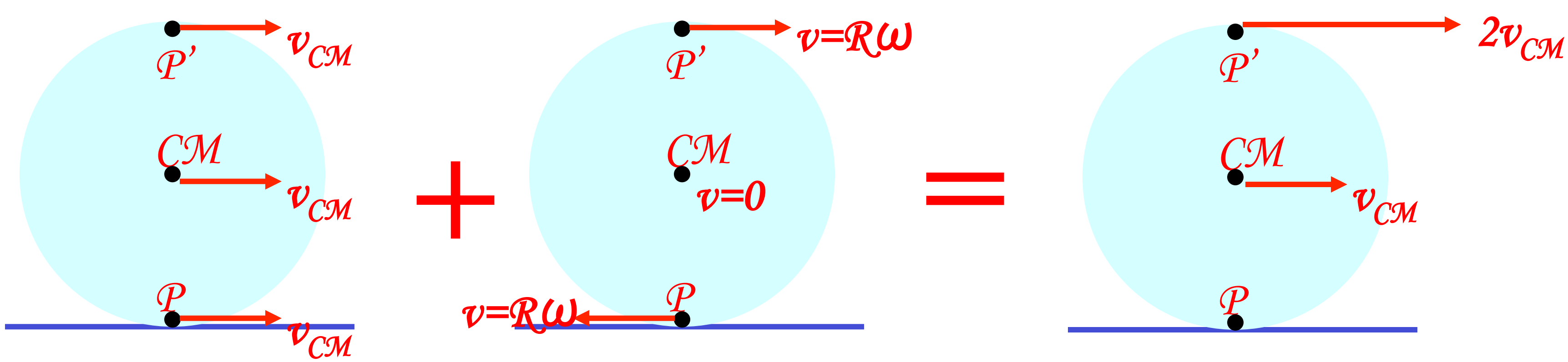

As we learned in rotational motion, all points in a rigid body moves at the same angular speed but at different linear speeds.

CM is moving at the same speed at all times.

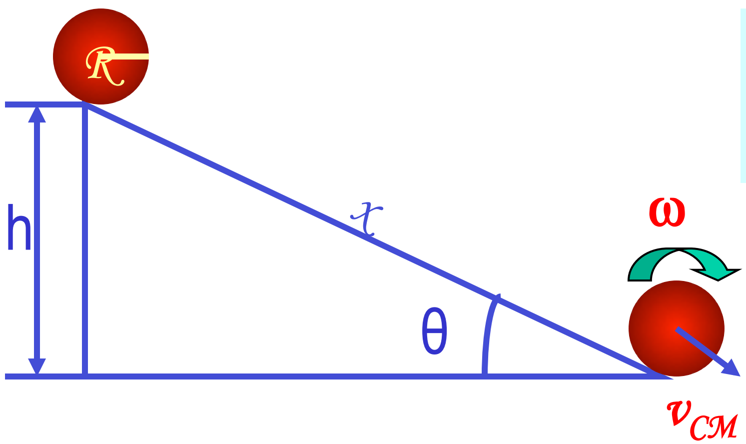
At any given time, the point that comes to P has 0 linear speed while the point at P' has twice the speed of CM

Why??

A rolling motion can be interpreted as the sum of Translation and Rotation



# Kinetic Energy of a Rolling Sphere




Let's consider a sphere with radius R rolling down the hill without slipping.

$$K = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} MR^2\omega^2$$

$$= \frac{1}{2} I_{CM}\left(\frac{v_{CM}}{R}\right)^2 + \frac{1}{2} Mv_{CM}^2$$

$$= \frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right)v_{CM}^2$$

Since $v_{CM} = R\omega$

What is the speed of the CM in terms of known quantities and how do you find this out?

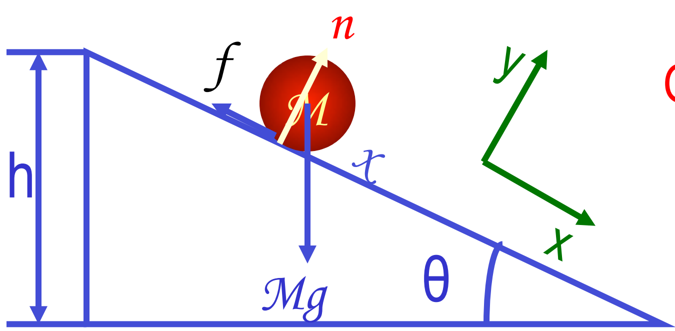
Since the kinetic energy at the bottom of the hill must be equal to the potential energy at the top of the hill

$$K = \frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right)v_{CM}^2 = Mgh$$

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM}/MR^2}}$$

# Ex. 10 – 16: Rolling Kinetic Energy

For solid sphere as shown in the figure, calculate the linear speed of the CM at the bottom of the hill and the magnitude of linear acceleration of the CM. Solve this problem using Newton's second law, the dynamic method.




What are the forces involved in this motion?

Gravitational Force, Frictional Force, Normal Force

Newton's second law applied to the CM gives

$$\sum F_x = Mg\sin\theta - f = Ma_{CM}$$
$$\sum F_y = n - Mg\cos\theta = 0$$

Since the forces $\mathcal{M}g$ and **n** go through the CM, their moment arm is 0 and do not contribute to torque, while the static friction $f$ causes torque

$$\tau_{CM} = fR = I_{CM}\alpha$$

We know that

$$I_{CM} = \frac{2}{5}MR^2$$
$$a_{CM} = R\alpha$$

We obtain

$$f = \frac{I_{CM}\alpha}{R} = \frac{\frac{2}{5}MR^2}{R}\left(\frac{a_{CM}}{R}\right) = \frac{2}{5}Ma_{CM}$$

Substituting $f$ in dynamic equations

$$Mg\sin\theta = \frac{7}{5}Ma_{CM} \qquad a_{CM} = \frac{5}{7}g\sin\theta$$