# PHYS 1443 – Section 001
# Lecture #19

*Wednesday, April 20, 2011*

*Dr. Jaehoon Yu*

- Kinetic Energy of a Rolling Body
- Work and Power in Rotational Motion
- Angular Momentum
- Angular Momentum and Torque
- Angular Momentum Conservation
- Similarity between Linear and Angular Quantities
- Conditions for Equilibrium

# Announcements

- Planetarium extra credit sheets
  - Tape ONLY one corner of the ticket stub on a sheet with your name and ID on it
    - I must be able to see the initial of the start lecturer of the show
  - Bring it by the beginning of the class Monday, May 2
- Colloquium today at 4pm in Sh101
  - Our own Dr. Musielak

# Physics Department
# The University of Texas at Arlington
# COLLOQUIUM

## QUANTUM DARK MATTER


## Dr. Zdzislaw Musielak

*Department of Physics*
*The University of Texas at Arlington*

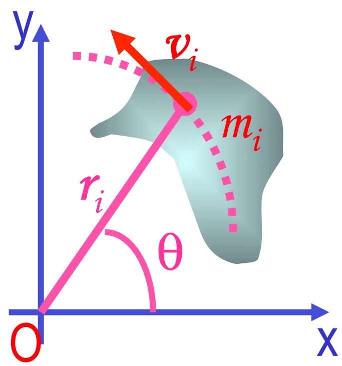

*4:00 pm Wednesday April 20, 2011 room 101 SH*


**Abstract:**

Astronomical observations show that most of the mass in the Universe is in the form of Dark Matter. Understanding the origin and nature of Dark Matter is one of the most urgent and challenging problems of modern science. Its solution will require new ideas that are likely to revolutionize physics, astrophysics, and other natural sciences. Standard proposals to explain Dark Matter that are based upon unification of the fundamental forces, new forces and new particles in the standard model will be briefly reviewed. A special emphasis will be given to the so-called quantum ('fuzzy') Dark Matter and a search for new fundamental equations of physics that may describe Dark Matter.


**Refreshments will be served at 3:30p.m in the Physics Library**

# Rotational Kinetic Energy




What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, $m_i$, moving at a tangential speed, $v_i$, is

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 \omega^2$$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

$$K_R = \sum_i K_i = \frac{1}{2} \sum_i m_i r_i^2 \omega^2 = \frac{1}{2} \left( \sum_i m_i r_i^2 \right) \omega^2$$

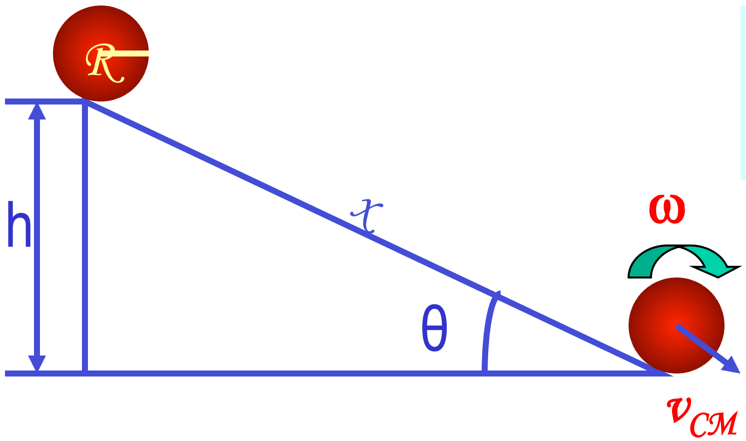
Since moment of Inertia, I, is defined as

$$I = \sum_i m_i r_i^2$$

The above expression is simplified as

$$K_R = \frac{1}{2} I \omega^2$$

# Kinetic Energy of a Rolling Sphere




Let's consider a sphere with radius R rolling down the hill without slipping.

$$KE = KE_{Rotation} + KE_{Linear} = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}^2$$

$$= \frac{1}{2} I_{CM}\left(\frac{v_{CM}}{R}\right)^2 + \frac{1}{2} M v_{CM}^2$$

$$= \frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right) v_{CM}^2$$

Since $v_{CM} = R\omega$

What is the speed of the CM in terms of known quantities and how do you find this out?

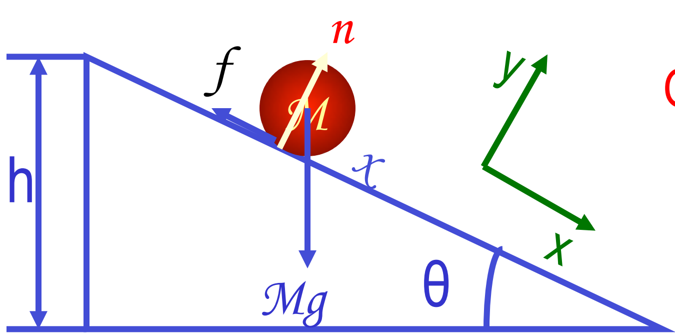
Since the kinetic energy at the bottom of the hill must be equal to the potential energy at the top of the hill

$$K = \frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right) v_{CM}^2 = Mgh$$

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM}/MR^2}}$$

# Ex. 10 – 16: Rolling Kinetic Energy

For solid sphere as shown in the figure, calculate the linear speed of the CM at the bottom of the hill and the magnitude of linear acceleration of the CM. Solve this problem using Newton's second law, the dynamic method.




What are the forces involved in this motion?

Gravitational Force, Frictional Force, Normal Force

Newton's second law applied to the CM gives

$$\sum F_x = Mg\sin\theta - f = Ma_{CM}$$

$$\sum F_y = n - Mg\cos\theta = 0$$

Since the forces $Mg$ and **n** go through the CM, their moment arm is 0 and do not contribute to torque, while the static friction $f$ causes torque

$$\tau_{CM} = fR = I_{CM}\alpha$$

We know that

$$I_{CM} = \frac{2}{5}MR^2$$

$$a_{CM} = R\alpha$$

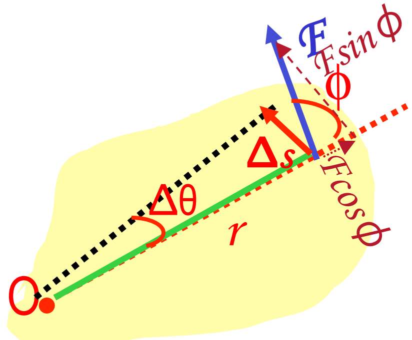
We obtain

$$f = \frac{I_{CM}\alpha}{R} = \frac{\frac{2}{5}MR^2}{R}\left(\frac{a_{CM}}{R}\right) = \frac{2}{5}Ma_{CM}$$

Substituting $f$ in dynamic equations

$$Mg\sin\theta = \frac{7}{5}Ma_{CM} \qquad a_{CM} = \frac{5}{7}g\sin\theta$$

# Work, Power, and Energy in Rotation




Let's consider the motion of a rigid body with a single external force $\mathcal{F}$ exerting on the point P, moving the object by $\Delta s$.
The work done by the force $\mathcal{F}$ as the object rotates through the infinitesimal distance $\Delta s = r\Delta\theta$ is

$$\Delta W = \vec{F} \cdot \Delta\vec{s} = (F\sin\phi) r\Delta\theta$$

What is $\mathcal{F}\sin\phi$? The tangential component of the force $\mathcal{F}$.

What is the work done by radial component $\mathcal{F}\cos\phi$? Zero, because it is perpendicular to the displacement.

Since the magnitude of torque is $r\mathcal{F}\sin\phi$,

$$\Delta W = (rF\sin\phi)\Delta\theta = \tau\Delta\theta$$

The rate of work, or power, becomes

$$P = \frac{\Delta W}{\Delta t} = \frac{\tau\Delta\theta}{\Delta t} = \tau\omega$$

How was the power defined in linear motion?

The rotational work done by an external force equals the change in rotational Kinetic energy.

$$\sum \tau = I\alpha = I\left(\frac{\Delta\omega}{\Delta t}\right) \Rightarrow \sum \tau\Delta\theta = I\omega\Delta\omega$$

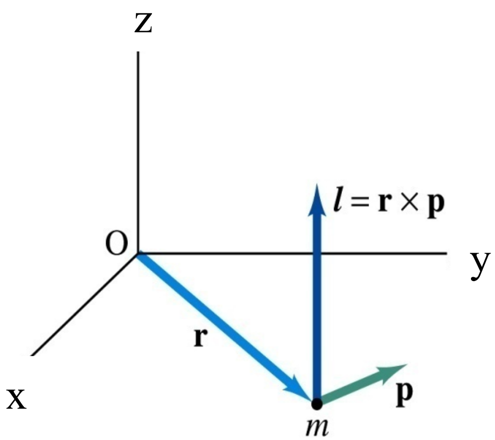
The work put in by the external force then

$$\Delta W = \int_{\omega_i}^{\omega_f} I\omega\, d\omega = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

Work-kinetic Energy Theorem

# Angular Momentum of a Particle

If you grab onto a pole while running, your body will rotate about the pole, gaining angular momentum. We've used the linear momentum to solve physical problems with linear motions, the angular momentum will do the same for rotational motions.




Let's consider a point-like object ( particle) with mass $m$ located at the vector location $\mathbf{r}$ and moving with linear velocity $\mathbf{v}$

The angular momentum $\mathbf{L}$ of this particle relative to the origin O is

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

What is the unit and dimension of angular momentum? $kg \cdot m^2/s \quad [ML^2T^{-1}]$

Note that $\mathbf{L}$ depends on origin O. Why? Because $\mathbf{r}$ changes

What else do you learn? The direction of $\mathbf{L}$ is +z.

Since $\mathbf{p}$ is $m\mathbf{v}$, the magnitude of $\mathbf{L}$ becomes $L = mvr\sin\phi = mr^2\sin\phi v/r = I\omega$

What do you learn from this? If the direction of linear velocity points to the origin of rotation, the particle does not have any angular momentum.

If the linear velocity is perpendicular to position vector, the particle moves exactly the same way as a point on a rim.

# Angular Momentum and Torque

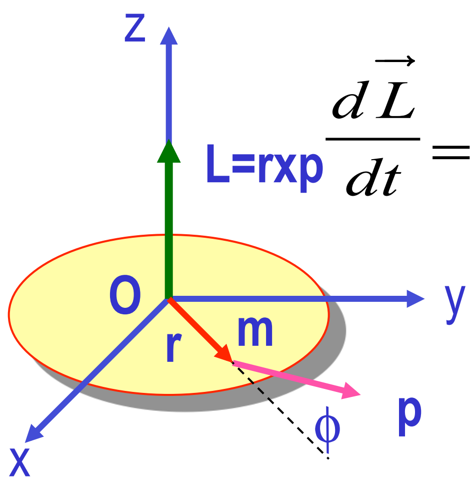
Can you remember how net force exerting on a particle and the change of its linear momentum are related?

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

Total external forces exerting on a particle is the same as the change of its linear momentum.

The same analogy works in rotational motion between torque and angular momentum.

Net torque acting on the particle is $\sum \vec{\tau} = \vec{r} \times \sum \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$




$$\frac{d\vec{L}}{dt} = \frac{d\left(\vec{r} \times \vec{p}\right)}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = 0 + \vec{r} \times \frac{d\vec{p}}{dt} = \sum \vec{\tau}$$

Why does this work?

Because $v$ is parallel to the linear momentum

Thus the torque-angular momentum relationship

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt}$$

**The net torque acting on a particle is the same as the time rate change of its angular momentum**

# Angular Momentum of a System of Particles

The total angular momentum of a system of particles about some point is the vector sum of the angular momenta of the individual particles

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \ldots\ldots + \vec{L}_n = \sum \vec{L}_i$$

Since the individual angular momentum can change, the total angular momentum of the system can change.

Both internal and external forces can provide torque to individual particles. However, the internal forces do not generate net torque due to Newton's third law.

Let's consider a two particle system where the two exert forces on each other.

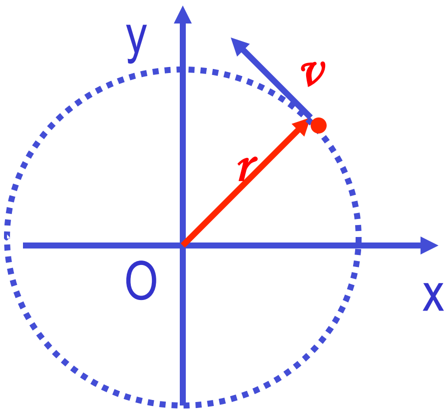
Since these forces are the action and reaction forces with directions lie on the line connecting the two particles, the vector sum of the torque from these two becomes 0.

Thus the time rate change of the angular momentum of a system of particles is equal to only the net external torque acting on the system

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

# Ex.11 – 1 for Angular Momentum

A particle of mass $m$ is moving on the xy plane in a circular path of radius r and linear velocity v about the origin O. Find the magnitude and the direction of the angular momentum with respect to O.




Using the definition of angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m\vec{r} \times \vec{v}$$

Since both the vectors, r and v, are on x-y plane and using right-hand rule, the direction of the angular momentum vector is +z (coming out of the screen)

The magnitude of the angular momentum is $\left|\vec{L}\right| = \left|m\vec{r} \times \vec{v}\right| = mrv\sin\phi = mrv\sin 90^\circ = mrv$

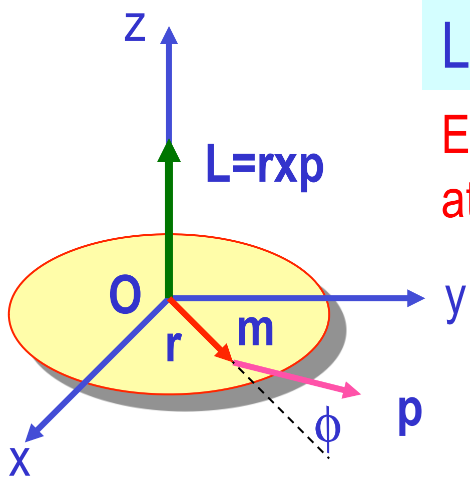
So the angular momentum vector can be expressed as $\vec{L} = mrv\vec{k}$

Find the angular momentum in terms of angular velocity **ω.**

Using the relationship between linear and angular speed

$$\vec{L} = mrv\vec{k} = mr^2\omega\vec{k} = mr^2\vec{\omega} = I\vec{\omega}$$

# Angular Momentum of a Rotating Rigid Body




Let's consider a rigid body rotating about a fixed axis

Each particle of the object rotates in the xy plane about the z-axis at the same angular speed, w

Magnitude of the angular momentum of a particle of mass $m_i$ about origin O is $m_i v_i r_i$

$$L_i = m_i r_i v_i = m_i r_i^2 \omega$$

Summing over all particle's angular momentum about z axis

$$L_z = \sum_i L_i = \sum_i \left( m_i r_i^2 \omega \right)$$

What do you see?

$$L_z = \sum_i \left( m_i r_i^2 \right) \omega = I\omega$$

Since $I$ is constant for a rigid body

$$\frac{dL_z}{dt} = I \frac{d\omega}{dt} = I\alpha$$

a is angular acceleration

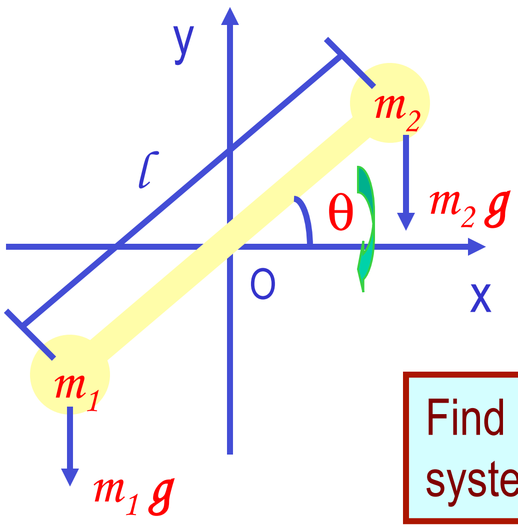
Thus the torque-angular momentum relationship becomes

$$\sum \tau_{ext} = \frac{dL_z}{dt} = I\alpha$$

The net external torque acting on a rigid body rotating about a fixed axis is equal to the moment of inertia about that axis multiplied by the object's angular acceleration with respect to that axis.

# Example for Rigid Body Angular Momentum

A rigid rod of mass $\mathcal{M}$ and length $\ell$ is pivoted without friction at its center. Two particles of mass $m_1$ and $m_2$ are attached to either end of the rod. The combination rotates on a vertical plane with an angular speed of ω. Find an expression for the magnitude of the angular momentum.




The moment of inertia of this system is

$$I = I_{rod} + I_{m_1} + I_{m_2} = \frac{1}{12}Ml^2 + \frac{1}{4}m_1 l^2 + \frac{1}{4}m_2 l^2$$

$$= \frac{l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)$$

$$L = I\omega = \frac{\omega l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)$$

Find an expression for the magnitude of the angular acceleration of the system when the rod makes an angle θ with the horizon.

If $m_1 = m_2$, no angular momentum because the net torque is 0.

If θ=+/−π/2, at equilibrium so no angular momentum.

First compute the net external torque

$$\tau_1 = m_1 g \frac{l}{2}\cos\theta \qquad \tau_2 = -m_2 g \frac{l}{2}\cos\theta$$

$$\tau_{ext} = \tau_1 + \tau_2 = \frac{gl\cos\theta(m_1 - m_2)}{2}$$

Thus α becomes

$$\alpha = \frac{\sum \tau_{ext}}{I} = \frac{\frac{1}{2}(m_1 - m_2)gl\cos\theta}{\frac{l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)} = \frac{2(m_1 - m_2)\cos\theta}{\left(\frac{1}{3}M + m_1 + m_2\right)}\frac{g}{l}$$