PHYS 3313 – Section 001
Lecture #13

Monday, March 23, 2015
Dr. Jaehoon Yu

- Bohr Radius
- Bohr's Hydrogen Model and Its Limitations
- Characteristic X-ray Spectra

Announcements

- Homework #3
  - End of chapter problems on CH4: 5, 14, 17, 21, 23 and 45
  - Due: Monday, March 30
- Quiz Wednesday, April 1
  - At the beginning of the class
  - Covers CH4.1 through what we finish Monday, March 30
  - BYOF with the same rule as before
- Colloquium at 4pm this Wednesday, in SH101

Uncertainties

- Statistical Uncertainty: A naturally occurring uncertainty due to the number of measurements
  - Usually estimated by taking the square root of the number of measurements or samples, $\sqrt{N}$
- Systematic Uncertainty: Uncertainty occurring due to unintended biases or unknown sources
  - Biases made by personal measurement habits
  - Some sources that could impact the measurements
- In any measurement, the uncertainties provide the significance to the measurement

Bohr's Quantized Radius of Hydrogen

- The angular momentum is $\left|\vec{L}\right| = \left|\vec{r} \times \vec{p}\right| = mvr = n\hbar$

- So the speed of an orbiting e can be written $v_e = \dfrac{n\hbar}{m_e r}$
- From the Newton's law for a circular motion

$$F_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = \frac{m_e v_e^2}{r} \Rightarrow v_e = \frac{e}{\sqrt{4\pi\varepsilon_0 m_e r}}$$

- So from above two equations, we can get

$$v_e = \frac{n\hbar}{m_e r} = \frac{e}{\sqrt{4\pi\varepsilon_0 m_e r}} \Rightarrow r = \frac{4\pi\varepsilon_0 n^2\hbar^2}{m_e e^2}$$

Bohr Radius

- The radius of the hydrogen atom for the n^{th} stationary state is

$$r_n = \frac{4\pi\varepsilon_0\hbar^2 n^2}{m_e e^2} = a_0 n^2$$

Where the **Bohr radius** for a given stationary state is:

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} = \frac{\left(1.055\times10^{-34}\,J\cdot s\right)^2}{\left(8.99\times10^9\,N\cdot m^2/C^2\right)\cdot\left(9.11\times10^{-31}\,kg\right)\cdot\left(1.6\times10^{-19}\,C\right)^2} = 0.53\times10^{-10}\,m$$

- The smallest diameter of the hydrogen atom is

$$d = 2r_1 = 2a_0 \approx 10^{-10}\,m \approx 1\,\mathring{A}$$

  - OMG!! The fundamental length!!

- $n = 1$ gives its lowest energy state (called the **"ground" state**)

Ex. 4.6 Justification for non-relativistic treatment of orbital e

- Are we justified for the non-relativistic treatment of the orbital electrons?
  - When do we apply relativistic treatment?
    - When v/c>0.1
- Orbital speed: $v_e = \frac{e}{\sqrt{4\pi\varepsilon_0 m_e r}}$
- Thus

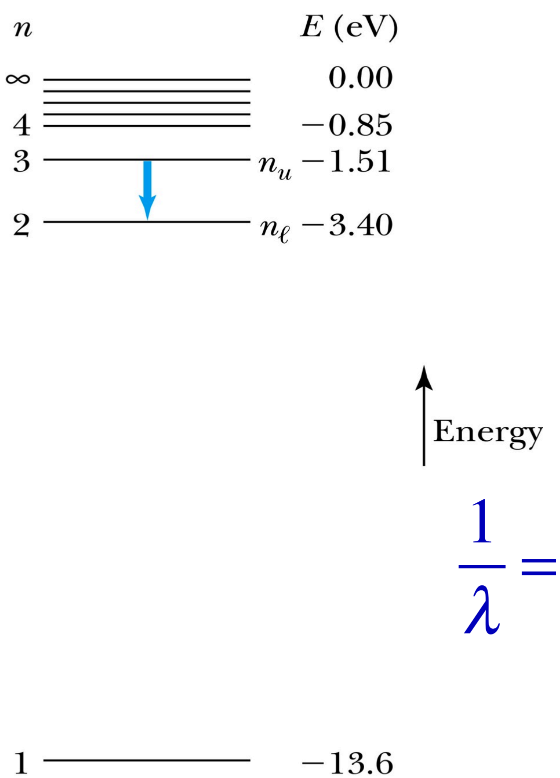
$$v_e = \frac{\left(1.6\times10^{-16}\right)\cdot\left(9\times10^{9}\right)}{\sqrt{\left(9.1\times10^{-31}\right)\cdot\left(0.5\times10^{-10}\right)}} \approx 2.2\times10^{6}\left(m/s\right) < 0.01c$$

The Hydrogen Atom

- Recalling the total E of an e in an atom, the n[th] stationary states En

$$E_n = -\frac{e^2}{8\pi\varepsilon_0 r_n} = -\frac{e^2}{8\pi\varepsilon_0 a_0 n^2} = -\frac{E_1}{n^2} \qquad E_0 = -\frac{e^2}{8\pi\varepsilon_0 a_0} = \frac{-\left(8.99\times10^9\, N\cdot m^2/C^2\right)\cdot\left(1.6\times10^{-19} C\right)^2}{2\left(0.53\times10^{-10} m\right)} = -13.6eV$$

where E_0 is the ground state energy



- Emission of light occurs when the atom is in an excited state and decays to a lower energy state ($n_u \longrightarrow n_\ell$).

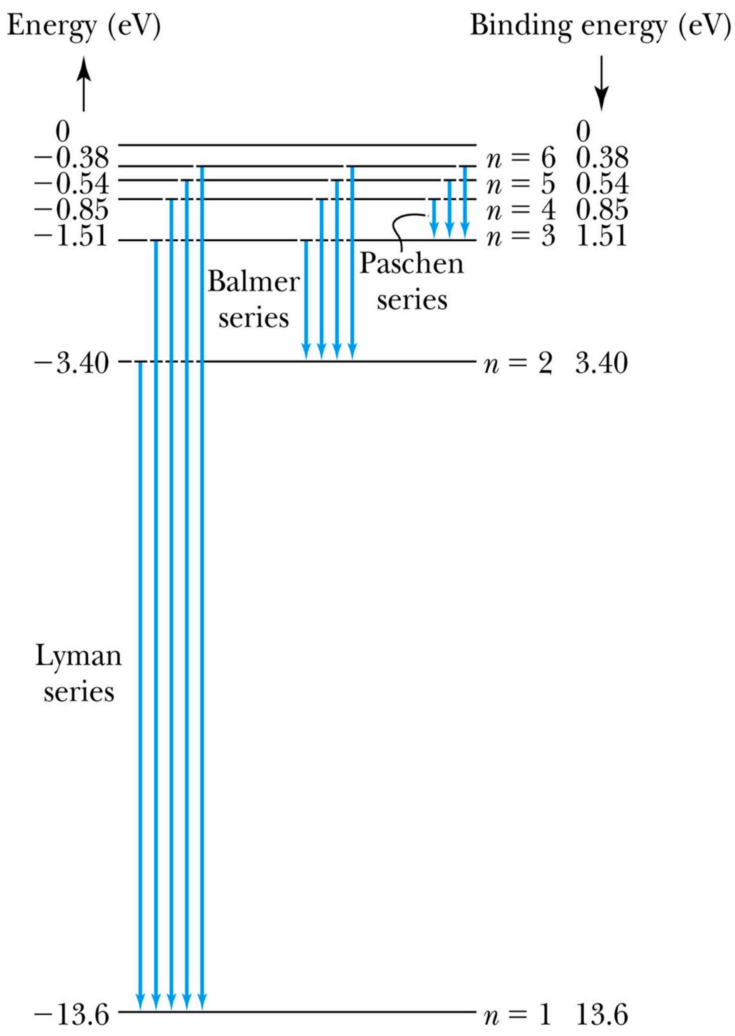
$$hf = E_u - E_l$$

where f is the frequency of a photon.

$$\frac{1}{\lambda} = \frac{f}{c} = \frac{E_u - E_l}{hc} = \frac{E_0}{hc}\left(\frac{1}{n_l^2} - \frac{1}{n_u^2}\right) = R_\infty\left(\frac{1}{n_l^2} - \frac{1}{n_u^2}\right)$$

R_∞ is the **Rydberg constant.** $R_\infty = E_0/hc$

Transitions in the Hydrogen Atom



- **Lyman series:** The atom will remain in the excited state for a short time before emitting a photon and returning to a lower stationary state. All hydrogen atoms exist in $n = 1$ (invisible).
- **Balmer series:** When sunlight passes through the atmosphere, hydrogen atoms in water vapor absorb the wavelengths (visible).

Fine Structure Constant

- The electron's speed on an orbit in the Bohr model:

$$v_e = \frac{n\hbar}{m_e r_n} = \frac{n\hbar}{m_e \frac{4\pi\varepsilon_0 n^2\hbar^2}{m_e e^2}} = \frac{1}{n}\frac{e^2}{4\pi\varepsilon_0\hbar}$$

- On the ground state, $v_1 = 2.2 \times 10^6$ m/s ~ less than 1% of the speed of light
- The ratio of v_1 to c is the **fine structure constant, α**.

$$\alpha \equiv \frac{v_1}{c} = \frac{\hbar}{m a_0 c} = \frac{e^2}{4\pi\varepsilon_0\hbar c} =$$

$$\frac{\left(8.99\times10^9\, N\cdot m^2/C^2\right)\cdot\left(1.6\times10^{-19}\,C\right)^2}{\left(1.055\times10^{-34}\,J\cdot s\right)\cdot\left(3\times10^8\,m/s\right)} \approx \frac{1}{137}$$

The Correspondence Principle

Classical electrodynamics + Bohr's atomic model

Determine the properties of radiation

Need a principle to relate the new modern results with classical ones.

Bohr's correspondence principle → In the limits where classical and quantum theories should agree, the quantum theory must produce the classical results.

The Correspondence Principle

- The frequency of the radiation emitted $f_{classical}$ is equal to the orbital frequency f_{orb} of the electron around the nucleus.

$$f_{classical} = f_{obs} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{v}{r} = \frac{1}{2\pi r}\frac{e}{\sqrt{4\pi\varepsilon_0 m_e r}} = \frac{1}{2\pi}\left(\frac{e^2}{4\pi\varepsilon_0 m_e r^3}\right)^{1/2} = \frac{m_e e^4}{4\varepsilon_0^2 h^3}\frac{1}{n^3}$$

- The frequency of the photon in the transition from *n* + 1 to *n* is

$$f_{Bohr} = \frac{E_0}{h}\left(\frac{1}{(n)^2} - \frac{1}{(n+1)^2}\right) = \frac{E_0}{h}\frac{n^2+2n+1-n^2}{n^2(n+1)^2} = \frac{E_0}{h}\left[\frac{2n+1}{n^2(n+1)^2}\right]$$

- For large *n the classical limit*, $f_{Bohr} \approx \frac{2nE_0}{hn^4} = \frac{2E_0}{hn^3}$

Substitute E_0: $f_{Bohr} = \frac{2E_0}{hn^3} = \frac{2}{hn^3}\left(\frac{e^2}{8\pi\varepsilon_0 a_0}\right) = \frac{m_e e^4}{4\varepsilon_0^2 h^3}\frac{1}{n^3} = f_{Classical}$

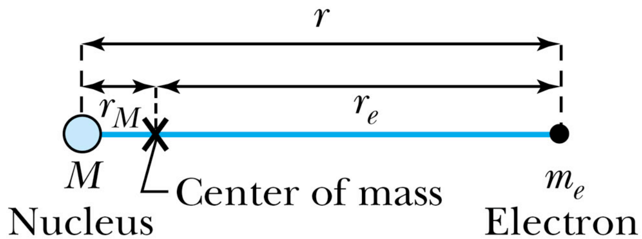
So the frequency of the radiated E between classical theory and Bohr model agrees in large n case!!

Importance of Bohr's Model

- Demonstrated the need for Plank's constant in understanding the atomic structure
- Assumption of quantized angular momentum which led to quantization of other quantities, r, v and E as follows
- Orbital Radius: $$r_n = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} n^2 = a_0 n^2$$
- Orbital Speed: $$v = \frac{n\hbar}{mr_n} = \frac{\hbar}{ma_0}\frac{1}{n}$$
- Energy levels: $$E_n = \frac{e^2}{8\pi\varepsilon_0 a_0 n^2} = \frac{E_0}{n^2}$$

Successes and Failures of the Bohr Model

- The electron and hydrogen nucleus actually revolve about their mutual center of mass ➔ reduced mass correction!!



- All we need is to replace m_e with atom's **reduced mass**.

$$\mu_e = \frac{m_e M}{m_e + M} = \frac{m_e}{1 + m_e/M}$$

- The Rydberg constant for infinite nuclear mass, R_∞ is replaced by *R*.

$$R = \frac{\mu_e}{m_e} R_\infty = \frac{1}{1 + m_e/M} R_\infty = \frac{\mu_e e^4}{4\pi c\hbar^3 \left(4\pi\varepsilon_0\right)^2}$$

For H: $R_H = 1.096776 \times 10^7 m^{-1}$

Limitations of the Bohr Model

The Bohr model was a great step of the new quantum theory, but it had its limitations.

1) Works only to single-electron atoms
   - Works even for ions ➔ What would change?
   - The charge of the nucleus $\frac{1}{\lambda} = Z^2 R\left(\frac{1}{n_l^2} - \frac{1}{n_u^2}\right)$
2) Could not account for the intensities or the fine structure of the spectral lines
   - Fine structure is caused by the electron spin
   - Under a magnetic field, the spectrum splits by the spin
3) Could not explain the binding of atoms into molecules