PHYS 3313 – Section 001
Lecture #15

Monday, March 30, 2015
*Dr. **Jaehoon Yu***

- Wave Motion and Properties
- Wave Packets and Packet Envelops
- Superposition of Waves
- Electron Double Slit Experiment
- Wave-Particle Duality
- The Uncertainty Principle
- The Schrodinger Wave Equation

Announcements

- Quiz this Wednesday, April 1
  - At the beginning of the class
  - Covers CH4.1 through what we finish Monday, March 30
  - BYOF with the same rule as before
- Colloquium at 4pm Wednesday, in SH101

Wave Motion

- de Broglie matter waves suggest a further description. The displacement of a traveling wave is

$$\Psi(x,t) = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

- This is a solution to the wave equation

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2}$$

- Define the wave number *k* and the angular frequency ω as:

$$k \equiv \frac{2\pi}{\lambda} \text{ and } \omega = \frac{2\pi}{T} \qquad \lambda = vT$$

- The wave function can be rewritten: $\Psi(x,t) = A\sin[kx - \omega t]$

Wave Properties

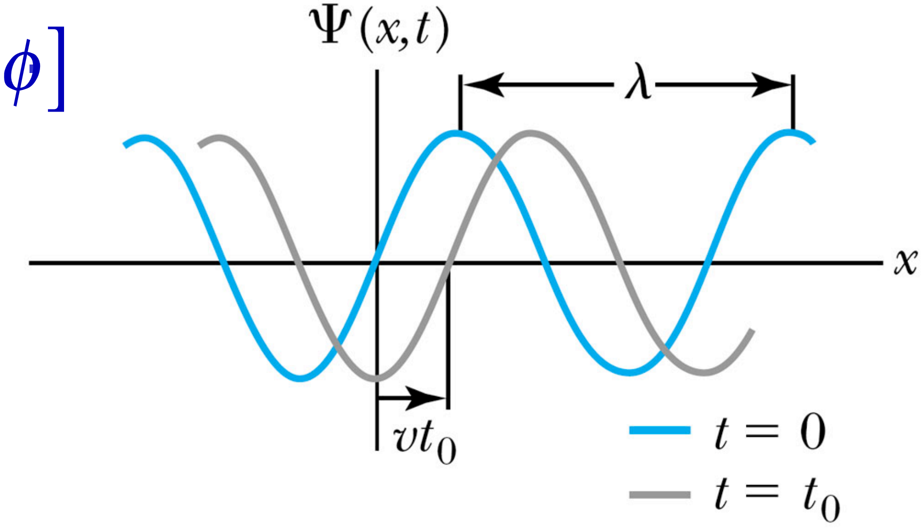
- The phase velocity is the velocity of a point on the wave that has a given phase (for example, the crest) and is given by

$$v_{ph} = \frac{\lambda}{T} = \frac{\lambda}{2\pi}\frac{2\pi}{T} = \frac{\omega}{k}$$

- The phase constant ϕ shifts the wave:

$$\Psi(x,t) = A\sin[kx - \omega t + \phi]$$
$$= A\cos[kx - \omega t]$$

(When $\phi=\pi/2$)



Principle of Superposition

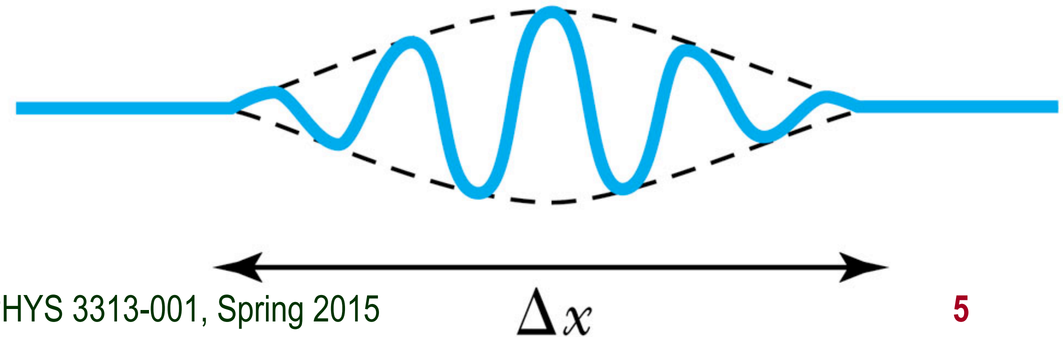
- When two or more waves traverse the same region, they act independently of each other.
- Combining two waves of the same amplitude yields:

$$\Psi(x,t) = \Psi_1(x,t)+\Psi_2(x,t) = A\cos(k_1x-\omega_1t)+A\cos(k_2x-\omega_2t) =$$

$$2A\cos\left(\frac{k_1-k_2}{2}x-\frac{\omega_1-\omega_2}{2}t\right)\cos\left(\frac{k_1+k_2}{2}x-\frac{\omega_1+\omega_2}{2}t\right) = 2A\cos\left(\frac{\Delta k}{2}x-\frac{\Delta\omega}{2}t\right)\cos(k_{\mathrm{av}}x-\omega_{\mathrm{av}}t)$$

- The combined wave oscillates within an envelope that denotes the maximum displacement of the combined waves.
- When combining many waves with different amplitudes and frequencies, a pulse, or **wave packet**, can be formed, which can move at a **group velocity**:

$$u_{\mathrm{gr}} = \frac{\Delta\omega}{\Delta k}$$



Fourier Series

- The sum of many waves that form a wave packet is called a **Fourier series**:

$$\Psi(x,t)=\sum_i A_i \sin\left[k_i x-\omega_i t\right]$$

- Summing an infinite number of waves yields the Fourier integral:

$$\Psi(x,t)=\int \tilde{A}(k)\cos\left[kx-\omega t\right]dk$$

Wave Packet Envelope

- The superposition of two waves yields a wave number and angular frequency of the wave packet envelope.

$$\cos\left(\frac{\Delta k}{2}x - \frac{\Delta \omega}{2}t\right)$$

- The range of wave numbers and angular frequencies that produce the wave packet have the following relations:

$$\Delta k \Delta x = 2\pi \qquad \Delta \omega \Delta t = 2\pi$$

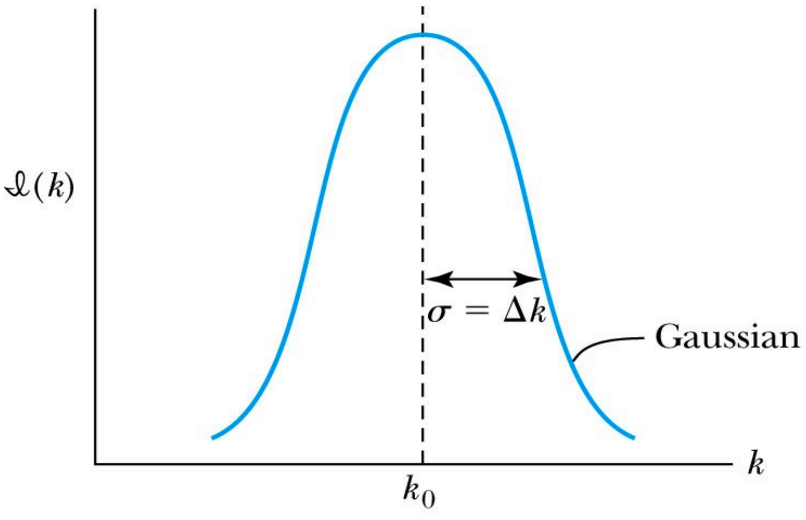
- A **Gaussian wave packet** has similar relations:

$$\Delta k \Delta x = \frac{1}{2} \qquad \Delta \omega \Delta t = \frac{1}{2}$$

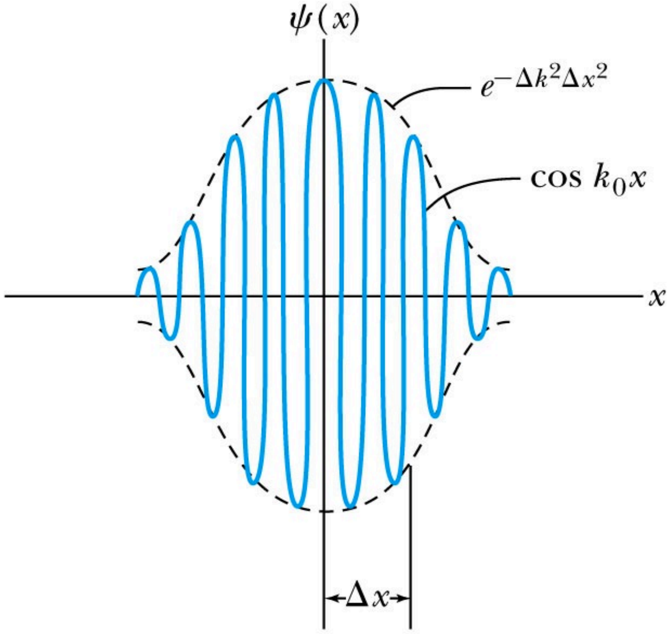
- The localization of the wave packet over a small region to describe a particle requires a large range of wave numbers. Conversely, a small range of wave numbers cannot produce a wave packet localized within a small distance.

Gaussian Function

- A Gaussian wave packet describes the envelope of a pulse wave. $$\Psi(x,0)=\Psi(x)=Ae^{-\Delta k^2x^2}\cos(k_0x)$$



(a)



(b)

- The group velocity is $u_{\text{gr}}=\dfrac{d\omega}{dk}$

Dispersion

- Considering the group velocity of a de Broglie wave packet yields:

$$E^2 = p^2c^2 + m^2c^4$$

$$2EdE = 2pc^2dp \Rightarrow \frac{dE}{dp} = \frac{pc^2}{E} = u_{\text{gr}}$$

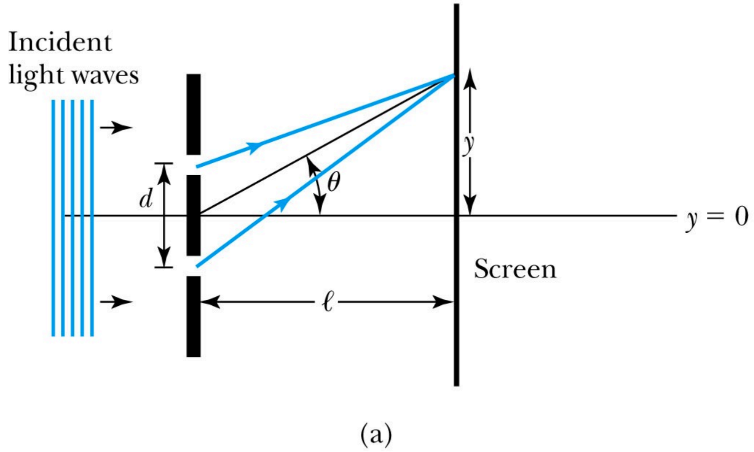
- The relationship between the phase velocity and the group velocity is

$$u_{\text{gr}} = \frac{d\omega}{dk} = \frac{d}{dk}\left(v_{ph}k\right) = v_{ph} + k\frac{dv_{ph}}{dk}$$

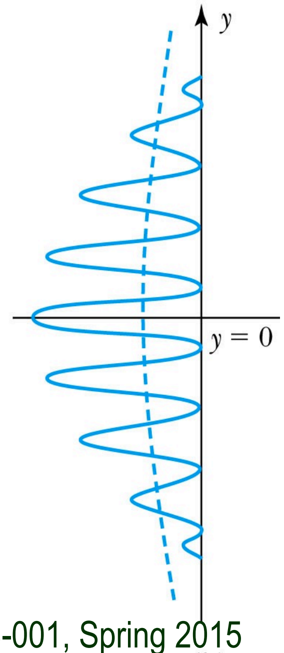
- Hence the group velocity may be greater or less than the phase velocity. A medium is called **nondispersive** when the phase velocity is the same for all frequencies and equal to the group velocity.

Waves or Particles?

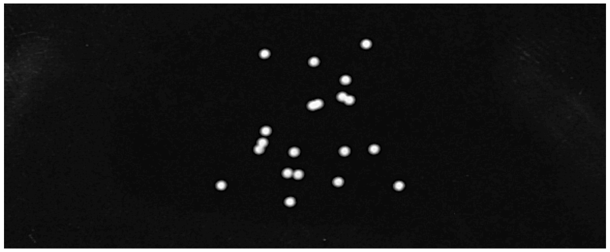
- Young’s double-slit diffraction experiment demonstrates the wave property of light.
- However, dimming the light results in single flashes on the screen representative of particles.



(a)

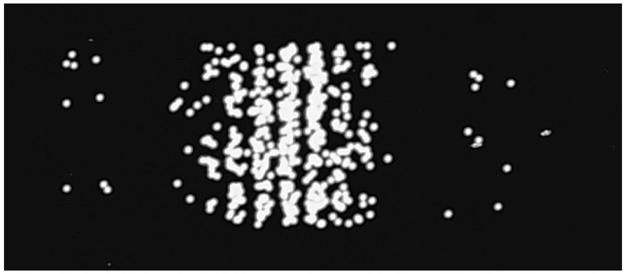


(b)

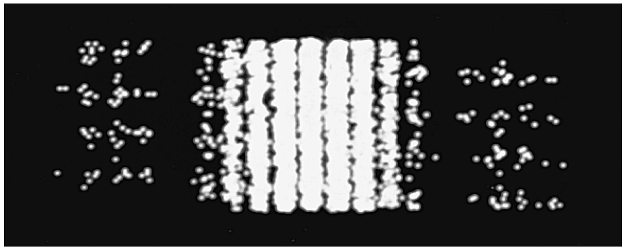


(a) 20 counts

(b) 100 counts



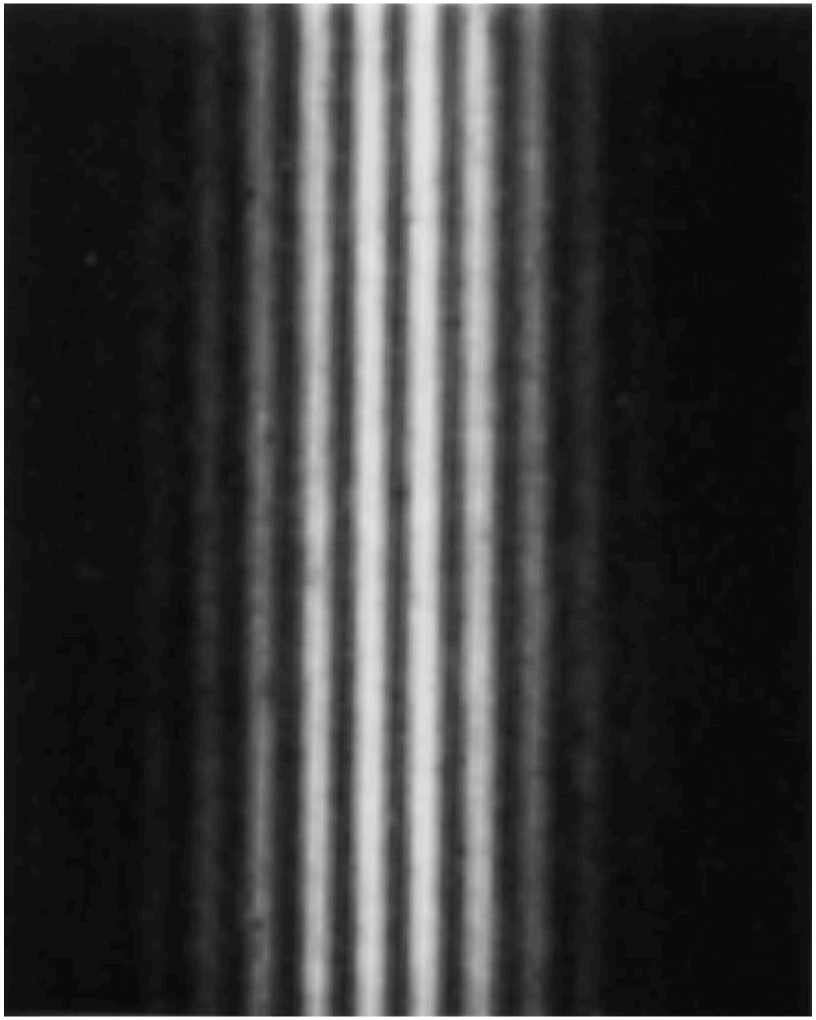
(c) 500 counts



(d) ~4000 counts

Electron Double-Slit Experiment

- C. Jönsson of Tübingen, Germany, succeeded in 1961 in showing double-slit interference effects for **electrons** by constructing very narrow slits and using relatively large distances between the slits and the observation screen.
- This experiment demonstrated that precisely the same behavior occurs for both light (waves) and electrons (particles).



Wave particle duality solution

- The solution to the wave particle duality of an event is given by the following principle.
- **Bohr's principle of complementarity**: It is not possible to describe physical observables simultaneously in terms of both particles and waves.
- **Physical observables** are the quantities such as position, velocity, momentum, and energy that can be experimentally measured. In any given instance we must use either the particle description or the wave description.

Uncertainty Principles

- It is impossible to measure simultaneously, with no uncertainty, the precise values of *k* and *x* for the same particle. The wave number *k* may be rewritten as

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{h/p} = p\frac{2\pi}{h} = \frac{p}{\hbar}$$

- For the case of a Gaussian wave packet we have

$$\Delta k \Delta x = \frac{\Delta p}{\hbar}\Delta x = \frac{1}{2} \quad \Rightarrow \Delta p_x \Delta x = \frac{\hbar}{2}$$

Thus for a single particle, we have Heisenberg's **uncertainty principle**:

$$\Delta p_x \Delta x \geq \frac{\hbar}{2}$$

Momentum-position

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Energy-time

Probability, Wave Functions, and the Copenhagen Interpretation

- The wave function determines the likelihood (or probability) of finding a particle at a particular position in space at a given time.

$$P(y)dy = \left|\Psi(y,t)^2\right|dy$$

- The total probability of finding the particle is 1. Forcing this condition on the wave function is called normalization.

$$\int_{-\infty}^{+\infty} P(y)dy = \int_{-\infty}^{+\infty} \left|\Psi(y,t)^2\right|dy = 1$$

The Copenhagen Interpretation

- Bohr's interpretation of the wave function consisted of 3 principles:
  1) The uncertainty principle of Heisenberg
  2) The complementarity principle of Bohr
  3) The statistical interpretation of Born, based on probabilities determined by the wave function
- Together these three concepts form a logical interpretation of the physical meaning of quantum theory. According to the Copenhagen interpretation, physics depends on the outcomes of measurement.

Particle in a Box

- A particle of mass m is trapped in a one-dimensional box of width ℓ.
- The particle is treated as a wave.
- The box puts boundary conditions on the wave. The wave function must be zero at the walls of the box and on the outside.
- In order for the probability to vanish at the walls, we must have an integral number of half wavelengths in the box.
  - $$\frac{n\lambda}{2} = \ell \quad \text{or} \quad \lambda_n = \frac{2\ell}{n} \qquad (n = 1,2,3.....)$$
- The energy of the particle is $$E = KE = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2}$$
- The possible wavelengths are quantized which yields the energy:

$$E_n = \frac{h^2}{2m\lambda^2} = \frac{h^2}{2m}\left(\frac{n}{2\ell}\right)^2 = n^2\frac{h^2}{8m\ell^2} \qquad (n = 1,2,3.....)$$

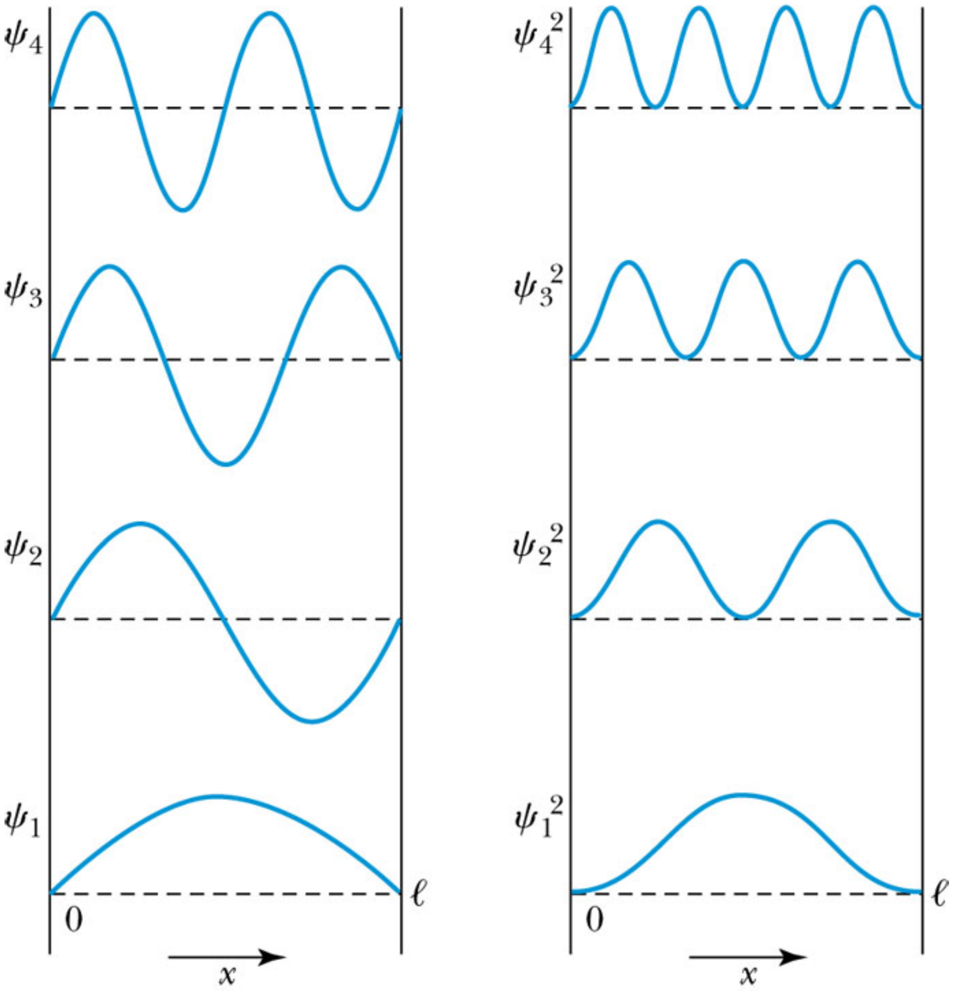
- The possible energies of the particle are quantized.

Probability of the Particle

- The probability of observing the particle between *x* and *x* + *dx* in each state is

$$P_n dx \propto \left|\Psi_n(x)\right|^2 dx$$

- Note that $E_0 = 0$ is not a possible energy level.
- The concept of energy levels, as first discussed in the Bohr model, has surfaced in a natural way by using waves.



The Schrödinger Wave Equation

- The Schrödinger wave equation in its time-dependent form for a particle of energy *E* moving in a potential *V* in one dimension is

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2}+V\Psi(x,t)$$

- The extension into three dimensions is

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}\right)+V\Psi(x,y,z,t)$$

- where $i=\sqrt{-1}$ is an imaginary number

General Solution of the Schrödinger Wave Equation

- The **general form** of the **solution of the Schrödinger wave equation is given by:**

$$\Psi(x,t) = Ae^{i(kx-\omega t)} = A\left[\cos(kx-\omega t) + i\sin(kx-\omega t)\right]$$

- which also describes a wave moving in the *x* direction. In general the amplitude may also be complex. *This is called the wave function of the particle.*
- The wave function is also **not** restricted to being real. Notice that the sine term has an imaginary number. Only the physically measurable quantities must be real. These include the probability, momentum and energy.

Normalization and Probability

- The probability $P(x)\,dx$ of a particle being between x and $X + dx$ was given in the equation

$$P(x)dx = \Psi^*(x,t)\Psi(x,t)dx$$

- Here Ψ^* denotes the complex conjugate of Ψ
- The probability of the particle being between x_1 and x_2 is given by

$$P = \int_{x_1}^{x_2} \Psi^* \Psi \, dx$$

- The wave function must also be normalized so that the probability of the particle being somewhere on the x axis is 1.

$$\int_{-\infty}^{+\infty} \Psi^*(x,t)\Psi(x,t)dx = 1$$