PHYS 3313 – Section 001
Lecture #16

Wednesday, April 1, 2015
Dr. Jaehoon Yu

- Probability of Particle
- Schrodinger Wave Equation and Solutions
- Normalization and Probability

Special Project #4

- Prove that the wave function $\Psi = A[\sin(kx-\omega t) + i\cos(kx-\omega t)]$ is a good solution for the time-dependent Schrödinger wave equation. Do NOT use the exponential expression of the wave function. (10 points)
- Determine whether or not the wave function $\Psi = Ae^{-\alpha|x|}$ satisfy the time-dependent Schrödinger wave equation. (10 points)
- Due for this special project is Wednesday, Apr. 8.
- You MUST have your own answers!

The Schrödinger Wave Equation

- Erwin Schrödinger and Werner Heinsenberg proposed quantum theory in 1920
  - The two proposed very different forms of equations
  - Heinserberg: Matrix based framework
  - Schrödinger: Wave mechanics, similar to the classical wave equation
- Paul Dirac and Schrödinger later on proved that the two give identical results
- The probabilistic nature of quantum theory is contradictory to the direct cause and effect seen in classical physics and makes it difficult to grasp!

The Time-dependent Schrödinger Wave Equation

- The Schrödinger wave equation in its **time-dependent** form for a particle of energy *E* moving in a potential *V* in one dimension is

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2}+V\Psi(x,t)$$

- The extension into three dimensions is

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}\right)+V\Psi(x,y,z,t)$$

- where $i=\sqrt{-1}$ is an imaginary number

Ex 6.1: Wave equation and Superposition

The wave equation must be linear so that we can use the superposition principle. Prove that the wave function in Schrodinger equation is linear by showing that it is satisfied for the wave equation $\Psi(x,t)=a\Psi_1(x,t)+b\Psi_2(x,t)$ where a and b are constants and $\Psi_1(x,t)$ and $\Psi_2(x,t)$ describe two waves each satisfying the Schrodinger Eq.

$$\Psi = a\Psi_1 + b\Psi_2 \qquad i\hbar\frac{\partial\Psi_1}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_1}{\partial x^2} + V\Psi_1 \qquad i\hbar\frac{\partial\Psi_2}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_2}{\partial x^2} + V\Psi_2$$

$$\frac{\partial\Psi}{\partial t} = \frac{\partial}{\partial t}\left(a\Psi_1 + b\Psi_2\right) = a\frac{\partial\Psi_1}{\partial t} + b\frac{\partial\Psi_2}{\partial t}$$

$$\frac{\partial\Psi}{\partial x} = \frac{\partial}{\partial x}\left(a\Psi_1 + b\Psi_2\right) = a\frac{\partial\Psi_1}{\partial x} + b\frac{\partial\Psi_2}{\partial x} \Rightarrow \frac{\partial^2\Psi}{\partial x^2} = \frac{\partial}{\partial x}\left(a\frac{\partial\Psi_1}{\partial x} + b\frac{\partial\Psi_2}{\partial x}\right) = a\frac{\partial^2\Psi_1}{\partial x^2} + b\frac{\partial^2\Psi_2}{\partial x^2}$$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \xrightarrow{\text{Rearrange terms}} i\hbar\frac{\partial\Psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} - V\Psi = \left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - V\right)\Psi = 0$$

$$i\hbar\frac{\partial\Psi}{\partial t} = i\hbar\left(a\frac{\partial\Psi_1}{\partial t} + b\frac{\partial\Psi_2}{\partial t}\right) = -\frac{\hbar^2}{2m}\left(a\frac{\partial^2\Psi_1}{\partial x^2} + b\frac{\partial^2\Psi_2}{\partial x^2}\right) + V\left(a\Psi_1 + b\Psi_2\right)$$

$$a\left(i\hbar\frac{\partial\Psi_1}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\Psi_1}{\partial x^2} - V\Psi_1\right) = -b\left(i\hbar\frac{\partial\Psi_2}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2\Psi_2}{\partial x^2} - V\Psi_2\right) = 0$$

General Solution of the Schrödinger Wave Equation

- The general form of the solution of the Schrödinger wave equation is given by:

$$\Psi(x,t) = Ae^{i(kx-\omega t)} = A\left[\cos(kx-\omega t) + i\sin(kx-\omega t)\right]$$

- which also describes a wave propagating in the *x* direction. In general the amplitude may also be complex. *This is called the* ***wave function of the particle****.*
- The wave function is also **not** restricted to being real. Only the physically measurable quantities (or **observables**) must be real. These include the probability, momentum and energy.

Ex 6.2: Solution for Wave Equation

Show that $Ae^{i(kx-\omega t)}$ satisfies the time-dependent Schrodinger wave Eq.

$$\Psi = Ae^{i(kx-\omega t)} \Rightarrow \frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial t}\left(Ae^{i(kx-\omega t)}\right) = -iA\omega e^{i(kx-\omega t)} = -i\omega\Psi$$

$$\frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial x}\left(Ae^{i(kx-\omega t)}\right) = iAke^{i(kx-\omega t)} = ik\Psi$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial}{\partial x}(ik\Psi) = ik\frac{\partial}{\partial x}(\Psi) = ik\left(iAke^{i(kx-\omega t)}\right) = -Ak^2e^{i(kx-\omega t)} = -k^2\Psi$$

$$i\hbar\frac{\partial \Psi}{\partial t} = i\hbar(-i\omega\Psi) = \hbar\omega\Psi = -\frac{\hbar^2}{2m}\left(-k^2\Psi\right) + V\Psi$$

$$\left(\hbar\omega - \frac{\hbar^2k^2}{2m} - V\right)\Psi = 0$$

$$= \left(E - \frac{p^2}{2m} - V\right) = 0$$

The Energy: $E = hf = h\left(\frac{\omega}{2\pi}\right) = \hbar\omega$

The wave number: $k = \frac{2\pi}{\lambda} = \frac{2\pi}{h/p} = \frac{2\pi p}{h} = \frac{p}{\hbar}$ $\Rightarrow$ The momentum: $p = \hbar k$

From the energy conservation: $E = K + V = \frac{p^2}{2m} + V$ $\Rightarrow$ $E - \frac{p^2}{2m} - V = 0$

So $Ae^{i(kx-\omega t)}$ is a good solution and satisfies Schrodinger Eq.

Ex 6.3: Bad Solution for Wave Equation

Determine $\Psi(x,t)=A\sin(kx-\omega t)$ is an acceptable solution for the time-dependent Schrodinger wave Eq.

$$\Psi = A\sin(kx-\omega t) \Rightarrow \frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial t}\left(A\sin(kx-\omega t)\right) = -A\omega\cos(kx-\omega t)$$

$$\frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial x}\left(A\sin(kx-\omega t)\right) = kA\cos(kx-\omega t)$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial}{\partial x}\left(kA\cos(kx-\omega t)\right) = -k^2 A\sin(kx-\omega t) = -k^2\Psi$$

$$i\hbar\left(-\omega\cos(kx-\omega t)\right) = -\frac{\hbar^2}{2m}\left(-k^2\sin(kx-\omega t)\right) + V\sin(kx-\omega t)$$

$$-i\hbar\omega\cos(kx-\omega t) = \left(\frac{\hbar^2 k^2}{2m}+V\right)\sin(kx-\omega t)$$

$$\Rightarrow -iE\cos(kx-\omega t) = \left(\frac{p^2}{2m}+V\right)\sin(kx-\omega t)$$

This is not true in all x and t. So $\Psi(x,t)=A\sin(kx-\omega t)$ is not an acceptable solution for Schrodinger Eq.