PHYS 3313 – Section 001
Lecture #17

Monday, April 6, 2015
Dr. Jaehoon Yu

- Normalization and Probability
- Time Independent Schrodinger Equation

Announcements

- Research paper template has been placed onto the class web page link to research
- Homework #4
  - End of chapter problems on CH5: 8, 10, 16, 24, 26, 36 and 47
  - Due Monday, Apr. 13
- Quiz #4 at the beginning of the class Monday, Apr. 13
  - Covers CH 5.4 through what we finish this Wednesday
- Colloquium this Wed.

Reminder: Special Project #4

- Prove that the wave function $\Psi=A[\sin(kx-\omega t)+i\cos(kx-\omega t)]$ is a good solution for the time-dependent Schrödinger wave equation. Do NOT use the exponential expression of the wave function. (10 points)
- Determine whether or not the wave function $\Psi=Ae^{-\alpha|x|}$ satisfy the time-dependent Schrödinger wave equation. (10 points)
- Due for this special project is Wednesday, Apr. 8.
- You MUST have your own answers!

Normalization and Probability

- The probability $P(x)\,dx$ of a particle being between x and $X + dx$ was given by the equation

$$P(x)dx = \Psi^*(x,t)\Psi(x,t)dx$$

- Here Ψ^* denotes the complex conjugate of Ψ
- The probability of the particle being between x_1 and x_2 is given by

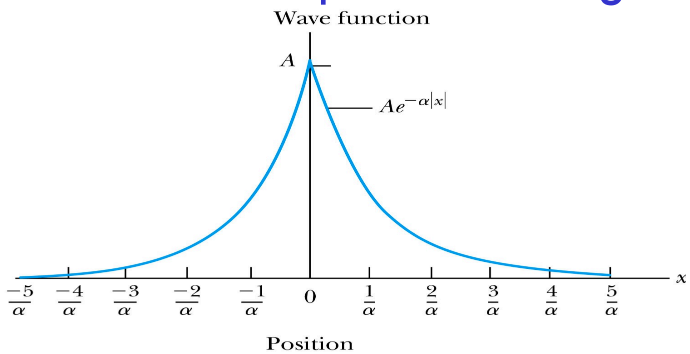
$$P = \int_{x_1}^{x_2} \Psi^* \Psi \, dx$$

- The wave function must also be normalized so that the probability of the particle being somewhere on the x axis is 1.

$$\int_{-\infty}^{+\infty} \Psi^*(x,t)\Psi(x,t)dx = 1$$

Ex 6.4: Normalization

Consider a wave packet formed by using the wave function that $Ae^{-\alpha|x|}$, where A is a constant to be determined by normalization. Normalize this wave function and find the probabilities of the particle being between 0 and $1/\alpha$, and between $1/\alpha$ and $2/\alpha$.



$$\Psi = Ae^{-\alpha|x|}$$

Probability density

$$\int_{-\infty}^{+\infty} \Psi^* \Psi dx = \int_{-\infty}^{+\infty} \left(Ae^{-\alpha|x|}\right)^* \left(Ae^{-\alpha|x|}\right) dx = \int_{-\infty}^{+\infty} \left(A^* e^{-\alpha|x|}\right)\left(Ae^{-\alpha|x|}\right) dx =$$

$$= \int_{-\infty}^{+\infty} A^2 e^{-2\alpha|x|} dx = 2\int_0^{+\infty} A^2 e^{-2\alpha x} dx = \left.\frac{2A^2}{-2\alpha} e^{-2\alpha x}\right|_0^{+\infty} = 0 + \frac{A^2}{\alpha} = 1$$

$$A = \sqrt{\alpha}$$

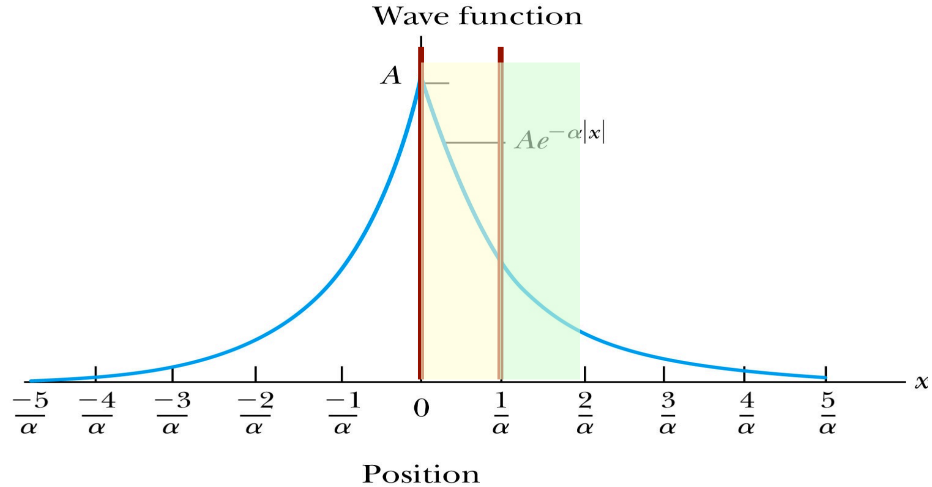
Normalized Wave Function

$$\Psi = \sqrt{\alpha}\, e^{-\alpha|x|}$$

Ex 6.4: Normalization, cont'd

Using the wave function, we can compute the probability for a particle to be with 0 to 1/α and 1/α to 2/α.

$$\Psi = \sqrt{\alpha}\, e^{-\alpha|x|}$$



For 0 to 1/α:

$$P = \int_0^{1/\alpha} \Psi^* \Psi dx = \int_0^{1/\alpha} \alpha e^{-2\alpha x} dx = \frac{\alpha}{-2\alpha} e^{-2\alpha x} \Big|_0^{1/\alpha} = -\frac{1}{2}\left(e^{-2} - 1\right) \approx 0.432$$

For 1/α to 2/α:

$$P = \int_{1/\alpha}^{2/\alpha} \Psi^* \Psi dx = \int_{1/\alpha}^{2/\alpha} \alpha e^{-2\alpha x} dx = \frac{\alpha}{-2\alpha} e^{-2\alpha x} \Big|_{1/\alpha}^{2/\alpha} = -\frac{1}{2}\left(e^{-4} - e^{-2}\right) \approx 0.059$$

How about 2/α:to ∞?

Properties of Valid Wave Functions

Boundary conditions

1) To avoid infinite probabilities, the wave function must be finite everywhere.
2) To avoid multiple values of the probability, the wave function must be single valued.
3) For finite potentials, the wave function and its derivatives must be continuous. This is required because the second-order derivative term in the wave equation must be single valued. (There are exceptions to this rule when V is infinite.)
4) In order to normalize the wave functions, they must approach zero as x approaches infinity.

Solutions that do not satisfy these properties do not generally correspond to physically realizable circumstances.

Time-Independent Schrödinger Wave Equation

- The potential in many cases will not depend explicitly on time.
- The dependence on time and position can then be separated in the Schrödinger wave equation. Let, $\Psi(x,t)=\psi(x)f(t)$

which yields: $$i\hbar\psi(x)\frac{\partial f(t)}{\partial t}=-\frac{\hbar^2 f(t)}{2m}\frac{\partial^2\psi(x)}{\partial x^2}+V(x)\psi(x)f(t)$$

Now divide by the wave function: $$i\hbar\frac{1}{f(t)}\frac{\partial f(t)}{\partial t}=-\frac{\hbar^2 f(t)}{2m}\frac{1}{\psi(x)}\frac{\partial^2\psi(x)}{\partial x^2}+V(x)$$

- The *left side* of this last equation depends only on time, and *the right side* depends only on spatial coordinates. Hence each side must be equal to a constant. The time dependent side is

$$i\hbar\frac{1}{f}\frac{df}{dt}=B$$

Time-Independent Schrödinger Wave Equation(con't)

- We integrate both sides and find: $i\hbar\int\frac{df}{f}=\int B\,dt \Rightarrow i\hbar\ln f = Bt + C$

  where *C* is an integration constant that we may choose to be 0. Therefore

$$\ln f = \frac{Bt}{i\hbar}$$

  This determines *f* to be $f(t) = e^{Bt/i\hbar} = e^{-iBt/\hbar}$. Comparing this to the time dependent portion of the free particle wave function $e^{-i\omega t} = e^{-iBt/\hbar}$

$$\Rightarrow B = \hbar\omega = E \qquad i\hbar\frac{1}{f(t)}\frac{\partial f(t)}{\partial t} = E$$

- This is known as the **time-independent Schrödinger wave equation**, and it is a fundamental equation in quantum mechanics.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

Stationary State

- Recalling the separation of variables: $\Psi(x,t)=\psi(x)f(t)$ and with $f(t)=e^{-i\omega t}$ the wave function can be written as: $\Psi(x,t)=\psi(x)e^{-i\omega t}$
- The probability density becomes:

$$\Psi^*\Psi=\psi^2(x)\left(e^{i\omega t}e^{-i\omega t}\right)=\psi^2(x)$$

- The probability distributions are constant in time. This is a standing wave phenomena that is called the stationary state.

Comparison of Classical and Quantum Mechanics

- Newton’s second law and Schrödinger’s wave equation are both differential equations.
- Newton’s second law can be derived from the Schrödinger wave equation, so the latter is the more fundamental.
- Classical mechanics only appears to be more precise because it deals with macroscopic phenomena. The underlying uncertainties in macroscopic measurements are just too small to be significant.