# PHYS 3313 – Section 001
# Lecture #8

*Monday, Feb. 11, 2019*
*Dr. Jaehoon Yu*

- The Relativistic Doppler Effect
- Relativistic Momentum and Energy
- Relationship Between Relativistic Quantities
- Binding Energy

# Announcements

- Homework #2
  - CH3 end of the chapter problems: 2, 19, 27, 36, 41, 47 and 57
  - Due Wednesday, Feb. 22
- Reminder: Quiz #2 Monday, Feb. 18
  - Beginning of the class
  - Covers CH1.1 – what we finish this Wednesday, Feb. 13
  - You can bring your calculator but it must not have any relevant formula pre-input
  - BYOF: You may bring a one 8.5x11.5 sheet (front and back) of handwritten formulae and values of constants for the exam
  - No derivations, word definitions, or solutions of any problems !
    - Lorentz velocity addition NOT allowed!!
    - Maxwell's equations NOT allowed!!
  - No additional formulae or values of constants will be provided!

# Reminder: Special Project #3

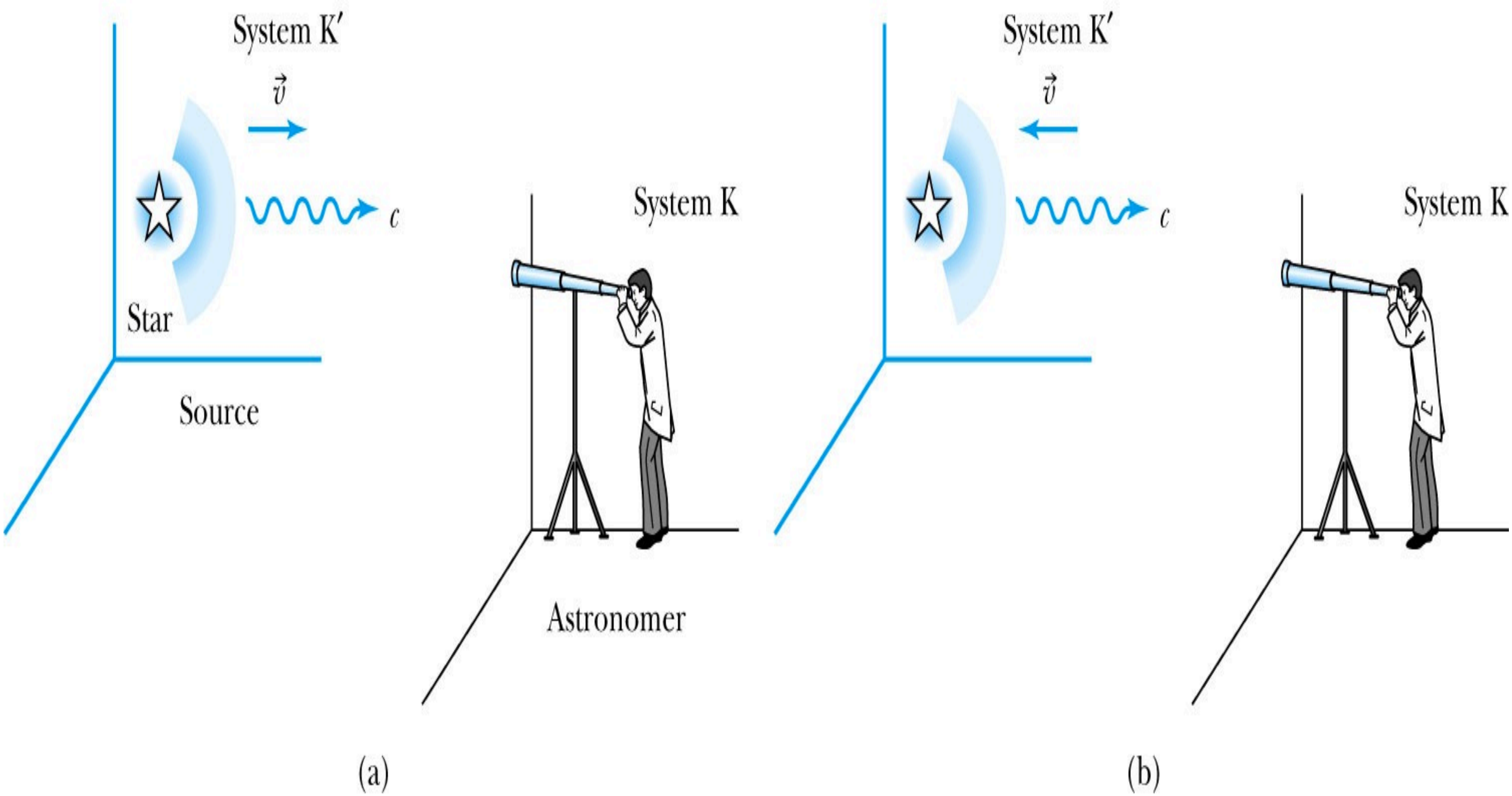
1. Derive the three Lorentz velocity transformation equations, starting from Lorentz coordinate transformation equations. (10 points)
2. Derive the three reverse Lorentz velocity transformation equations, starting from Lorentz coordinate transformation equations. (10 points)
3. Prove that the space-time invariant quantity $s^2=x^2-(ct)^2$ is indeed invariant, i.e. $s^2=s'^2$, in Lorentz Transformation. (5 points)
4. You must derive each one separately starting from the Lorentz spatial coordinate transformation equations to obtain any credit.
   - Just simply switching the signs and primes will NOT be sufficient!
   - Must take the simplest form of the equations, using $\beta$ and $\gamma$.
5. You MUST have your own, independent handwritten answers to the above three questions even if you worked together with others. All those who share the answers will get 0 credit if copied.

- Due for the submission is this Wednesday, Feb. 13!

# The Doppler Effect

- The Doppler effect of <u>sound</u>
  - *Increased frequency* of sound as the source approaches the receiver
  - *Decreased frequency* as the source recedes.
- The same change in sound frequency occurs when the source is fixed and the receiver is moving.
  - The change in frequency of the sound wave depends on whether the source or the receiver is moving.
- Does this violate the principle of relativity?
  - No
  - Why not?
  - Sound wave is in a special frame of media such as air, water, or a steel plate in order to propagate;
- Light does not need a medium to propagate!

# Recall the Doppler Effect



# The Relativistic Doppler Effect

Consider a source of light (a star) and a receiver (an astronomer) approaching one another with a relative velocity *v*.

1) Consider the receiver in system K and the light source in system K' moving toward the receiver with velocity *v*.
2) The source emits *n* waves throughout the time interval *T*.
3) Because the speed of light is always *c* and the source is moving with velocity *v*, the total distance between the front and rear of the wave emitted during the time interval *T* is:

$$\text{Length of the wave train} = cT - vT$$

# The Relativistic Doppler Effect (con't)

Because there are *n* waves emitted by the source in time period T, the wavelength measured by the stationary receiver is

$$\lambda = \frac{cT - vT}{n}$$

And the resulting frequency measured by the receiver is

$$f = \frac{c}{\lambda} = \frac{cn}{cT - vT}$$

The number of waves emitted in the moving frame of the source is n=$f_0 T'_0$ with the proper time T'$_0$=T/$\gamma$ we obtain the measured frequency by the receiver as

$$f = \frac{cf_0 T/\gamma}{cT - vT} = \frac{1}{1 - v/c}\frac{f_0}{\gamma} = \frac{\sqrt{1-\beta^2}}{1-\beta} f_0 = \sqrt{\frac{(1-\beta)(1+\beta)}{(1-\beta)^2}} f_0 = \sqrt{\frac{1+\beta}{1-\beta}} f_0$$

# Results of Relativistic Doppler Effect

When source/receiver is approaching with $\beta = v/c$ the resulting frequency is

$$f = \sqrt{\frac{1+\beta}{1-\beta}} f_0$$

**Higher than the actual source's frequency, blue shift!!**

When source/receiver is receding with $\beta = v/c$ the resulting frequency is

$$f = \sqrt{\frac{1-\beta}{1+\beta}} f_0$$

**Lower than the actual source's frequency, red shift!!**

If we use $+\beta$ for approaching source/receiver and $-\beta$ for receding source/receiver, relativistic Doppler Effect can be expressed

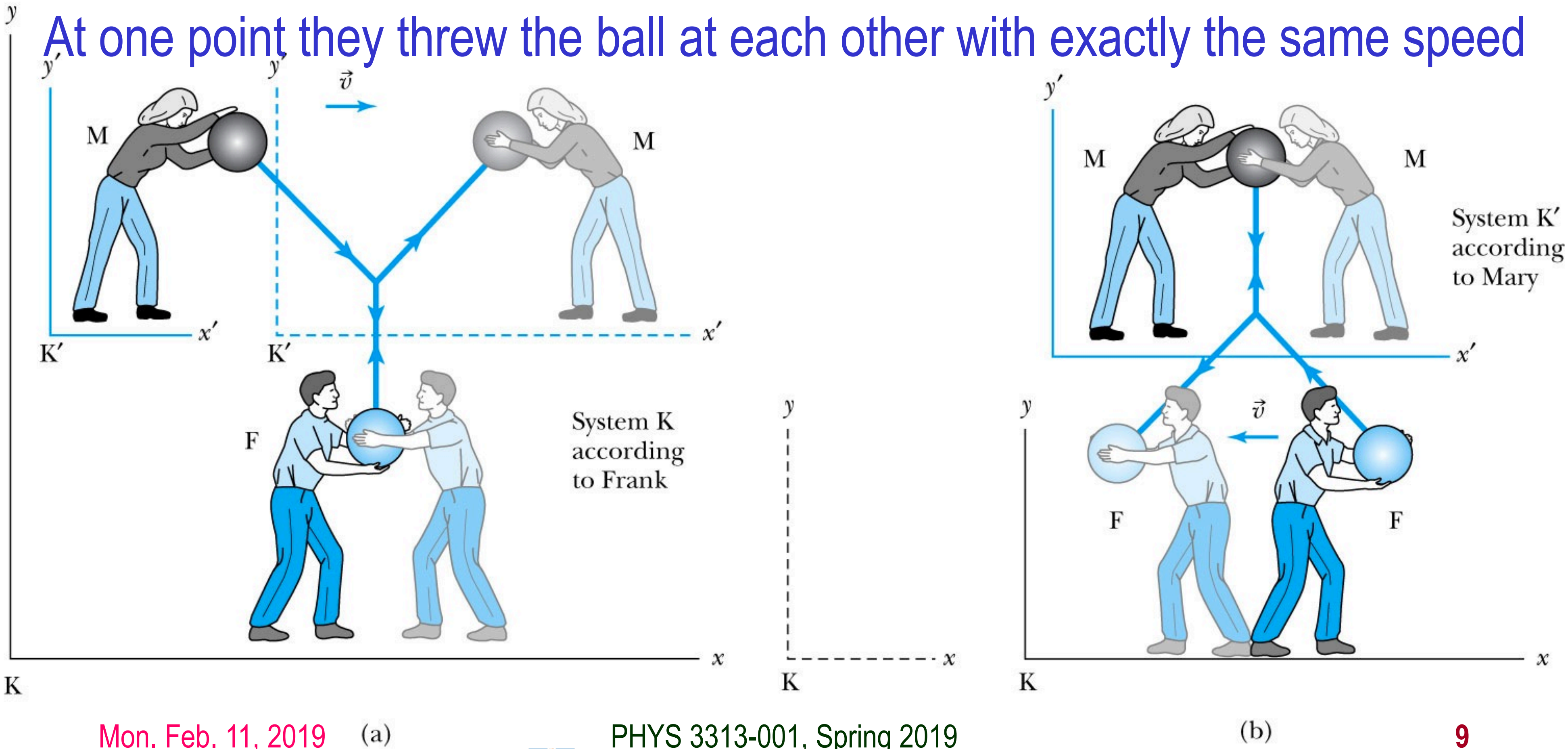
$$\boxed{f = \sqrt{\frac{1+\beta}{1-\beta}} f_0}$$

**For more generalized case**

$$\boxed{f = \frac{1+\beta\cos\theta'}{\sqrt{1-\beta^2}} f_0}$$

# Relativistic Momentum

- The most fundamental principle used here is the momentum conservation!
- Frank is at rest in system K holding a ball of mass *m*.
- Mary holds a similar ball in system K' that is moving in the *x* direction with velocity *v* with respect to system K.
- At one point they threw the ball at each other with exactly the same speed



# Relativistic Momentum

- If we use the definition of momentum, the momentum of the ball thrown by Frank is entirely in the *y* direction

$$p_{Fy} = mu_0$$

- The change of momentum as observed by Frank is

$$\Delta p_F = \Delta p_{Fy} = -2mu_0$$

- Mary measures the initial velocity of her own ball to be

$$u'_{Mx} = 0 \text{ and } u'_{My} = -u_0.$$

- In order to determine the velocity of Mary's ball as measured by Frank we use the velocity transformation equations:

# Reminder: Lorentz Velocity Transformations

In addition to the previous relations, the **Lorentz velocity transformations** for $v'_x$, $v'_y$, and $v'_z$ can be obtained by switching primed and unprimed and changing $v$ to $-v$.(the velocity of the moving frame!!)

$$\mathrm{v}'_x = \frac{v_x - v}{1 - \left(v/c^2\right) v_x}$$

$$\mathrm{v}'_y = \frac{v_y}{\gamma\left[1 - \left(v/c^2\right) v_x\right]}$$

$$\mathrm{v}'_z = \frac{v_z}{\gamma\left[1 - \left(v/c^2\right) v_x\right]}$$

# Relativistic Momentum

- If we use the definition of momentum, the momentum of the ball thrown by Frank is entirely in the *y* direction

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- Mary measures the initial velocity of her own ball to be

$$u'_{Mx} = 0 \text{ and } u'_{My} = -u_0.$$

- In order to determine the velocity of Mary's ball as measured by Frank we use the velocity transformation equations:

$$u_{Mx} = v \qquad\qquad u_{My} = -u_0\sqrt{1 - v^2/c^2}$$

# Relativistic Momentum

Before the collision, the momentum of Mary's ball as measured by Frank (in the **F**ixed frame) with the Lorentz velocity transformation becomes

$$p_{Mx} = mv \qquad p_{My} = -mu_0\sqrt{1-v^2/c^2}$$

For a perfectly elastic collision, the momentum after the collision is

$$p_{Mx} = mv \qquad p_{My} = +mu_0\sqrt{1-v^2/c^2}$$

Thus the change in momentum of Mary's ball according to Frank is

$$\Delta p_M = \Delta p_{My} = 2mu_0\sqrt{1-\beta^2} \neq -\Delta p_{Fy}$$

OMG! The linear momentum is not conserved even w/o an external force!!

What do we do?

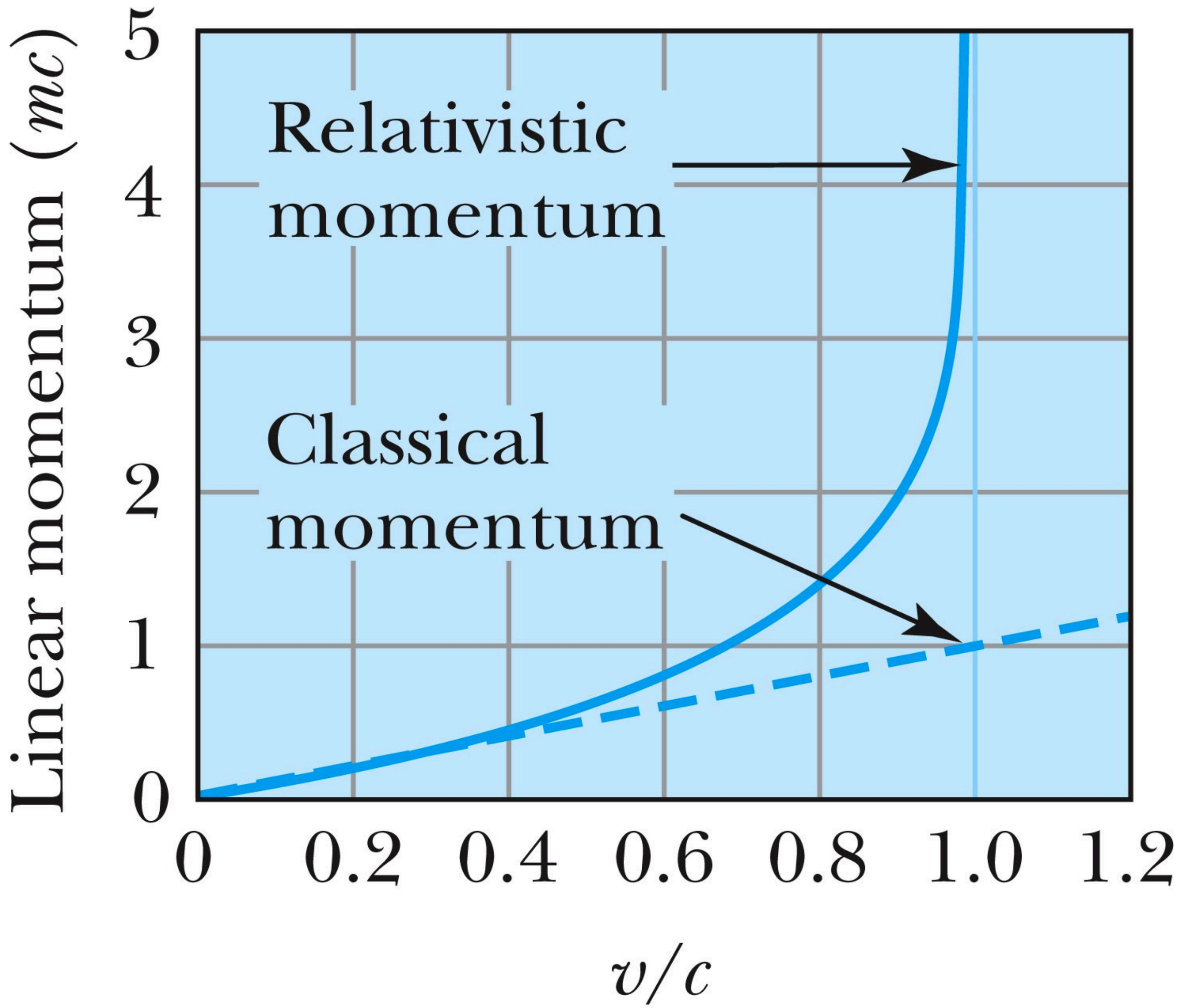
➔Redefine the momentum in a fashion

$$\vec{p} = m\frac{d\left(\gamma_u \vec{r}\right)}{dt} = m\gamma_u \vec{u}$$

➔Something has changed. Mass is now, m$\gamma$!! The relativistic mass!!

➔Mass as the fundamental property of matter is called the "rest mass", $m_0$!

# Relativistic and Classical Linear Momentum



# How do we keep momentum conserved in a relativistic case?

Redefine the classical momentum in the form:

$$\vec{p} = \Gamma(u)m\vec{u} = \frac{1}{\sqrt{1-u^2/c^2}}m\vec{u}$$

This $\Gamma(u)$ is different than the $\gamma$ factor since it uses the particle's speed u

➔ What? How does this make sense?

➔ Well the particle itself is moving at a relativistic speed, thus that must impact the measurements by the observer in the rest frame!!

Now, the agreed form of the momentum in all frames is ($\tau$ is the proper time):

$$\vec{p} = m\frac{d\vec{r}}{d\tau} = m\frac{d\vec{r}}{dt}\frac{dt}{d\tau} = m\vec{u}\gamma = \frac{1}{\sqrt{1-u^2/c^2}}m\vec{u}$$

Resulting in the new relativistic definition of the momentum: $\vec{p} = m\gamma\vec{u}$

When u→0, this formula becomes that of the classical.

What can the speed **u** be to maintain the relativistic momentum to 1% of classical momentum?

# Relativistic Energy

- Due to the new idea of relativistic mass, we must now redefine the concepts of work and energy.
  - Modify Newton's second law to include our new definition of linear momentum, and the force becomes:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\gamma m\vec{u}\right) = \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1-u^2/c^2}}\right)$$

- The work *W* done by a force **F** to move a particle from rest to a certain kinetic energy is

$$W = K = \int \frac{d}{dt}\left(\gamma m\vec{u}\right)\cdot\vec{u}\,dt$$

- Resulting relativistic kinetic energy becomes

$$K = \int_0^{\gamma u} um \cdot d\left(\gamma u\right) = \gamma mc^2 - mc^2 = \left(\gamma - 1\right)mc^2$$

- Why doesn't this look anything like the classical KE?

# Relativistic KE at u<<c

- All relativistic quantities must become that of classical quantity at very low speed, u<<c
- How does $K = (\gamma - 1)mc^2$ become $K = \frac{1}{2}mu^2$ ?
- When u<<c, one can expand $\gamma$ factor as follows

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}} = \left(1-\left(\frac{u}{c}\right)^2\right)^{-1/2} = 1-\left(-\frac{1}{2}\right)\left(\frac{u}{c}\right)^2 + \left(-\frac{1}{2}\right)^2\left(\frac{u}{c}\right)^4 + \cdots \sim 1-\left(-\frac{1}{2}\right)\left(\frac{u}{c}\right)^2$$

- Thus,

$$K = (\gamma - 1)mc^2 = \left(1-\left(-\frac{1}{2}\right)\left(\frac{u}{c}\right)^2 - 1\right)mc^2 = \frac{1}{2}\left(\frac{u}{c}\right)^2 mc^2 = \frac{1}{2}mu^2$$

# Big note on Relativistic KE

- Only $K = (\gamma - 1)mc^2$ is RIGHT!

- $K = \frac{1}{2}mu^2$ and $K = \frac{1}{2}\gamma mu^2$ are WRONG!

# Total Energy and Rest Energy

Rewriting the relativistic kinetic energy, we obtain:

$$\gamma mc^2 = \frac{mc^2}{\sqrt{1-u^2/c^2}} = K + mc^2$$

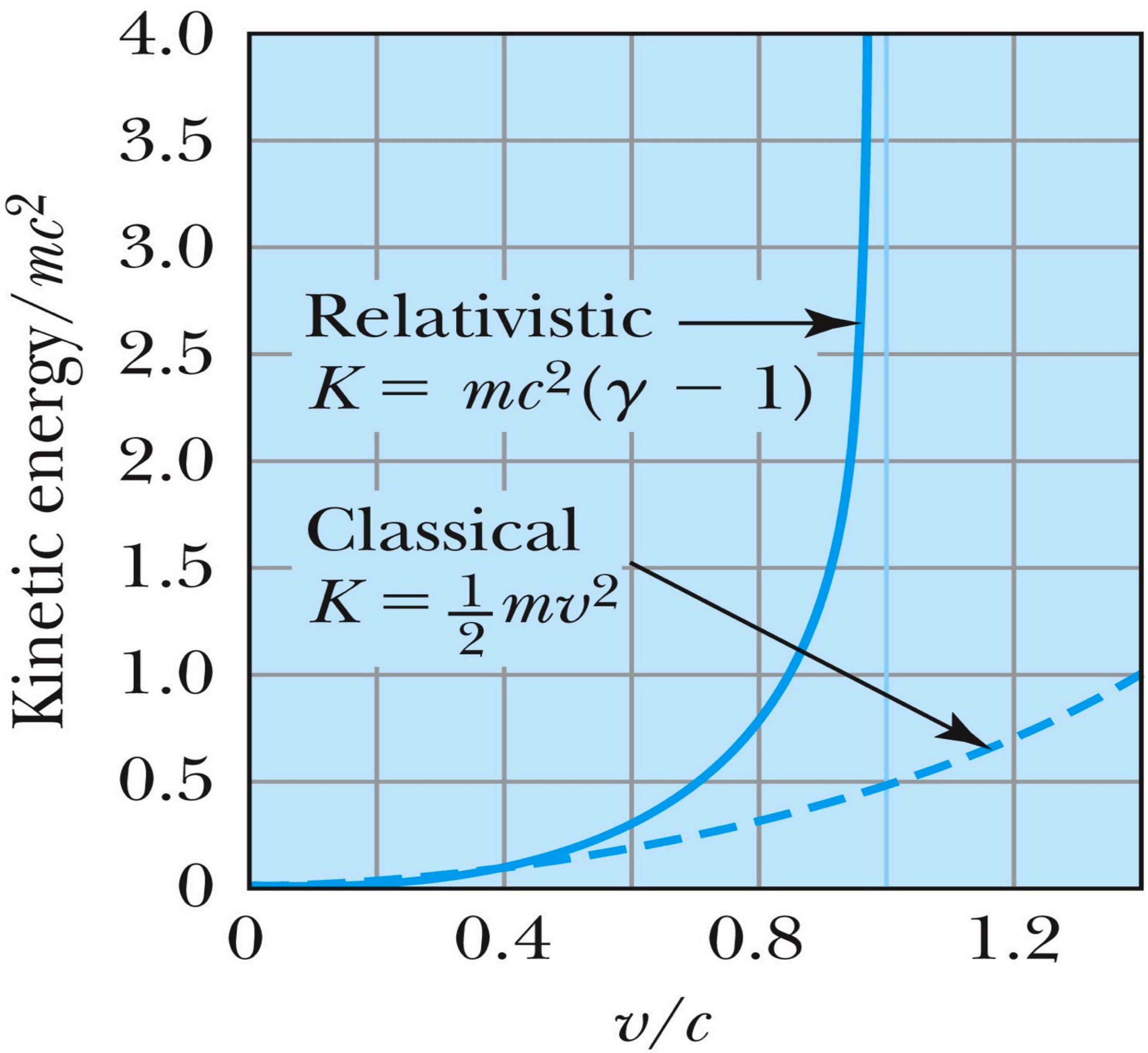
The term $mc^2$ is called the rest energy and is denoted by $E_0$.

$$E_0 = mc^2$$

The sum of the kinetic energy and rest energy is interpreted as the total energy of the particle. (note that u is the speed of the particle)

$$E_{Tot} = \gamma mc^2 = \frac{mc^2}{\sqrt{1-u^2/c^2}} = \frac{E_0}{\sqrt{1-u^2/c^2}} = K + E_0$$

# Relativistic and Classical Kinetic Energies



# Relationship of Energy and Momentum

$$p = \gamma m u = \frac{mu}{\sqrt{1 - u^2/c^2}}$$

We square this formula, multiply by $c^2$, and rearrange the terms.

$$p^2c^2 = \gamma^2 m^2 u^2 c^2 = \gamma^2 m^2 c^4 \left(\frac{u^2}{c^2}\right) = \gamma^2 m^2 c^4 \beta^2$$

$$\beta^2 = 1 - \frac{1}{\gamma^2} \Rightarrow p^2c^2 = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right) = \gamma^2 m^2 c^4 - m^2 c^4$$

Rewrite

$$p^2c^2 = E^2 - E_0^2$$

Rewrite

$$E^2 = p^2c^2 + E_0^2 = p^2c^2 + m^2c^4$$

# Massless Particles have a speed equal to the speed of light c

- Recall that a photon has "zero" rest mass and the equation from the last slide reduces to: E = pc and we may conclude that:

$$E = \gamma mc^2 = pc = \gamma muc$$

- Thus the speed, u, of a massless particle must be c since, as $m \to 0$, $\gamma \to \infty$ and it follows that: u = c.

# Units of Work, Energy and Mass

- The work done in accelerating a charge through a potential difference V is $W = qV$.
  - For a proton, with the charge $e = 1.602 \times 10^{-19}$ C being accelerated across a potential difference of 1 V, the work done is

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$$

$$W = (1.602 \times 10^{-19})(1 \text{ V}) = 1.602 \times 10^{-19} \text{ J}$$

- eV is also used as a unit of energy.

# Other Units

1) Rest energy of a particle:
   Example: Rest energy, $E_0$, of proton

$$E_0(proton) = m_p c^2 = (1.67\times10^{-27}\,kg)\cdot(3.00\times10^{8}\,m/s) = 1.50\times10^{-10}\,J$$

$$= 1.50\times10^{-10}\,J\cdot\frac{1eV}{1.602\times10^{-19}\,J} = 9.38\times10^{8}\,eV$$

**2) Atomic mass unit** (amu): Example: carbon-12

$$M\left({}^{12}C\ \text{atom}\right) = \frac{12\,g/mole}{6.02\times10^{23}\,atoms/mole}$$

$$= 1.99\times10^{-23}\,g/atom$$

$$M\left({}^{12}C\ \text{atom}\right) = 1.99\times10^{-26}\,kg/atom = 12\,u/atom$$

What is 1u in eV?

# Binding Energy

- The potential energy associated with the force keeping a system together → $E_B$.
- The difference between the rest energy of the individual particles and the rest energy of the combined bound system.

$$M_{\text{bound system}}c^2 + E_B = \sum_i m_i c^2$$

$$E_B = \sum_i m_i c^2 - M_{\text{bound system}}c^2$$

# Examples 2.13 and 2.15

- Ex. 2.13: A proton with 2-GeV kinetic energy hits another proton with 2 GeV KE in a head on collision. (proton rest mass = $938MeV/c^2$)
  - Compute v, β, p, K and E for each of the initial protons
  - What happens to the kinetic energy?
- Ex. 2.15: What is the minimum kinetic energy the protons must have in the head-on collision in the reaction $p+p \rightarrow \pi^+ + d$, in order to produce the positively charged pion ($139.6MeV/c^2$ ) and a deuteron.($1875.6MeV/c^2$).