PHYS 3313 – Section 001
Lecture #19

Wednesday, April 3, 2019

Dr. Jaehoon Yu

- Momentum Operator
- Position and Energy Operators
- Infinite Square-well Potential
- Quantized Energy
- Finite Square-well Potential
- Penetration Depth
- 3D Infinite Potential

Announcements

- Bring out SP#5
- Homework #5
  - CH6 end of chapter problems: 34, 46 and 65
  - CH7 end of chapter problems: 7, 9, 17 and 29
  - Due Monday, Apr. 15
- Colloquium at 4pm today in SH101
  - Dr. Antonia Hubbard of Northwestern University

Physics Department
The University of Texas at Arlington
COLLOQUIUM

1st Flight Engineering Results and Future Dark Matter Searches with the Micro-X Sounding Rocket

Antonia Hubbard
Postdoctoral Fellow
Northwestern University

Wednesday April 3, 2019
4:00 p.m. Room 101 Science Hall

Abstract

The Micro-X Microcalorimeter X-Ray Imaging Rocket is a sounding rocket mission that launched on July 22, 2018. This was the first operation of high resolution Transition Edge Sensors in space, opening up sensitivity to new physics. Micro-X is designed to observe Supernova Remnants and BSM X-ray interactions, like those proposed from keV-scale sterile neutrino dark matter. I will present the engineering results of the first flight, with special emphasis on the successful performance of the cryostat and electronics within the challenging conditions of a sounding rocket flight. While a rocket pointing error led to minimal time on-target, the science instrument operated as expected, and data from this flight will be used to establish background flux limits and as calibration data in preparation for future flights. I will additionally present the future of the instrument for dark matter searches, and the unique sensitivity it holds in resolving the 3.5 keV line controversy.

Refreshments will be served at 3:30 p.m. in the Physics Lounge

Reminder: Special Project #6

- Derive the following using trigonometric identities

$$A\cos(k_1x-\omega_1t)+A\cos(k_2x-\omega_2t)=$$

$$2A\cos\left(\frac{k_1-k_2}{2}x-\frac{\omega_1-\omega_2}{2}t\right)\cos\left(\frac{k_1+k_2}{2}x-\frac{\omega_1+\omega_2}{2}t\right)=$$

- 10 points total for this derivation
- Due for this special project is Monday, Apr. 8.
- You MUST have your own answers!

Special Project #7

- Show that the Schrodinger wave equation becomes Newton's second law in the classical limit. (15 points)
- Deadline Wednesday, Apr. 17
- You MUST have your own answers!

Momentum Operator

- To find the expectation value of *p*, we first need to represent *p* in terms of *x* and *t*. Consider the derivative of the wave function of a free particle with respect to *x*:

$$\frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial x}\left[e^{i(kx-\omega t)}\right] = ike^{i(kx-\omega t)} = ik\Psi$$

With $k = p/\hbar$ we have $\dfrac{\partial \Psi}{\partial x} = i\dfrac{p}{\hbar}\Psi$

This yields $$p\left[\Psi(x,t)\right] = -i\hbar\frac{\partial \Psi(x,t)}{\partial x}$$

- This suggests we define the momentum operator as $\hat{p} = -i\hbar\dfrac{\partial}{\partial x}$
- The expectation value of the momentum is

$$\langle p\rangle = \int_{-\infty}^{+\infty}\Psi^*(x,t)\hat{p}\Psi(x,t)\,dx = -i\hbar\int_{-\infty}^{+\infty}\Psi^*(x,t)\frac{\partial \Psi(x,t)}{\partial x}dx$$

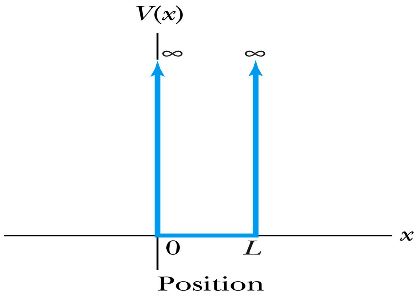
Position and Energy Operators

- The position *x* **is its own operator** as seen above.
- The time derivative of the free-particle wave function is
$$\frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial t}\left[e^{i(kx-\omega t)}\right] = -i\omega e^{i(kx-\omega t)} = -i\omega\Psi$$
Substituting $\omega = E/\hbar$ yields $E\left[\Psi(x,t)\right] = i\hbar\frac{\partial \Psi(x,t)}{\partial t}$
- So the energy operator is $\hat{E} = i\hbar\frac{\partial}{\partial t}$
- The expectation value of the energy is
$$\langle E \rangle = \int_{-\infty}^{+\infty} \Psi^*(x,t)\hat{E}\Psi(x,t)\,dx = i\hbar\int_{-\infty}^{+\infty} \Psi^*(x,t)\frac{\partial \Psi(x,t)}{\partial t}dx$$

Infinite Square-Well Potential

- The simplest such system is that of a particle trapped in a box with infinitely hard walls that the particle cannot penetrate. This potential is called an infinite square well potential and is given by

$$V(x) = \begin{cases} \infty & x \le 0, x \ge L \\ 0 & 0 < x < L \end{cases}$$



- The wave function must be zero where the potential is infinite.
- Where the potential is zero inside the box, the time independent Schrödinger wave equation $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$ becomes $\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi$ where $k = \sqrt{2mE/\hbar^2}$.
- The general solution is $\psi(x) = A\sin kx + B\cos kx$.

Quantization

- Since the wave function must be continuous, the boundary conditions of the potential dictate that the wave function must be zero at x = 0 and x = L. These yield B=0 for valid solutions for, and for **integer values** of n such that $kL = n\pi$ ➔ k=nπ/L
- The wave function is now $$\psi_n(x) = A\sin\left(\frac{n\pi x}{L}\right)$$
- We normalize the wave function $\int_{-\infty}^{+\infty}\psi_n^*(x)\psi_n(x)dx = 1$

$$A^2\int_{-\infty}^{+\infty}\sin^2\left(\frac{n\pi x}{L}\right)dx = A^2\int_0^L\sin^2\left(\frac{n\pi x}{L}\right)dx = 1$$

$$\text{Recall } \int\sin^2(ax)dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax)$$

Details of the computation

$$\frac{2}{L}\left\langle \sin^2\left(\frac{n\pi x}{L}\right)\right\rangle = \frac{2}{L}\cdot\frac{1}{2\pi}\int_0^{2\pi}\left(\sin^2 y\right)dy$$

$$\text{Let } y = \frac{n\pi x}{L}$$

$$= \frac{2}{L}\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\left(1-\cos 2y\right)dy$$

$$= \frac{2}{L}\cdot\frac{1}{2\pi}\cdot\frac{1}{2}\left[y-\frac{1}{2}\sin 2y\right]_0^{2\pi} = \frac{2}{L}\cdot\frac{1}{2\pi}\cdot\frac{1}{2}\left[2\pi-0-0\right]$$

$$= \frac{2}{L}\cdot\frac{1}{\cancel{2}\cancel{\pi}}\cdot\frac{1}{2}\cdot\cancel{2}\cancel{\pi} = \frac{\cancel{2}}{L}\cdot\frac{1}{\cancel{2}} = \frac{1}{L}$$

Quantization, cnt'd

- Since the wave function must be continuous, the boundary conditions of the potential dictate that the wave function must be zero at $x = 0$ and $x = L$. These yield B=0 for valid solutions for, and for **integer values** of n such that $kL = n\pi$ ➔ $k=n\pi/L$
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$$A^2\int_{-\infty}^{+\infty}\sin^2\left(\frac{n\pi x}{L}\right)dx = A^2\int_0^L\sin^2\left(\frac{n\pi x}{L}\right)dx = 1$$

- The normalized wave function becomes

Recall $\int \sin^2(ax)dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax)$

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

- These functions are identical to those obtained for a vibrating string with fixed ends (a standing wave!!)

Quantized Energy

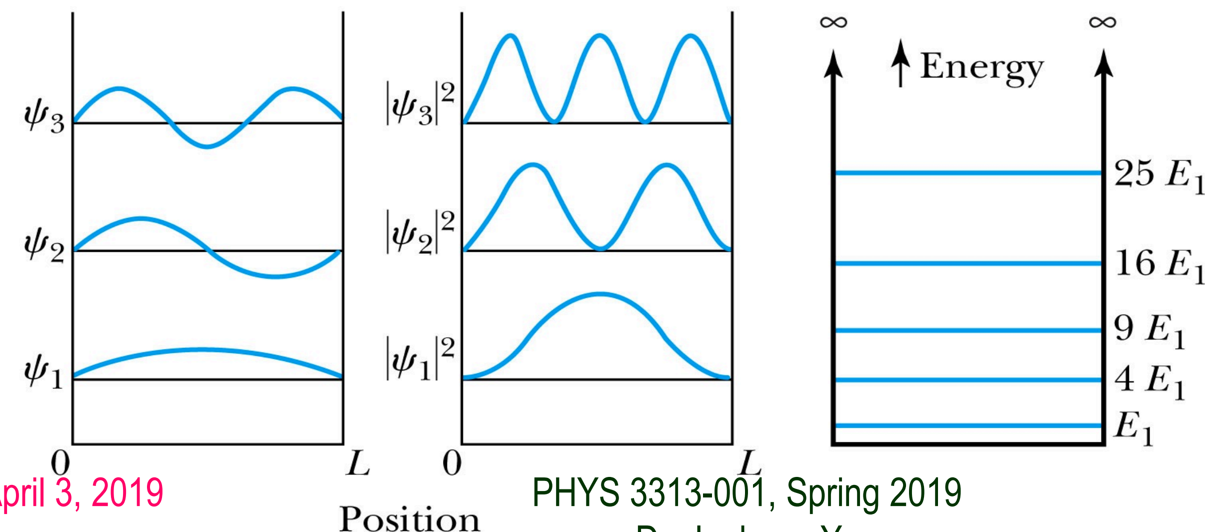
- The quantized wave number now becomes $k_n(x) = \frac{n\pi}{L} = \sqrt{\frac{2mE_n}{\hbar^2}}$
- Solving for the energy yields

$$E_n = n^2 \frac{\pi^2 \hbar^2}{2mL^2} \quad (n = 1, 2, 3, \cdots)$$

- Note that the energy depends on the integer values of *n*. Hence the energy is quantized and nonzero.
- The special case of *n* = 1 is called the **ground state energy**.

$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

$\psi_n^* \psi_n = |\psi_n|^2 = \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right)$



$$E_3 = \frac{9\pi^2\hbar^2}{2mL^2} = 9E_1$$

$$E_2 = \frac{2\pi^2\hbar^2}{mL^2} = 4E_1$$

$$E_1 = \frac{\pi^2\hbar^2}{2mL^2}$$

How does this correspond to Classical Mech.?

- What is the probability of finding a particle in a box of length L? $\frac{1}{L}$
- Bohr's **correspondence principle** says that QM and CM must correspond to each other! When?
  - When n becomes large, the QM approaches to CM
- So when n→∞, the probability of finding a particle in a box of length L is

$$P(x)=\psi_n^*(x)\psi_n(x)=|\psi_n(x)|^2=\frac{2}{L}\lim_{n\to\infty}\sin^2\left(\frac{n\pi x}{L}\right)\approx\frac{2}{L}\left\langle\sin^2\left(\frac{n\pi x}{L}\right)\right\rangle=\frac{2}{L}\cdot\frac{1}{2}=\frac{1}{L}$$

- Which is identical to the CM probability!!
- One can also see this from the plot of P!