PHYS 3313 – Section 001
Lecture #20

Monday, April 8, 2019
Dr. Jaehoon Yu

- Finite Square-well Potential
- Penetration Depth
- 3D Infinite Potential
- Simple Harmonic Oscillator
- Barriers and Tunneling
- Alpha Particle Decay

Announcements

- Bring out SP#6
- Reminder: Homework #5
  - CH6 end of chapter problems: 34, 46 and 65
  - CH7 end of chapter problems: 7, 9, 17 and 29
  - Due Monday, Apr. 15
- There will be a quiz coming Monday, Apr. 15
  - Beginning of the class
  - Covers: CH 6.2 to what we finish this Wednesday
  - BOYF: You may bring a one 8.5x11.5 sheet (front and back) of handwritten formulae and values of constants for the exam
  - No derivations, word definitions or setups or solutions of any problems!
    - No Maxwell's equations!
  - No additional formulae or values of constants will be provided!

Reminder: Special Project #7

- Show that the Schrodinger wave equation becomes Newton's second law in the classical limit. (15 points)
- Deadline Wednesday, Apr. 17
- You MUST have your own answers!

Research Projects

Each of the 9 research groups has an assigned research topic and the date and sequence of the presentations

Group Number	Research Topic	Reseasrch Group Members	Presentation Date and Order
1	Black-body Radiation	Z. Burns, N. Chapagain, A. Contreras, T. Doe, C. Nelson, B. Schuyler	#3, Wed. April 24, 2019
2	Michelson-Morley Experiment	S. Boucher, J. Breen, R. Contreras, A. Richey, G. Sabine, B. Taylor	#5, Wed. April 24, 2019
3	The Photo-Electric Effect	I. De Anda, M. Hanna, O. Jagtap, M. Kamerer, C. Morales, T. Nguyen	#4, Mon. April 22, 2019
4	The Brownian Motion	I. Busch, M. Hail, T. Maxfield, J. Perez, D. Rademacher, P. Williams	#4, Wed. April 24, 2019
5	Compton Effect	A. Adebayo, E. Alasadi, A. Chaid, T. Freeman, K. Karki, C. Newhouse	#2, Mon. April 22, 2019
6	Discovery of Electron	C. Garces, E. Glazier, R. Guerra, C. Leferink, E. Ralston, J. Scantlin	#2, Wed. April 24, 2019
7	Rutherford Scattering	M. Bui, A. Cole, J. Curtis, C. Kizer Pugh, A. Losh, I. Tucker	#1, Mon. April 22, 2019
8	Super-Conductivity	M. Aquino, J. Bradford, S. Graf, S. Kapoor, M. Liu, M. Smith	#1, Wed. April 24, 2019
9	The Discovery of Radioactivity	Y. Aryal, B. Garza, G. Hodges, C. Orr, S. Simmonds, R. Wood	#3, Mon. April 22, 2019

Research Presentations

- Each of the 9 research groups makes a 10+2min presentation
  - 10min presentation + 2min Q&A
  - All presentations must be in power point
  - I must receive all final presentation files **by 8pm, Sunday, Apr. 21, 2019**
    - No changes are allowed afterward
  - The representative of the group makes the presentation followed by all group members' participation in the Q&A session
- Date and time:
  - In class Monday and Wednesday, Apr. 22 and 24, 2019
- Important metrics
  - Contents of the presentation: 60%
    - Inclusion of all important points as mentioned in the report
    - The quality of the research and making the right points
  - Quality of the presentation itself: 15%
  - Presentation manner: 10%
  - Q&A handling: 10%
  - Staying in the allotted presentation time: 5%
  - Judging participation and sincerity: 5%

Expectation Values & Operators

- Expectation value for any function g(x)

$$\langle g(x) \rangle = \int_{-\infty}^{+\infty} \Psi(x,t)^* g(x) \Psi(x,t) dx$$

- Position operator is the same as itself, x
- Momentum Operator

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

- Energy Operator

$$\hat{E} = i\hbar \frac{\partial}{\partial t}$$

Ex 6.8: Expectation values inside a box

Determine the expectation values for x, x^2, p and p^2 of a particle in an infinite square well of width L for the first excited state.

What is the wave function of the first excited state? n=? 2

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \qquad \psi_{n=2}(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$$

See Appendix 3

$$\langle x\rangle_{n=2} = \int_{-\infty}^{+\infty}\psi_{n=2}^*(x)\,x\,\psi_{n=2}(x) = \frac{2}{L}\int_0^L x\sin^2\left(\frac{2\pi x}{L}\right)dx = \frac{L}{2}$$

$$\langle x^2\rangle_{n=2} = \frac{2}{L}\int_0^L x^2\sin^2\left(\frac{2\pi x}{L}\right)dx = 0.32L^2$$

$$\langle p\rangle_{n=2} = \frac{2}{L}\int_0^L \sin\left(\frac{2\pi x}{L}\right)(-i\hbar)\frac{\partial}{\partial x}\left[\sin\left(\frac{n\pi x}{L}\right)\right]dx = -i\hbar\frac{2}{L}\frac{2\pi}{L}\int_0^L \sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right)dx = 0$$

$$\langle p^2\rangle_{n=2} = \frac{2}{L}\int_0^L \sin\left(\frac{2\pi x}{L}\right)(-i\hbar)^2\frac{\partial^2}{\partial x^2}\left[\sin\left(\frac{2\pi x}{L}\right)\right]dx = \hbar^2\frac{2}{L}\left(\frac{2\pi}{L}\right)^2\int_0^L \sin^2\left(\frac{2\pi x}{L}\right)dx = \frac{4\pi^2\hbar^2}{L^2}$$

$$E_2 = \frac{4\pi^2\hbar^2}{2mL^2} = \frac{\langle p^2\rangle_{n=2}}{2m}$$

Ex 6.9: Proton Transition Energy

A typical diameter of a nucleus is about 10^{-14}m. Use the infinite square-well potential to calculate the transition energy from the first excited state to the ground state for a proton confined to the nucleus.

The energy of the state n is $E_n = n^2 \frac{\pi^2 \hbar^2}{2mL^2}$

What is n for the ground state? n=1

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2} = \frac{\pi^2 \hbar^2 c^2}{2mc^2 L^2} = \frac{1}{mc^2} \frac{\pi^2 \cdot (197.3eV \cdot nm)^2}{2 \cdot (10^{-5} nm)} = \frac{1.92 \times 10^{15} eV^2}{938.3 \times 10^6 eV} = 2.0 MeV$$

What is n for the 1st excited state? n=2

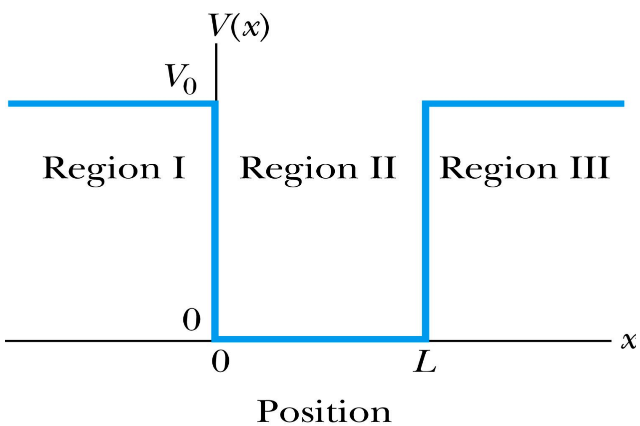
$$E_2 = 2^2 \frac{\pi^2 \hbar^2}{2mL^2} = 8.0 MeV$$

So the proton transition energy is

$$\Delta E = E_2 - E_1 = 6.0 MeV$$

Finite Square-Well Potential

- The finite square-well potential is $V(x) = \begin{cases} V_0 & x \le 0, \\ 0 & 0 < x < L \\ V_0 & x \ge L \end{cases}$
- The Schrödinger equation outside the finite well in regions I and III is $-\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} = (E - V_0)$ for regions I and III, or using $\alpha^2 = 2m(V_0 - E)/\hbar^2$ yields $\frac{d^2\psi}{dx^2} = \alpha^2\psi$. The solution to this differential eq. has exponentials of the form $e^{\alpha x}$ and $e^{-\alpha x}$. In the region x > L, we reject the positive exponential and in the region x < 0, we reject the negative exponential. Why?



$$\psi_I(x) = Ae^{\alpha x} \quad \text{region I}, x < 0$$

$$\psi_{III}(x) = Ae^{-\alpha x} \quad \text{region III}, x > L$$

This is because the wave function should be 0 as x→±infinity.

Finite Square-Well Solution

- Inside the square well, where the potential *V* is zero and the particle is free, the wave equation becomes $\frac{d^2\psi}{dx^2} = -k^2\psi$ where $k = \sqrt{2mE/\hbar^2}$
- Instead of a sinusoidal solution we can write

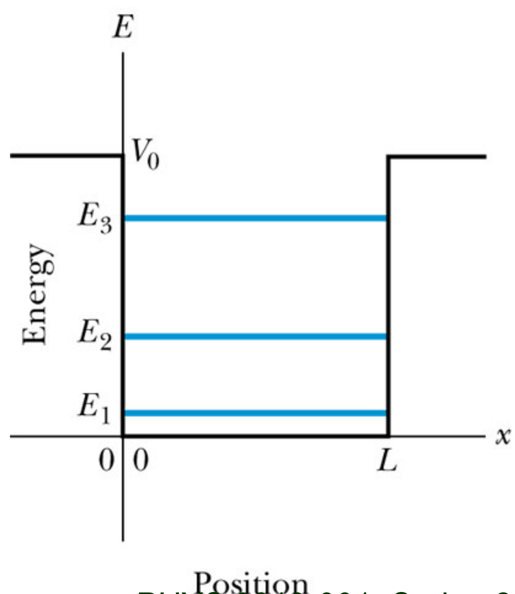
$$\psi_{II}(x) = Ce^{ikx} + De^{-ikx} \quad \text{region II, } 0 < x < L$$

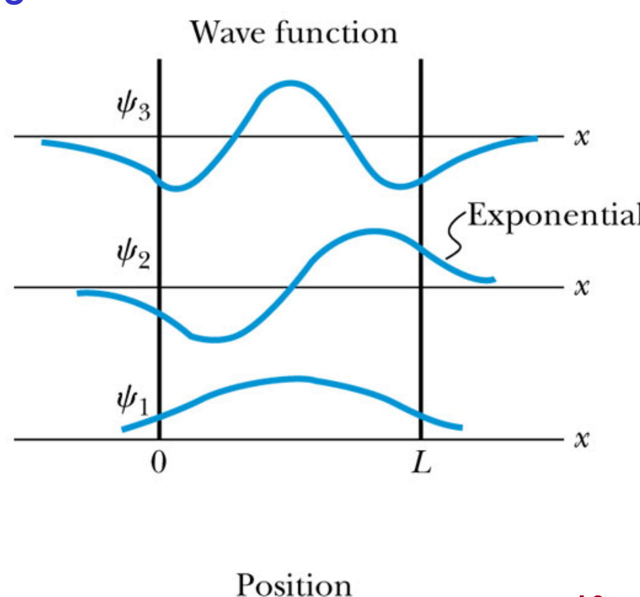
- The boundary conditions require that

$$\psi_I = \psi_{II} \text{ at } x = 0 \text{ and } \psi_{II} = \psi_{III} \text{ at } x = L$$

and the wave function must be smooth where the regions meet.

- Note that the wave function is nonzero outside of the box.
- Non-zero at the boundary either..
- What would the energy look like?





Penetration Depth

- The penetration depth is the distance outside the potential well where the probability significantly decreases. It is given by

$$\delta x \approx \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m\left(V_0 - E\right)}}$$

- It should not be surprising to find that the penetration distance that violates classical physics is proportional to Planck's constant.

Three-Dimensional Infinite-Potential Well

- The wave function must be a function of all three spatial coordinates.
- We begin with the conservation of energy $E = K + V = \frac{p^2}{2m} + V$
- Multiply this by the wave function to get

$$E\psi = \left(\frac{p^2}{2m} + V\right)\psi = \frac{p^2}{2m}\psi + V\psi$$

- Now consider momentum as an operator acting on the wave function. In this case, the operator must act twice on each dimension. Given:

$$p^2 = p_x^2 + p_y^2 + p_z^2 \qquad \hat{p}_x\psi = -i\hbar\frac{\partial\psi}{\partial x} \quad \hat{p}_y\psi = -i\hbar\frac{\partial\psi}{\partial y} \quad \hat{p}_z\psi = -i\hbar\frac{\partial\psi}{\partial z}$$

- The three dimensional Schrödinger wave equation is

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + V\psi = E\psi$$

Rewrite ⇒

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

Ex 6.10: Expectation values inside a box

Consider a free particle inside a box with lengths L_1, L_2 and L_3 along the x, y, and z axes, respectively, as shown in the figure. The particle is constrained to be inside the box. Find the wave functions and energies. Then find the ground energy and wave function and the energy of the first excited state for a cube of sides L.

What are the boundary conditions for this situation?

Particle is free, so x, y and z wave functions are independent from each other!

Each wave function must be 0 at the wall! Inside the box, potential V is 0.

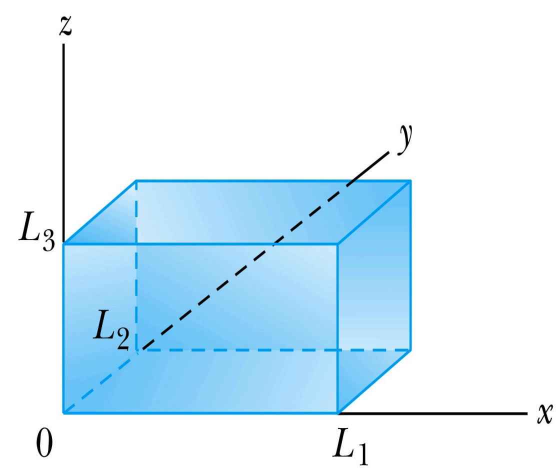
$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi \Rightarrow -\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$

A reasonable solution is

$$\psi(x,y,z) = A\sin(k_1 x)\sin(k_2 y)\sin(k_3 z)$$

Using the boundary conditions

$$\psi = 0 \text{ at } x = L_1 \Rightarrow k_1 L_1 = n_1\pi \Rightarrow k_1 = n_1\pi / L_1$$



So the wave numbers are $k_1 = \frac{n_1\pi}{L_1}$ $k_2 = \frac{n_2\pi}{L_2}$ $k_3 = \frac{n_3\pi}{L_3}$

Ex 6.10: Expectation values inside a box

Consider a free particle inside a box with lengths L_1, L_2 and L_3 along the x, y, and z axes, respectively, as shown in figure. The particle is constrained to be inside the box. Find the wave functions and energies. Then find the ground energy and wave function and the energy of the first excited state for a cube of sides L.

The energy can be obtained through the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi = E\psi$$

$$\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}\left(A\sin(k_1x)\sin(k_2y)\sin(k_3z)\right) = k_1A\cos(k_1x)\sin(k_2y)\sin(k_3z)$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(A\sin(k_1x)\sin(k_2y)\sin(k_3z)\right) = -k_1^2A\sin(k_1x)\sin(k_2y)\sin(k_3z) = -k_1^2\psi$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi = \frac{\hbar^2}{2m}\left(k_1^2+k_2^2+k_3^2\right)\psi = E\psi$$

$$E = \frac{\hbar^2}{2m}\left(k_1^2+k_2^2+k_3^2\right) = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_1^2}{L_1^2}+\frac{n_2^2}{L_2^2}+\frac{n_3^2}{L_3^2}\right)$$

What is the ground state energy?

$E_{1,1,1}$ when $n_1=n_2=n_3=1$, how much?

When are the energies the same for different combinations of n_i?

Degeneracy*

- Analysis of the Schrödinger wave equation in three dimensions introduces three quantum numbers that quantize the energy.
- A quantum state is degenerate when there is more than one wave function for a given energy.
- Degeneracy results from particular properties of the potential energy function that describes the system. A perturbation of the potential energy, such as the spin under a B field, can remove the degeneracy.

***Mirriam-webster: having two or more states or subdivisions**