PHYS 1443 – Section 001
Lecture #7

Thursday, June 8, 2006
Dr. Jaehoon Yu

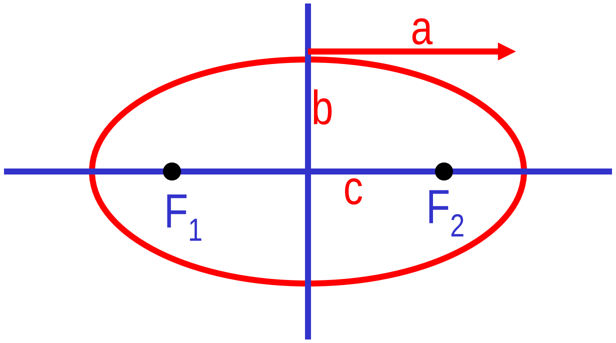
- Kepler's Laws
- Motion in Accelerated Frames
- Work done by a constant force
- Scalar Product of Vectors
- Work done by a varying force
- Work and Kinetic Energy Theorem
- Potential Energy

Today's homework is HW #4, due 7pm, Monday, June 12!!

Announcements

- Mid-term exam
  - 8:00 – 10am, Thursday, June 15, in class
  - CH 1 – 8 or 9?

Kepler's Laws & Ellipse



Ellipses have two different axis, major (long) and minor (short) axis, and two focal points, F_1 & F_2

a is the length of a semi-major axis

b is the length of a semi-minor axis

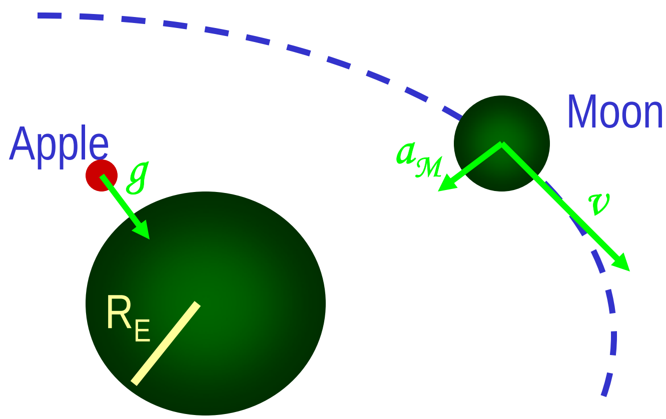
Kepler lived in Germany and discovered the law's governing planets' movement some 70 years before Newton, by analyzing data.

1. All planets move in elliptical orbits with the Sun at one focal point.
2. The radius vector drawn from the Sun to a planet sweeps out equal area in equal time intervals. (*Angular momentum conservation*)
3. The square of the orbital period of any planet is proportional to the cube of the semi-major axis of the elliptical orbit.

Newton's laws explain the cause of the above laws. Kepler's third law is a direct consequence of law of gravitation being inverse square law.

The Law of Gravity and Motions of Planets

- Newton assumed that the law of gravitation applies the same whether it is the apple on the surface of the Moon or of the Earth.
- The interacting bodies are assumed to be point like particles.



Newton predicted that the ratio of the Moon's acceleration a_M to the apple's acceleration g would be

$$\frac{a_M}{g} = \frac{(1/r_M)^2}{(1/R_E)^2} = \left(\frac{R_E}{r_M}\right)^2 = \left(\frac{6.37\times10^6}{3.84\times10^8}\right)^2 = 2.75\times10^{-4}$$

Therefore the centripetal acceleration of the Moon, a_M, is

$$a_M = 2.75\times10^{-4}\times9.80 = 2.70\times10^{-3}\,m/s^2$$

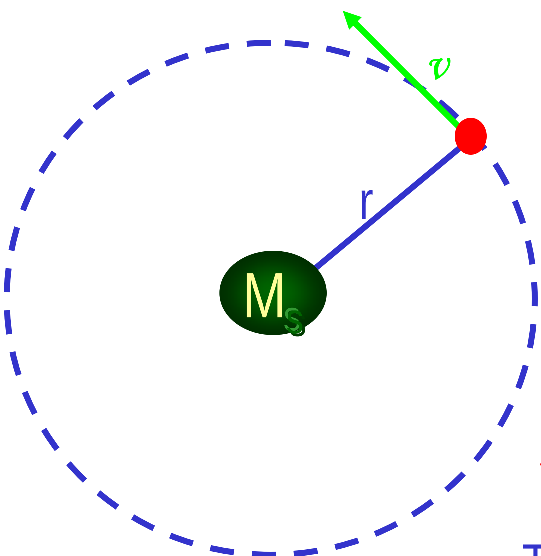
Newton also calculated the Moon's orbital acceleration a_M from the knowledge of its distance from the Earth and its orbital period, T=27.32 days=2.36x10^{6}s

$$a_M = \frac{v^2}{r_M} = \frac{(2\pi r_M/T)^2}{r_M} = \frac{4\pi^2 r_M}{T^2} = \frac{4\pi^2\times3.84\times10^8}{(2.36\times10^6)^2} = 2.72\times10^{-3}\,m/s^2 \approx \frac{9.80}{(60)^2}$$

This means that the distance to the Moon is about 60 times that of the Earth's radius, and its acceleration is reduced by the square of the ratio. This proves that the inverse square law is valid.

Kepler's Third Law

It is crucial to show that Keper's third law can be predicted from the inverse square law for circular orbits.



Since the gravitational force exerted by the Sun is radially directed toward the Sun to keep the planet on a near circular path, we can apply Newton's second law

$$\frac{GM_sM_P}{r^2} = \frac{M_p v^2}{r}$$

Since the orbital speed, v, of the planet with period T is $v = \frac{2\pi r}{T}$

The above can be written $\frac{GM_sM_P}{r^2} = \frac{M_p (2\pi r/T)^2}{r}$

Solving for T one can obtain $T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3 = K_s r^3$ and $K_s = \left(\frac{4\pi^2}{GM_s}\right) = 2.97 \times 10^{-19}\ s^2/m^3$

This is Kepler's third law. It's also valid for the ellipse for r being the length of the semi-major axis. The constant K_s is independent of mass of the planet.

Example of Kepler's Third Law

Calculate the mass of the Sun using the fact that the period of the Earth's orbit around the Sun is 3.16×10^{7}s, and its distance from the Sun is 1.496×10^{11}m.

Using Kepler's third law.

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3 = K_s r^3$$

The mass of the Sun, M_s, is

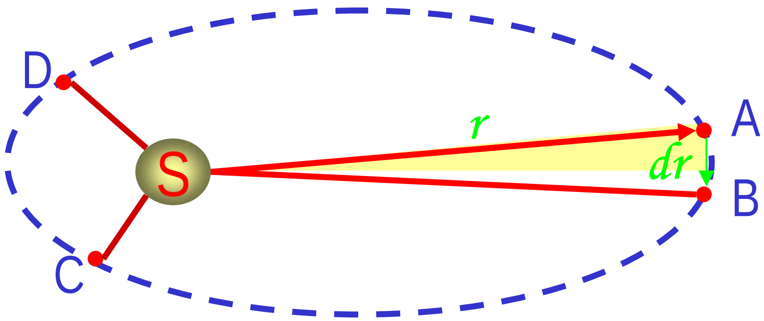
$$M_s = \left(\frac{4\pi^2}{GT^2}\right) r^3$$

$$= \left(\frac{4\pi^2}{6.67\times10^{-11}\times\left(3.16\times10^{7}\right)^2}\right)\times\left(1.496\times10^{11}\right)^3$$

$$= 1.99\times10^{30}\,kg$$

Kepler's Second Law and Angular Momentum Conservation

Consider a planet of mass M_p moving around the Sun in an elliptical orbit.



Since the gravitational force acting on the planet is always toward radial direction, it is a *central force*

Therefore the torque acting on the planet by this force is always 0.

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times F\hat{r} = 0$$

Since torque is the time rate change of angular momentum *L*, the angular momentum is constant.

$$\vec{\tau} = \frac{d\vec{L}}{dt} = 0 \qquad \vec{L} = const$$

Because the gravitational force exerted on a planet by the Sun results in no torque, the angular momentum L of the planet is constant.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times M_p\vec{v} = M_p\vec{r} \times \vec{v} = const$$

Since the area swept by the motion of the planet is

$$dA = \frac{1}{2}\left|\vec{r} \times d\vec{r}\right| = \frac{1}{2}\left|\vec{r} \times \vec{v}dt\right| = \frac{L}{2M_p}dt \Rightarrow \frac{dA}{dt} = \frac{L}{2M_p} = const$$

This is Keper's second law which states that the radius vector from the Sun to a planet sweeps out equal areas in equal time intervals.

Motion in Accelerated Frames

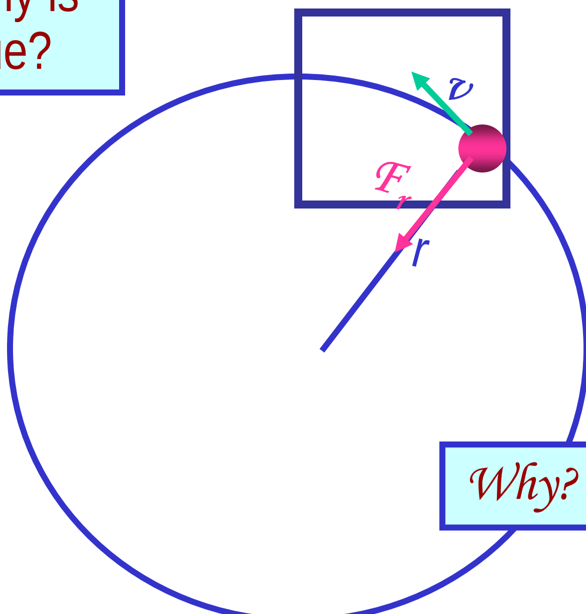
Newton's laws are valid only when observations are made in an inertial frame of reference. What happens in a non-inertial frame?

Fictitious forces are needed to apply Newton's second law in an accelerated frame.

This force does not exist when the observations are made in an inertial reference frame.

What does this mean and why is this true?

Let's consider a free ball inside a box under uniform circular motion.



How does this motion look like in an inertial frame (or frame outside a box)?

We see that the box has a radial force exerted on it but none on the ball directly

How does this motion look like in the box?

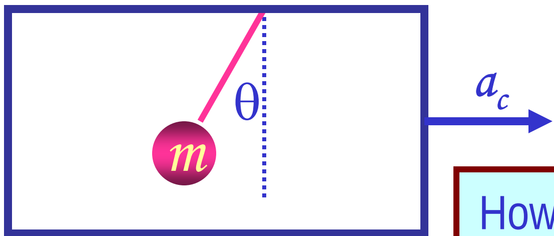
The ball is tumbled over to the wall of the box and feels that it is getting force that pushes it toward the wall.

Why?

According to Newton's first law, the ball wants to continue on its original movement but since the box is turning, the ball feels like it is being pushed toward the wall relative to everything else in the box.

Example of Motion in Accelerated Frames

A ball of mass m is hung by a cord to the ceiling of a boxcar that is moving with an acceleration a. What do the inertial observer at rest and the non-inertial observer traveling inside the car conclude? How do they differ?

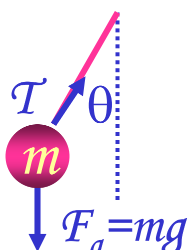


This is how the ball looks like no matter which frame you are in.

How do the free-body diagrams look for two frames?

How do the motions interpreted in these two frames? Any differences?

Inertial Frame



$$\sum \vec{F} = \vec{F}_g + \vec{T}$$

$$\sum F_x = ma_x = ma_c = T\sin\theta$$

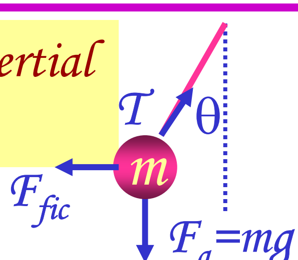
$$\sum F_y = T\cos\theta - mg = 0$$

$$T = \frac{mg}{\cos\theta}$$

$$a_c = g\tan\theta$$

For an inertial frame observer, the forces being exerted on the ball are only T and F_g. The acceleration of the ball is the same as that of the box car and is provided by the x component of the tension force.

Non-Inertial Frame



$$\sum \vec{F} = \vec{F}_g + \vec{T} + \vec{F}_{fic}$$

$$\sum F_x = T\sin\theta - F_{fic} = 0$$

$$F_{fic} = ma_{fic} = T\sin\theta$$

$$\sum F_y = T\cos\theta - mg = 0$$

$$T = \frac{mg}{\cos\theta}$$

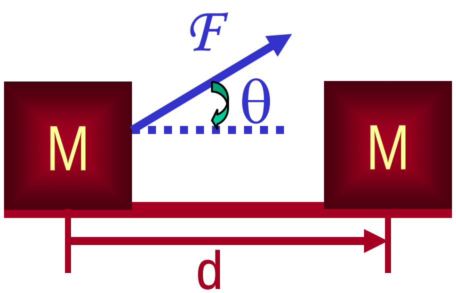
$$a_{fic} = g\tan\theta$$

In the non-inertial frame observer, the forces being exerted on the ball are T, F_g, and F_{fic}. For some reason the ball is under a force, F_{fic}, that provides acceleration to the ball.

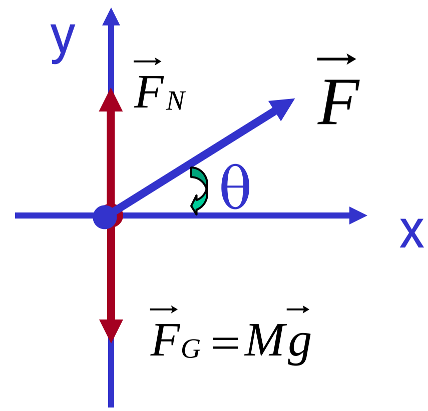
While the mathematical expression of the acceleration of the ball is identical to that of inertial frame observer's, the cause of the force is dramatically different.

Work Done by a Constant Force

Work in physics is done only when a sum of forces exerted on an object made a motion to the object.



Free Body Diagram



Which force did the work? Force $\vec{F}$

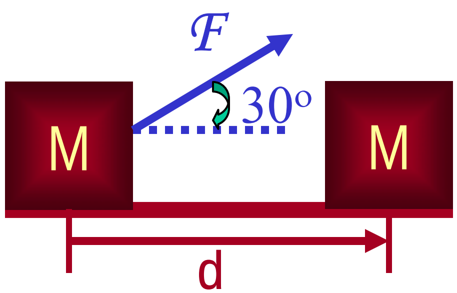
How much work did it do? $W = \left(\sum \vec{F}\right) \cdot \vec{d} = Fd\cos\theta$ Unit? $N \bullet m = J$ (for Joule)

What does this mean? Physical work is done only by the component of the force along the movement of the object.

Work is an <u>energy transfer</u>!!

Example of Work w/ Constant Force

A man cleaning a floor pulls a vacuum cleaner with a force of magnitude F=50.0N at an angle of 30.0° with East. Calculate the work done by the force on the vacuum cleaner as the vacuum cleaner is displaced by 3.00m to East.



$$W = \left(\sum \vec{F}\right)\cdot\vec{d} = \left|\left(\sum \vec{F}\right)\right|\left|\vec{d}\right|\cos\theta$$

$$W = 50.0\times 3.00\times\cos 30^{\circ} = 130J$$

Does work depend on mass of the object being worked on?

Yes

Why don't I see the mass term in the work at all then?

It is reflected in the force. If the object has smaller mass, its would take less force to move it the same distance as the heavier object.   So it would take less work. Which makes perfect sense, doesn't it?

Scalar Product of Two Vectors

- Product of magnitude of the two vectors and the cosine of the angle between them $\vec{A}\bullet\vec{B} \equiv \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$
- Operation is commutative $\vec{A}\bullet\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta = \left|\vec{B}\right|\left|\vec{A}\right|\cos\theta = \vec{B}\bullet\vec{A}$
- Operation follows distribution law of multiplication $\vec{A}\bullet\left(\vec{B}+\vec{C}\right) = \vec{A}\bullet\vec{B}+\vec{A}\bullet\vec{C}$
- Scalar products of Unit Vectors $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \quad \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
- How does scalar product look in terms of components?

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \qquad \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} \qquad \vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$$

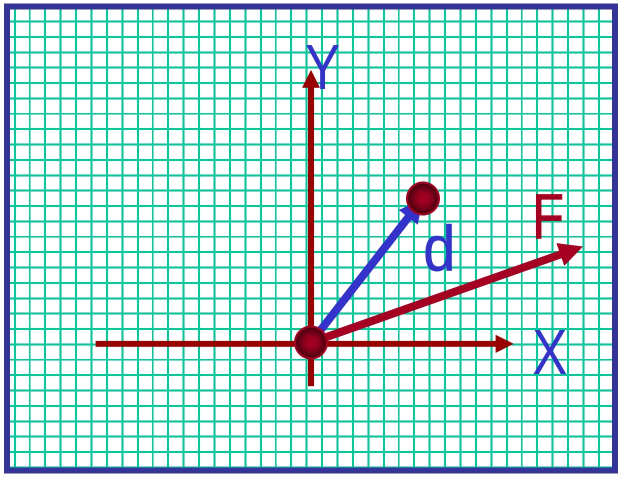
$$\vec{A}\bullet\vec{B} = \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)\bullet\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right) = \left(A_xB_x\,\hat{i}\bullet\hat{i} + A_yB_y\,\hat{j}\bullet\hat{j} + A_zB_z\,\hat{k}\bullet\hat{k}\right) + cross\ terms$$

cross terms =0

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Example of Work by Scalar Product

A particle moving in the xy plane undergoes a displacement **d**=(2.0**i**+3.0**j**)m as a constant force **F**=(5.0**i**+2.0**j**) N acts on the particle.



a) Calculate the magnitude of the displacement and that of the force.

$$\left|\vec{d}\right| = \sqrt{d_x^2 + d_y^2} = \sqrt{(2.0)^2 + (3.0)^2} = 3.6m$$

$$\left|\vec{F}\right| = \sqrt{F_x^2 + F_y^2} = \sqrt{(5.0)^2 + (2.0)^2} = 5.4N$$

b) Calculate the work done by the force F.

$$W = \vec{F} \cdot \vec{d} = \left(2.0\hat{i} + 3.0\hat{j}\right) \cdot \left(5.0\hat{i} + 2.0\hat{j}\right) = 2.0 \times 5.0\,\hat{i} \cdot \hat{i} + 3.0 \times 2.0\,\hat{j} \cdot \hat{j} = 10 + 6 = 16(J)$$

Can you do this using the magnitudes and the angle between **d** and **F**?

$$W = \vec{F} \cdot \vec{d} = \left|\vec{F}\right|\left|\vec{d}\right|\cos\theta$$