PHYS 1443 – Section 001
Lecture #13

Thursday, June 22, 2006
Dr. Jaehoon Yu

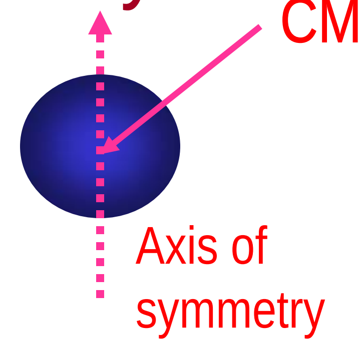
- CM and the Center of Gravity
- Fundamentals on Rotational Motion
- Rotational Kinematics
- Relationship between angular and linear quantities
- Rolling Motion of a Rigid Body
- Torque
- Torque and Vector Product
- Moment of Inertia

Announcements

- Reading assignments
  - CH. 11.6, 11.8, 11.9 and 11.10
- Last quiz next Wednesday
  - Early in the class
  - Covers Ch. 10 – what we cover next Tuesday

Center of Mass and Center of Gravity

The center of mass of any symmetric object lies on an axis of symmetry and on any plane of symmetry, if object's mass is evenly distributed throughout the body.

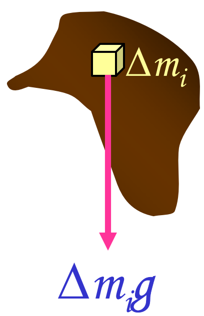


How do you think you can determine the CM of objects that are not symmetric?

One can use gravity to locate CM.

1. Hang the object by one point and draw a vertical line following a plum-bob.
2. Hang the object by another point and do the same.
3. The point where the two lines meet is the CM.

Center of Gravity



Since a rigid object can be considered as a collection of small masses, one can see the total gravitational force exerted on the object as

$$\vec{F}_g = \sum_i \vec{F}_i = \sum_i \Delta m_i \vec{g} = M\vec{g}$$

What does this equation tell you?

The net effect of these small gravitational forces is equivalent to a single force acting on a point (Center of Gravity) with mass M.

The CoG is the point in an object as if all the gravitational force is acting on!

Motion of a Group of Particles

We've learned that the CM of a system can represent the motion of a system. Therefore, for an isolated system of many particles in which the total mass M is preserved, the velocity, total momentum, acceleration of the system are

Velocity of the system

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{d}{dt}\left(\frac{1}{M}\sum m_i\vec{r}_i\right) = \frac{1}{M}\sum m_i\frac{d\vec{r}_i}{dt} = \frac{\sum m_i\vec{v}_i}{M}$$

Total Momentum of the system

$$\vec{p}_{CM} = M\vec{v}_{CM} = M\frac{\sum m_i\vec{v}_i}{M} = \sum m_i\vec{v}_i = \sum\vec{p} = \vec{p}_{tot}$$

Acceleration of the system

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{d}{dt}\left(\frac{1}{M}\sum m_i\vec{v}_i\right) = \frac{1}{M}\sum m_i\frac{d\vec{v}_i}{dt} = \frac{\sum m_i\vec{a}_i}{M}$$

External force exerting on the system

$$\sum\vec{F}_{ext} = M\vec{a}_{CM} = \sum m_i\vec{a}_i = \frac{d\vec{p}_{tot}}{dt}$$

What about the internal forces?

If net external force is 0

$$\sum\vec{F}_{ext} = 0 = \frac{d\vec{p}_{tot}}{dt}$$

$$\vec{p}_{tot} = \text{const}$$

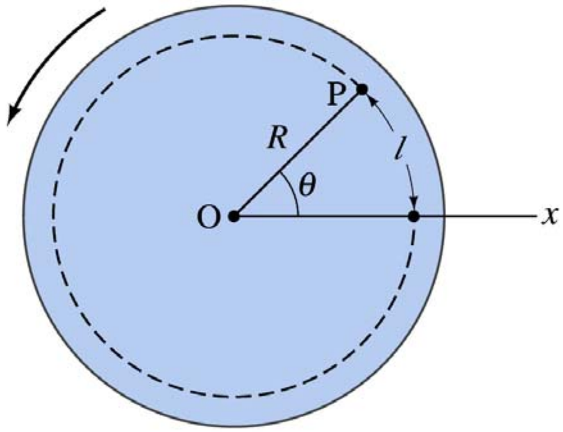
System's momentum is conserved.

Fundamentals on Rotation

Linear motions can be described as the motion of the center of mass with all the mass of the object concentrated on it.

Is this still true for rotational motions?

No, because different parts of the object have different linear velocities and accelerations.



Consider a motion of a rigid body – an object that does not change its shape – rotating about the axis protruding out of the slide.

The arc length is $l = R\theta$

Therefore the angle, θ, is $\theta = \frac{l}{R}$. *And the unit of the angle is in radian. It is dimensionless!!*

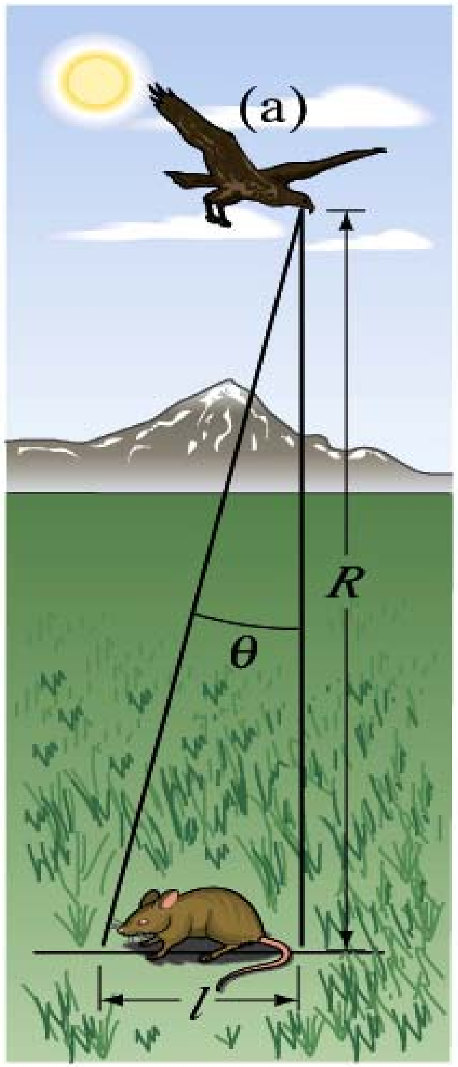
One radian is the angle swept by an arc length equal to the radius of the arc.

Since the circumference of a circle is 2πr, $360^\circ = 2\pi r / r = 2\pi$

The relationship between radian and degrees is $1\,\text{rad} = 360^\circ / 2\pi = 180^\circ / \pi \cong 180^\circ / 3.14 \cong 57.3^\circ$

Example 10 – 1

A particular bird's eyes can barely distinguish objects that subtend an angle no smaller than about 3×10^{-4} rad. (a) How many degrees is this? (b) How small an object can the bird just distinguish when flying at a height of 100m?



(a) One radian is $360^{o}/2\pi$. Thus

$$3\times10^{-4}\,rad = \left(3\times10^{-4}\,rad\right)\times\left(360^{\circ}/2\pi\,rad\right) = 0.017^{\circ}$$

(b) Since $l=r\theta$ and for small angle arc length is approximately the same as the chord length.

$$l = r\theta = 100m\times3\times10^{-4}\,rad = 3\times10^{-2}\,m = 3cm$$

Rotational Kinematics

The first type of motion we have learned in linear kinematics was under a constant acceleration. We will learn about the rotational motion under constant angular acceleration, because these are the simplest motions in both cases.

Just like the case in linear motion, one can obtain

Angular Speed under constant angular acceleration:

$$\omega_f = \omega_i + \alpha t$$

Angular displacement under constant angular acceleration:

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

One can also obtain

$$\omega_f^2 = \omega_i^2 + 2\alpha\left(\theta_f - \theta_i\right)$$

Angular Displacement, Velocity, and Acceleration

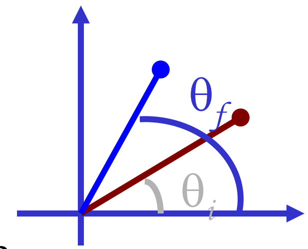
Using what we have learned in the previous slide, how would you define the angular displacement?

$$\Delta\theta = \theta_f - \theta_i$$

How about the average angular speed?

Unit? rad/s

$$\bar{\omega} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$



And the instantaneous angular speed?

Unit? rad/s

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

By the same token, the average angular acceleration

Unit? rad/s^2

$$\bar{\alpha} \equiv \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

And the instantaneous angular acceleration? Unit? rad/s^2

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

When rotating about a fixed axis, every particle on a rigid object rotates through the same angle and has the same angular speed and angular acceleration.

Example for Rotational Kinematics

A wheel rotates with a constant angular acceleration of 3.50 rad/s^2. If the angular speed of the wheel is 2.00 rad/s at t_i=0, a) through what angle does the wheel rotate in 2.00s?

Using the angular displacement formula in the previous slide, one gets

$$\theta_f - \theta_i = \omega t + \frac{1}{2}\alpha t^2$$

$$= 2.00 \times 2.00 + \frac{1}{2} 3.50 \times (2.00)^2 = 11.0 rad$$

$$= \frac{11.0}{2\pi} rev. = 1.75 rev.$$

Example for Rotational Kinematics cnt'd

What is the angular speed at t=2.00s?

Using the angular speed and acceleration relationship

$$\omega_f = \omega_i + \alpha t = 2.00 + 3.50 \times 2.00 = 9.00 rad/s$$

Find the angle through which the wheel rotates between t=2.00 s and t=3.00 s.

Using the angular kinematic formula

$$\theta_f - \theta_i = \omega t + \frac{1}{2}\alpha t^2$$

At t=2.00s

$$\theta_{t=2} = 2.00 \times 2.00 + \frac{1}{2} 3.50 \times 2.00 = 11.0 rad$$

At t=3.00s

$$\theta_{t=3} = 2.00 \times 3.00 + \frac{1}{2} 3.50 \times (3.00)^2 = 21.8 rad$$

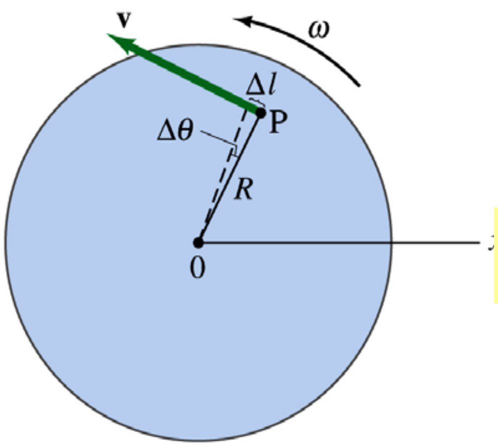
Angular displacement

$$\Delta\theta = \theta_3 - \theta_2 = 10.8 rad = \frac{10.8}{2\pi} rev. = 1.72 rev.$$

Relationship Between Angular and Linear Quantities

What do we know about a rigid object that rotates about a fixed axis of rotation?

Every particle (or masslet) in the object moves in a circle centered at the axis of rotation.



When a point rotates, it has both the linear and angular components in its motion.

What is the linear component of the motion you see?

Linear velocity along the tangential direction.

How do we related this linear component of the motion with angular component?

The direction of ω follows a right-hand rule.

The arc-length is $l = R\theta$ So the tangential speed v is $v = \frac{dl}{dt} = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} = r\omega$

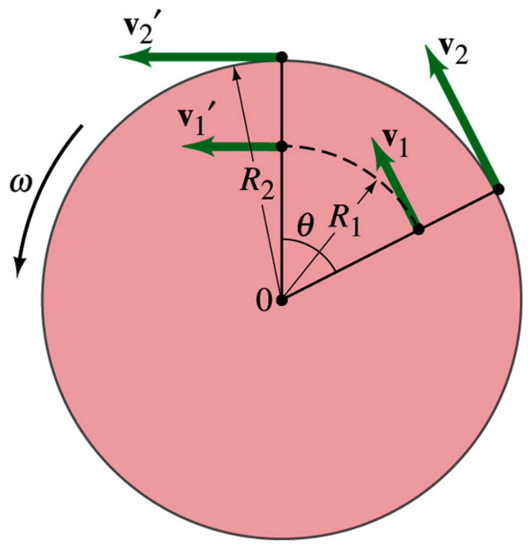
What does this relationship tell you about the tangential speed of the points in the object and their angular speed?:

Although every particle in the object has the same angular speed, its tangential speed differs proportional to its distance from the axis of rotation.

The farther away the particle is from the center of rotation, the higher the tangential speed.

Is the lion faster than the horse?

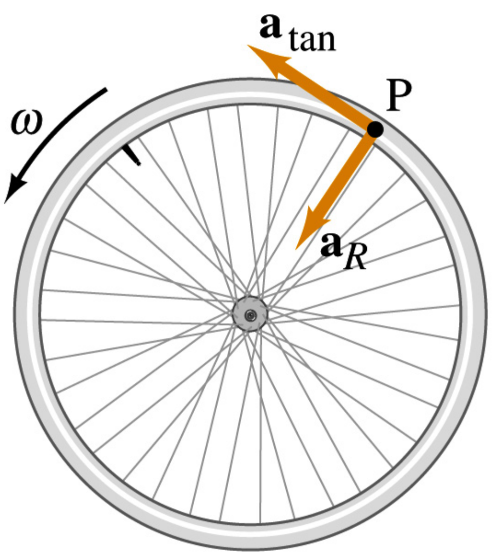
A rotating carousel has one child sitting on a horse near the outer edge and another child on a lion halfway out from the center. (a) Which child has the greater linear speed? (b) Which child has the greater angular speed?



(a) Linear speed is the distance traveled divided by the time interval. So the child sitting at the outer edge travels more distance within the given time than the child sitting closer to the center. Thus, the horse is faster than the lion.

(b) Angular speed is the angle traveled divided by the time interval. The angle both the children travel in the given time interval is the same. Thus, both the horse and the lion have the same angular speed.

How about the acceleration?



How many different linear accelerations do you see in a circular motion and what are they? Two

Tangential, a_t, and the radial acceleration, a_r.

Since the tangential speed v is $v = r\omega$

The magnitude of tangential acceleration a_t is $a_t = \frac{dv}{dt} = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} = r\alpha$

What does this relationship tell you?

Although every particle in the object has the same angular acceleration, its tangential acceleration differs proportional to its distance from the axis of rotation.

The radial or centripetal acceleration a_r is $a_r = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$

What does this tell you?

The father away the particle is from the rotation axis, the more radial acceleration it receives. In other words, it receives more centripetal force.

Total linear acceleration is $a = \sqrt{a_t^2 + a_r^2} = \sqrt{(r\alpha)^2 + (r\omega^2)^2} = r\sqrt{\alpha^2 + \omega^4}$

Example

(a) What is the linear speed of a child seated 1.2m from the center of a steadily rotating merry-go-around that makes one complete revolution in 4.0s? (b) What is her total linear acceleration?

First, figure out what the angular speed of the merry-go-around is.

$$\varpi = \frac{1rev}{4.0s} = \frac{2\pi}{4.0s} = 1.6 rad/s$$

Using the formula for linear speed

$$v = r\omega = 1.2m \times 1.6 rad/s = 1.9 m/s$$

Since the angular speed is constant, there is no angular acceleration.

Tangential acceleration is

$$a_t = r\alpha = 1.2m \times 0 rad/s^2 = 0 m/s^2$$

Radial acceleration is

$$a_r = r\varpi^2 = 1.2m \times (1.6 rad/s)^2 = 3.1 m/s^2$$

Thus the total acceleration is

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{0 + (3.1)^2} = 3.1 m/s^2$$

Example for Rotational Motion

Audio information on compact discs are transmitted digitally through the readout system consisting of laser and lenses. The digital information on the disc are stored by the pits and flat areas on the track. Since the speed of readout system is constant, it reads out the same number of pits and flats in the same time interval. In other words, the linear speed is the same no matter which track is played. a) Assuming the linear speed is 1.3 m/s, find the angular speed of the disc in revolutions per minute when the inner most (r=23mm) and outer most tracks (r=58mm) are read.

Using the relationship between angular and tangential speed $v = r\omega$

$r = 23mm \quad \omega = \frac{v}{r} = \frac{1.3m/s}{23mm} = \frac{1.3}{23\times10^{-3}} = 56.5rad/s$

$= 9.00rev/s = 5.4\times10^{2}rev/\min$

$r = 58mm \quad \omega = \frac{1.3m/s}{58mm} = \frac{1.3}{58\times10^{-3}} = 22.4rad/s \quad = 2.1\times10^{2}rev/\min$

b) The maximum playing time of a standard music CD is 74 minutes and 33 seconds. How many revolutions does the disk make during that time?

$\overline{\omega} = \frac{(\omega_i + \omega_f)}{2} = \frac{(540+210)rev/\min}{2} = 375rev/\min$

$\theta_f = \theta_i + \overline{\omega}t = 0 + \frac{375}{60}rev/s \times 4473s = 2.8\times10^{4}rev$

c) What is the total length of the track past through the readout mechanism?

$l = v_t\Delta t = 1.3m/s \times 4473s$

$= 5.8\times10^{3}m$

d) What is the angular acceleration of the CD over the 4473s time interval, assuming constant α?

$\alpha = \frac{(\omega_f - \omega_i)}{\Delta t} = \frac{(22.4-56.5)rad/s}{4473s}$

$= 7.6\times10^{-3}rad/s^{2}$

Rolling Motion of a Rigid Body

What is a rolling motion?

A more generalized case of a motion where the rotational axis moves together with the object

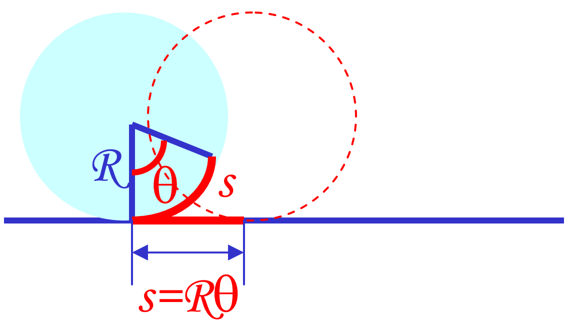
A rotational motion about the moving axis

To simplify the discussion, let's make a few assumptions

1. Limit our discussion on very symmetric objects, such as cylinders, spheres, etc
2. The object rolls on a flat surface

Let's consider a cylinder rolling without slipping on a flat surface

Under what condition does this "Pure Rolling" happen?



The total linear distance the CM of the cylinder moved is

$$s = R\theta$$

Thus the linear speed of the CM is

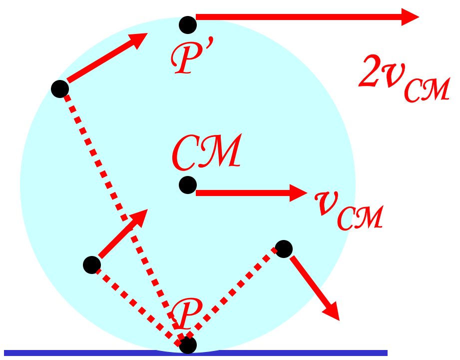
$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

Condition for "Pure Rolling"

More Rolling Motion of a Rigid Body

The magnitude of the linear acceleration of the CM is

$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha$$



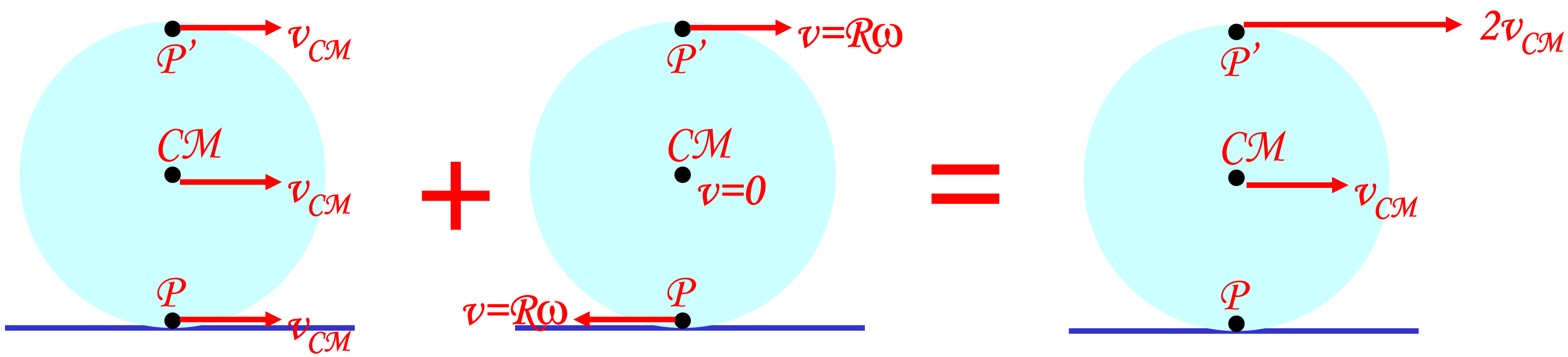
As we learned in the rotational motion, all points in a rigid body moves at the same angular speed but at a different linear speed.

CM is moving at the same speed at all times.

At any given time, the point that comes to P has 0 linear speed while the point at P' has twice the speed of CM

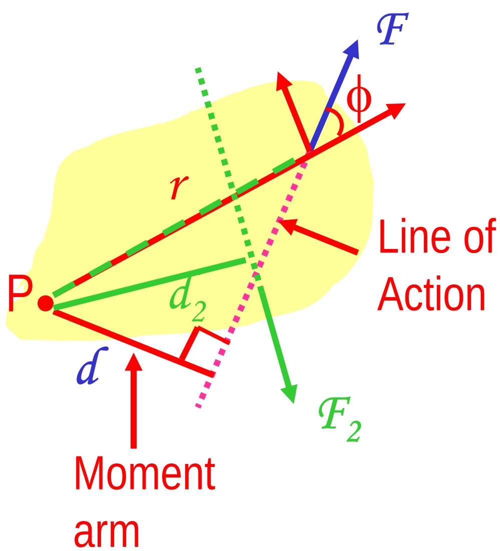
Why??

A rolling motion can be interpreted as the sum of Translation and Rotation



Torque

Torque is the tendency of a force to rotate an object about an axis. Torque, τ, is a vector quantity.



Consider an object pivoting about the point P by the force $\mathcal{F}$ being exerted at a distance r.

The line that extends out of the tail of the force vector is called the line of action.

The perpendicular distance from the pivoting point P to the line of action is called Moment arm.

Magnitude of torque is defined as the product of the force exerted on the object to rotate it and the moment arm.

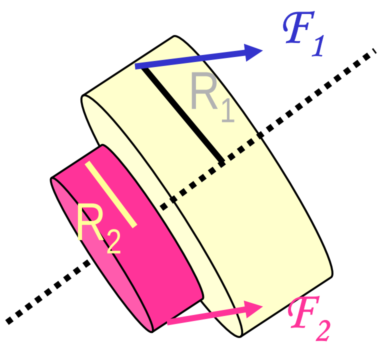
$$\tau \equiv rF\sin\phi = Fd$$

When there are more than one force being exerted on certain points of the object, one can sum up the torque generated by each force vectorially. The convention for sign of the torque is positive if rotation is in counter-clockwise and negative if clockwise.

$$\sum \tau = \tau_1 + \tau_2$$
$$= F_1 d_1 - F_2 d_2$$

Example for Torque

A one piece cylinder is shaped as in the figure with core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the picture. A rope wrapped around the drum whose radius is R_1 exerts force F_1 to the right on the cylinder, and another force exerts F_2 on the core whose radius is R_2 downward on the cylinder. A) What is the net torque acting on the cylinder about the rotation axis?



The torque due to F_1 $\tau_1 = -R_1F_1$ and due to F_2 $\tau_2 = R_2F_2$

So the total torque acting on the system by the forces is

$$\sum \tau = \tau_1 + \tau_2 = -R_1F_1 + R_2F_2$$

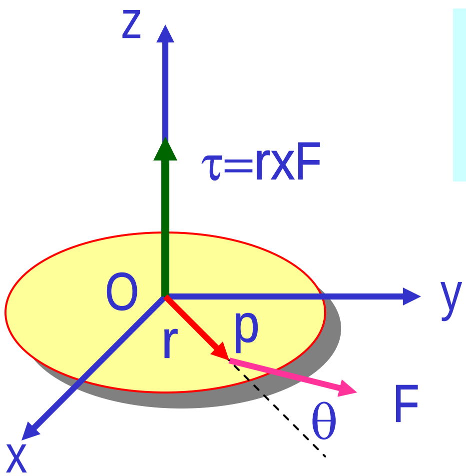
Suppose F_1=5.0 N, R_1=1.0 m, F_2= 15.0 N, and R_2=0.50 m. What is the net torque about the rotation axis and which way does the cylinder rotate from the rest?

Using the above result

$$\sum \tau = -R_1F_1 + R_2F_2$$
$$= -5.0 \times 1.0 + 15.0 \times 0.50 = 2.5 N \bullet m$$

The cylinder rotates in counter-clockwise.

Torque and Vector Product



Let's consider a disk fixed onto the origin O and the force $\mathcal{F}$ exerts on the point p. What happens?

The disk will start rotating counter clockwise about the Z axis

The magnitude of torque given to the disk by the force $\mathcal{F}$ is

$$\tau = Fr\sin\phi$$

But torque is a vector quantity, what is the direction?
How is torque expressed mathematically?

$$\vec{\tau} \equiv \vec{r} \times \vec{F}$$

What is the direction?

The direction of the torque follows the right-hand rule!!

The above operation is called
Vector product or Cross product

$$\vec{C} \equiv \vec{A} \times \vec{B}$$

$$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$$

What is the result of a vector product?

Another vector

What is another vector operation we've learned?

Scalar product

$$C \equiv \vec{A} \cdot \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$$

Result? A scalar

Properties of Vector Product

Vector Product is Non-commutative

What does this mean?

If the order of operation changes the result changes $\vec{A}\times\vec{B}\neq\vec{B}\times\vec{A}$

Following the right-hand rule, the direction changes

$$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$$

Vector Product of two parallel vectors is 0.

$$\left|\vec{C}\right|=\left|\vec{A}\times\vec{B}\right|=\left|\vec{A}\right|\left|\vec{B}\right|\sin\theta=\left|\vec{A}\right|\left|\vec{B}\right|\sin 0=0$$

Thus, $\vec{A}\times\vec{A}=0$

If two vectors are perpendicular to each other

$$\left|\vec{A}\times\vec{B}\right|=\left|\vec{A}\right|\left|\vec{B}\right|\sin\theta=\left|\vec{A}\right|\left|\vec{B}\right|\sin 90^\circ=\left|\vec{A}\right|\left|\vec{B}\right|=AB$$

Vector product follows distribution law

$$\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

The derivative of a Vector product with respect to a scalar variable is

$$\frac{d\left(\vec{A}\times\vec{B}\right)}{dt}=\frac{d\vec{A}}{dt}\times\vec{B}+\vec{A}\times\frac{d\vec{B}}{dt}$$

More Properties of Vector Product

The relationship between unit vectors, $\vec{i}$, $\vec{j}$ and $\vec{k}$

$$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = 0$$

$$\vec{i}\times\vec{j} = -\vec{j}\times\vec{i} = \vec{k}$$

$$\vec{j}\times\vec{k} = -\vec{k}\times\vec{j} = \vec{i}$$

$$\vec{k}\times\vec{i} = -\vec{i}\times\vec{k} = \vec{j}$$

Vector product of two vectors can be expressed in the following determinant form

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \vec{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \vec{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= \left(A_yB_z - A_zB_y\right)\vec{i} - \left(A_xB_z - A_zB_x\right)\vec{j} + \left(A_xB_y - A_yB_x\right)\vec{k}$$