PHYS 1443 – Section 001
Lecture #14

Monday, June 26, 2006

Dr. Jaehoon Yu

- Moment of Inertia
- Parallel Axis Theorem
- Torque and Angular Acceleration
- Rotational Kinetic Energy
- Work, Power and Energy in Rotation
- Angular Momentum & Its Conservation
- Similarity of Linear and Angular Quantities

Announcements

- Quiz Results
  - Average: 60.2
  - Top score: 100
- Last quiz this Wednesday
  - Early in the class
  - Covers Ch. 10 – what we cover tomorrow
- Final exam
  - Date and time: 8 – 10am, Friday, June 30
  - Location: SH103
  - Covers: Ch 9 – what we cover by Wednesday
  - No class this Thursday

Moment of Inertia

Rotational Inertia: Measure of resistance of an object to changes in its rotational motion. Equivalent to mass in linear motion.

For a group of particles $\boxed{I \equiv \sum_i m_i r_i^2}$

For a rigid body $\boxed{I \equiv \int r^2 dm}$

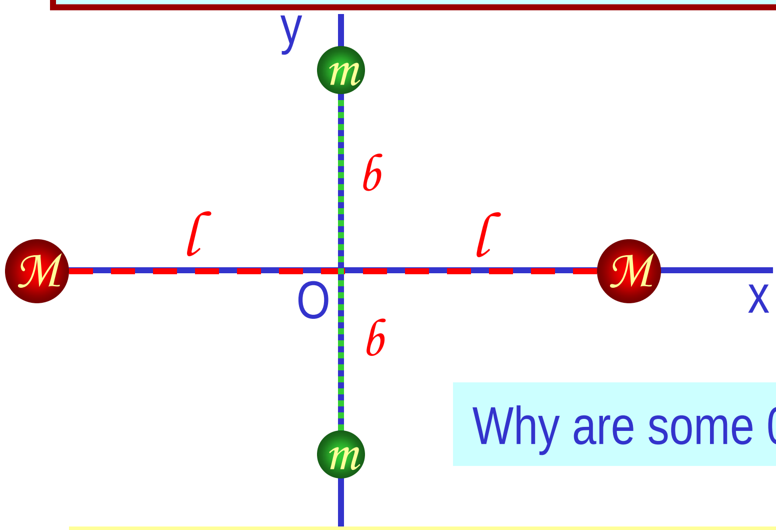
What are the dimension and unit of Moment of Inertia? $\left[ML^2\right]$ $kg \cdot m^2$

Determining Moment of Inertia is extremely important for computing equilibrium of a rigid body, such as a building.

Dependent on the axis of rotation!!!

Example for Moment of Inertia

In a system of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at angular speed ω.



Since the rotation is about y axis, the moment of inertia about y axis, I_y, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m \cdot 0^2 + m \cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2\right)\omega^2 = Ml^2\omega^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2\left(Ml^2 + mb^2\right) \qquad K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2 + 2mb^2\right)\omega^2 = \left(Ml^2 + mb^2\right)\omega^2$$

Calculation of Moments of Inertia

Moments of inertia for large objects can be computed, if we assume the object consists of small volume elements with mass, Δm_i.

The moment of inertia for the large rigid object is $I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$

It is sometimes easier to compute moments of inertia in terms of volume of the elements rather than their mass

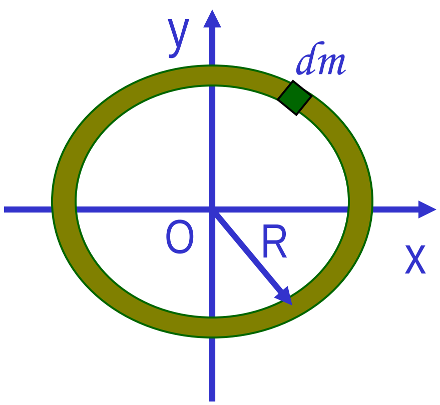
How can we do this?

Using the volume density, ρ, replace dm in the above equation with dV.

$$\rho = \frac{dm}{dV} \quad dm = \rho dV$$

The moments of inertia becomes $\boxed{I = \int \rho r^2 dV}$

Example: Find the moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.



The moment of inertia is

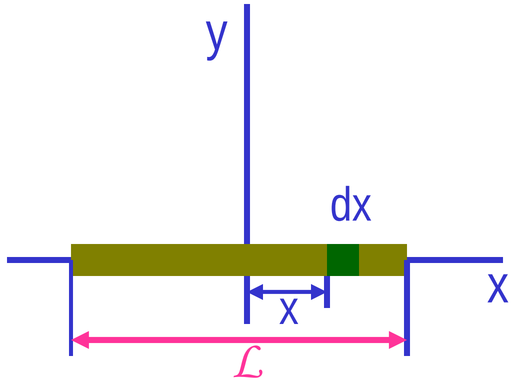
$$I = \int r^2 dm = R^2 \int dm = MR^2$$

What do you notice from this result?

The moment of inertia for this object is the same as that of a point of mass M at the distance R.

Example for Rigid Body Moment of Inertia

Calculate the moment of inertia of a uniform rigid rod of length $\mathcal{L}$ and mass M about an axis perpendicular to the rod and passing through its center of mass.



The line density of the rod is $\lambda = \frac{M}{L}$

so the masslet is $dm = \lambda dx = \frac{M}{L} dx$

The moment of inertia is

$$I = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

What is the moment of inertia when the rotational axis is at one end of the rod.

$$I = \int r^2 dm = \int_0^L \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_0^L$$

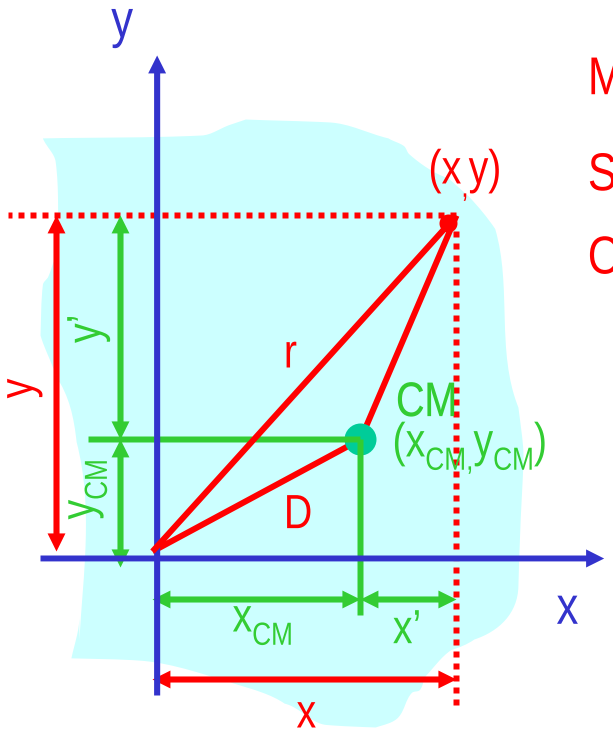
$$= \frac{M}{3L}\left[(L)^3 - 0\right] = \frac{M}{3L}\left(L^3\right) = \frac{ML^2}{3}$$

Will this be the same as the above? Why or why not?

Nope! Since the moment of inertia is resistance to motion, it makes perfect sense for it to be harder to move when it is rotating about the axis at one end.

Parallel Axis Theorem

Moments of inertia for highly symmetric object is easy to compute if the rotational axis is the same as the axis of symmetry. However if the axis of rotation does not coincide with axis of symmetry, the calculation can still be done in simple manner using **parallel-axis theorem.** $I = I_{CM} + MD^2$



Moment of inertia is defined $I = \int r^2 dm = \int \left(x^2 + y^2\right) dm \quad (1)$

Since x and y are $x = x_{CM} + x' \quad y = y_{CM} + y'$

One can substitute x and y in Eq. 1 to obtain

$$I = \int \left[\left(x_{CM} + x'\right)^2 + \left(y_{CM} + y'\right)^2\right] dm$$

$$= \left(x_{CM}^2 + y_{CM}^2\right)\int dm + 2x_{CM}\int x' dm + 2y_{CM}\int y' dm + \int \left(x'^2 + y'^2\right) dm$$

Since the x' and y' are the distances from CM, by definition $\int x' dm = 0 \quad \int y' dm = 0$

Therefore, the parallel-axis theorem

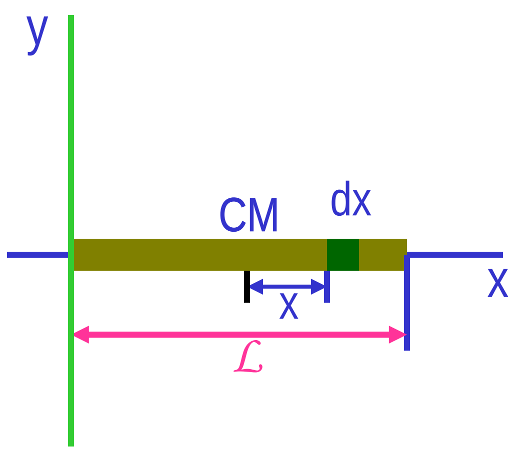
$$I = \left(x_{CM}^2 + y_{CM}^2\right)\int dm + \int \left(x'^2 + y'^2\right) dm = MD^2 + I_{CM}$$

What does this theorem tell you?

Moment of inertia of any object about any arbitrary axis are the same as the sum of moment of inertia for a rotation about the CM and that of the CM about the rotation axis.

Example for Parallel Axis Theorem

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis that goes through one end of the rod, using parallel-axis theorem.



The line density of the rod is $\lambda = \frac{M}{L}$

so the masslet is $dm = \lambda dx = \frac{M}{L}dx$

The moment of inertia about the CM

$$I_{CM} = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

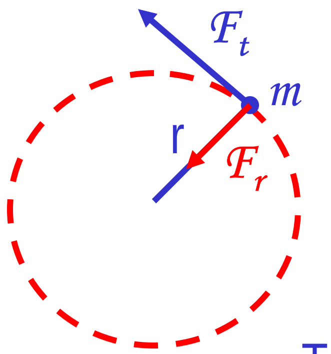
Using the parallel axis theorem

$$I = I_{CM} + D^2 M = \frac{ML^2}{12} + \left(\frac{L}{2}\right)^2 M = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

The result is the same as using the definition of moment of inertia.

Parallel-axis theorem is useful to compute moment of inertia of a rotation of a rigid object with complicated shape about an arbitrary axis

Torque & Angular Acceleration



Let's consider a point object with mass m rotating on a circle.

What forces do you see in this motion?

The tangential force F_t and radial force F_r

The tangential force F_t is $$F_t = ma_t = mr\alpha$$

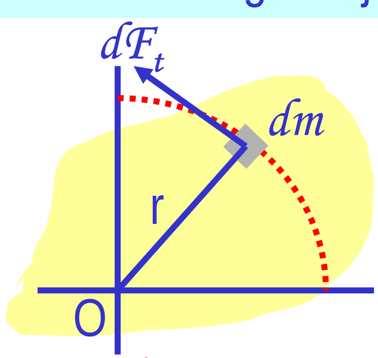
The torque due to tangential force F_t is $$\tau = F_t r = ma_t r = mr^2\alpha = I\alpha$$

What do you see from the above relationship? $$\tau = I\alpha$$

What does this mean? Torque acting on a particle is proportional to the angular acceleration.

What law do you see from this relationship? Analogs to Newton's 2nd law of motion in rotation.

How about a rigid object?



The external tangential force dF_t is $$dF_t = dma_t = dmr\alpha$$

The torque due to tangential force F_t is $$d\tau = dF_t r = \left(r^2 dm\right)\alpha$$

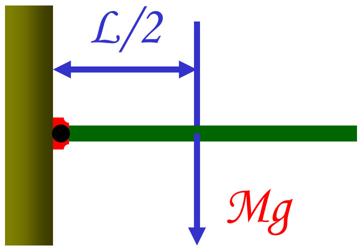
The total torque is $$\sum \tau = \alpha \int r^2 dm = I\alpha$$

What is the contribution due to radial force and why?

Contribution from radial force is 0, because its line of action passes through the pivoting point, making the moment arm 0.

Example for Torque and Angular Acceleration

A uniform rod of length $\mathcal{L}$ and mass $\mathcal{M}$ is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane. The rod is released from rest in the horizontal position. What are the initial angular acceleration of the rod and the initial linear acceleration of its right end?



The only force generating torque is the gravitational force $\mathcal{M}g$

$$\tau = Fd = F\frac{L}{2} = Mg\frac{L}{2} = I\alpha$$

Since the moment of inertia of the rod when it rotates about one end

$$I = \int_0^L r^2 dm = \int_0^L x^2 \lambda dx = \left(\frac{M}{L}\right)\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2}{3}$$

We obtain

$$\alpha = \frac{MgL}{2I} = \frac{MgL}{\frac{2ML^2}{3}} = \frac{3g}{2L}$$

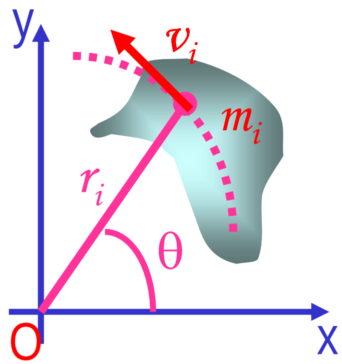
Using the relationship between tangential and angular acceleration

$$a_t = L\alpha = \frac{3g}{2}$$

What does this mean?

The tip of the rod falls faster than an object undergoing a free fall.

Rotational Kinetic Energy



What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, m_i, moving at a tangential speed, v_i, is

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 \omega^2$$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

$$K_R = \sum_i K_i = \frac{1}{2} \sum_i m_i r_i^2 \omega^2 = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2$$

Since moment of Inertia, I, is defined as

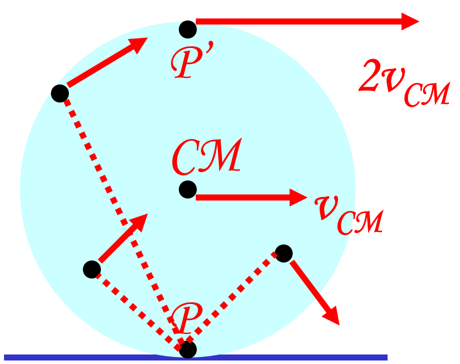
$$I = \sum_i m_i r_i^2$$

The above expression is simplified as

$$K_R = \frac{1}{2} I \omega^2$$

Total Kinetic Energy of a Rolling Body

What do you think the total kinetic energy of the rolling cylinder is?



Since it is a rotational motion about the point P, we can write the total kinetic energy

$$K = \frac{1}{2} I_P \omega^2$$

Where, I_P, is the moment of inertia about the point P.

Using the parallel axis theorem, we can rewrite

$$K = \frac{1}{2} I_P \omega^2 = \frac{1}{2}\left(I_{CM} + MR^2\right)\omega^2 = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} MR^2\omega^2$$

Since $v_{CM}=R\omega$, the above relationship can be rewritten as

$$K = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}^2$$

Rotational kinetic energy about the CM

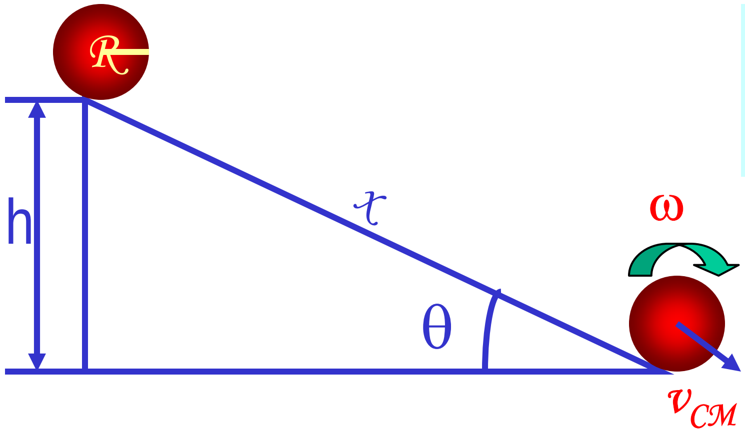
Translational Kinetic energy of the CM

What does this equation mean?

Total kinetic energy of a rolling motion is the sum of the rotational kinetic energy about the CM

And the translational kinetic of the CM

Kinetic Energy of a Rolling Sphere



Let's consider a sphere with radius R rolling down a hill without slipping.

$$K = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} MR^2\omega^2$$

$$= \frac{1}{2} I_{CM}\left(\frac{v_{CM}}{R}\right)^2 + \frac{1}{2} M v_{CM}^2$$

$$= \frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right) v_{CM}^2$$

Since $v_{CM} = R\omega$

What is the speed of the CM in terms of known quantities and how do you find this out?

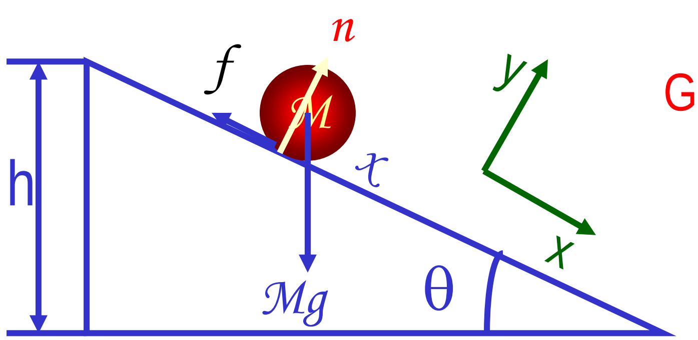
Since the kinetic energy at the bottom of the hill must be equal to the potential energy at the top of the hill

$$K = \frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right) v_{CM}^2 = Mgh$$

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM}/MR^2}}$$

Example for Rolling Kinetic Energy

For solid sphere as shown in the figure, calculate the linear speed of the CM at the bottom of the hill and the magnitude of linear acceleration of the CM. Solve this problem using Newton's second law, the dynamic method.



What are the forces involved in this motion?

Gravitational Force, Frictional Force, Normal Force

Newton's second law applied to the CM gives

$$\sum F_x = Mg\sin\theta - f = Ma_{CM}$$
$$\sum F_y = n - Mg\cos\theta = 0$$

Since the forces $\mathcal{M}g$ and n go through the CM, their moment arm is 0 and do not contribute to torque, while the static friction f causes torque

$$\tau_{CM} = fR = I_{CM}\alpha$$

We know that

$$I_{CM} = \frac{2}{5}MR^2$$

$$a_{CM} = R\alpha$$

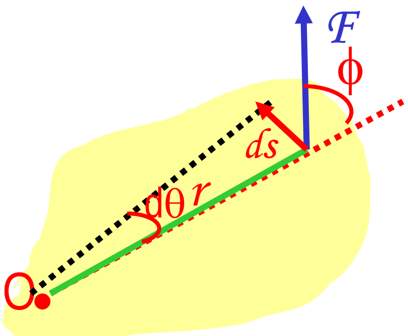
We obtain

$$f = \frac{I_{CM}\alpha}{R} = \frac{\frac{2}{5}MR^2}{R}\left(\frac{a_{CM}}{R}\right) = \frac{2}{5}Ma_{CM}$$

Substituting f in dynamic equations

$$Mg\sin\theta = \frac{7}{5}Ma_{CM} \qquad a_{CM} = \frac{5}{7}g\sin\theta$$

Work, Power, and Energy in Rotation



Let's consider a motion of a rigid body with a single external force *F* exerting on the point P, moving the object by *ds*.

The work done by the force *F* as the object rotates through the infinitesimal distance ds=rdθ is

$$dW = \vec{F}\cdot d\vec{s} = \left(F\cos(\pi-\phi)\right) r d\theta = \left(F\sin\phi\right) r d\theta$$

What is *F*sinφ?

The tangential component of force *F*.

What is the work done by radial component *F*cosφ?

Zero, because it is perpendicular to the displacement.

Since the magnitude of torque is r*F*sinφ,

$$dW = \left(rF\sin\phi\right) d\theta = \tau d\theta$$

The rate of work, or power becomes

$$P = \frac{dW}{dt} = \frac{\tau d\theta}{dt} = \tau\omega$$

How was the power defined in linear motion?

The rotational work done by an external force equals the change in rotational energy.

$$\sum \tau = I\alpha = I\left(\frac{d\omega}{dt}\right) = I\left(\frac{d\omega}{d\theta}\right)\left(\frac{d\theta}{dt}\right) = I\omega\left(\frac{d\omega}{d\theta}\right)$$

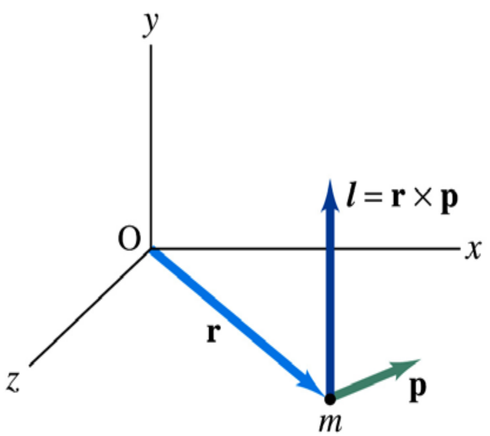
The work put in by the external force then

$$dW = \sum \tau d\theta = I\omega d\omega$$

$$W = \int_{\theta_i}^{\theta_f} \sum \tau d\theta = \int_{\omega_i}^{\omega_f} I\omega d\omega = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2$$

Angular Momentum of a Particle

If you grab onto a pole while running, your body will rotate about the pole, gaining angular momentum. We've used linear momentum to solve physical problems with linear motions, angular momentum will do the same for rotational motions.



Let's consider a point-like object (particle) with mass m located at the vector location r and moving with linear velocity v

The angular momentum L of this particle relative to the origin O is

$$\boxed{\vec{L} \equiv \vec{r} \times \vec{p}}$$

What is the unit and dimension of angular momentum? $kg \cdot m^2/s \quad [ML^2T^{-1}]$

Note that L depends on origin O. Why? Because r changes

What else do you learn? The direction of L is +z

Since p is mv, the magnitude of L becomes $L = mvr \sin\phi$

What do you learn from this?

If the direction of linear velocity points to the origin of rotation, the particle does not have any angular momentum.

If the linear velocity is perpendicular to position vector, the particle moves exactly the same way as a point on a rim.

Angular Momentum and Torque

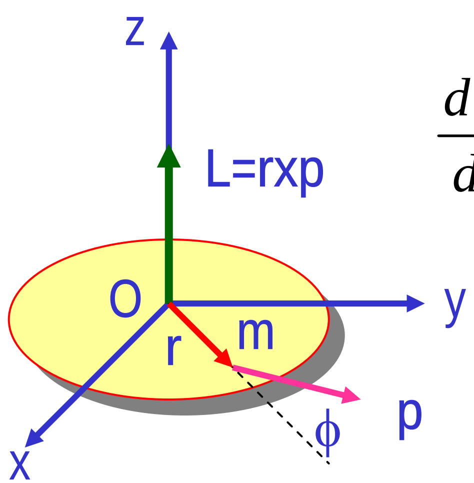
Can you remember how net force exerting on a particle and the change of its linear momentum are related?

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

Total external forces exerting on a particle is the same as the change of its linear momentum.

The same analogy works in rotational motion between torque and angular momentum.

Net torque acting on a particle is $\sum \vec{\tau} = \vec{r} \times \sum \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$



$$\frac{d\vec{L}}{dt} = \frac{d\left(\vec{r} \times \vec{p}\right)}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = 0 + \vec{r} \times \frac{d\vec{p}}{dt}$$

Why does this work?

Because v is parallel to the linear momentum

Thus the torque-angular momentum relationship

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt}$$

The net torque acting on a particle is the same as the time rate change of its angular momentum

Angular Momentum of a System of Particles

The total angular momentum of a system of particles about some point is the vector sum of the angular momenta of the individual particles

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \ldots\ldots + \vec{L}_n = \sum \vec{L}_i$$

Since the individual angular momentum can change, the total angular momentum of the system can change.

Both internal and external forces can provide torque to individual particles. However, the internal forces do not generate net torque due to Newton's third law.

Let's consider a two particle system where the two exert forces on each other.

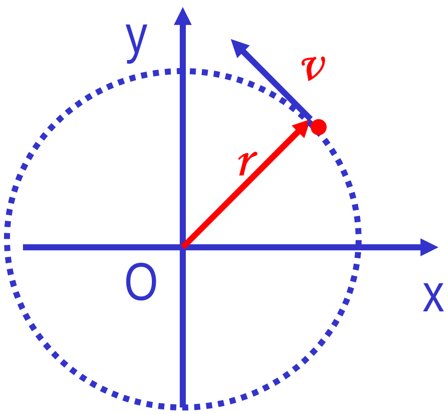
Since these forces are action and reaction forces with directions lie on the line connecting the two particles, the vector sum of the torque from these two becomes 0.

Thus the time rate change of the angular momentum of a system of particles is equal to the net external torque acting on the system

$$\boxed{\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}}$$

Example for Angular Momentum

A particle of mass m is moving on the xy plane in a circular path of radius r and linear velocity v about the origin O. Find the magnitude and direction of angular momentum with respect to O.



Using the definition of angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m\vec{r} \times \vec{v}$$

Since both the vectors, r and v, are on x-y plane and using right-hand rule, the direction of the angular momentum vector is +z (coming out of the screen)

The magnitude of the angular momentum is $\left|\vec{L}\right| = \left|m\vec{r} \times \vec{v}\right| = mrv\sin\phi = mrv\sin 90^\circ = mrv$

So the angular momentum vector can be expressed as $\vec{L} = mrv\vec{k}$

Find the angular momentum in terms of angular velocity ω.

Using the relationship between linear and angular speed

$$\vec{L} = mrv\vec{k} = mr^2\omega\vec{k} = mr^2\vec{\omega} = I\vec{\omega}$$