PHYS 1443 – Section 001
Lecture #15

Tuesday, June 27, 2006

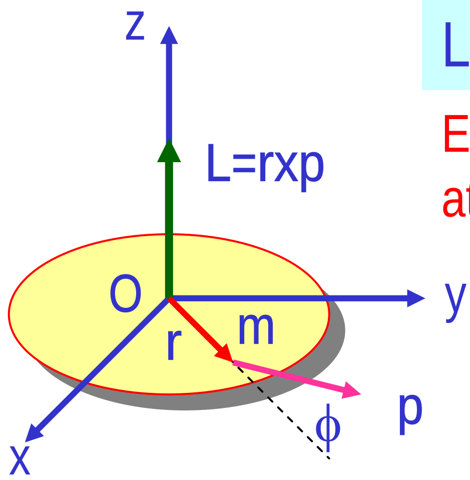
Dr. Jaehoon Yu

- Angular Momentum & Its Conservation
- Similarity of Linear and Angular Quantities
- Conditions for Equilibrium
- Mechanical Equilibrium
- How to solve equilibrium problems?
- Elastic properties of solids

Announcements

- Reading assignments
  - CH12 – 7 and 12 – 8
- Last quiz tomorrow
  - Early in the class
  - Covers Ch. 10 – 12
- Final exam
  - Date and time: 8 – 10am, Friday, June 30
  - Location: SH103
  - Covers: Ch 9 – what we cover tomorrow
  - No class this Thursday

Angular Momentum of a Rotating Rigid Body



Let's consider a rigid body rotating about a fixed axis

Each particle of the object rotates in the xy plane about the z-axis at the same angular speed, ω

Magnitude of the angular momentum of a particle of mass m_i about origin O is $m_i v_i r_i$

$$L_i = m_i r_i v_i = m_i r_i^2 \omega$$

Summing over all particle's angular momentum about z axis

$$L_z = \sum_i L_i = \sum_i \left(m_i r_i^2 \omega \right)$$

What do you see?

$$L_z = \sum_i \left(m_i r_i^2 \right) \omega = I\omega$$

Since I is constant for a rigid body

$$\frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha$$

α is angular acceleration

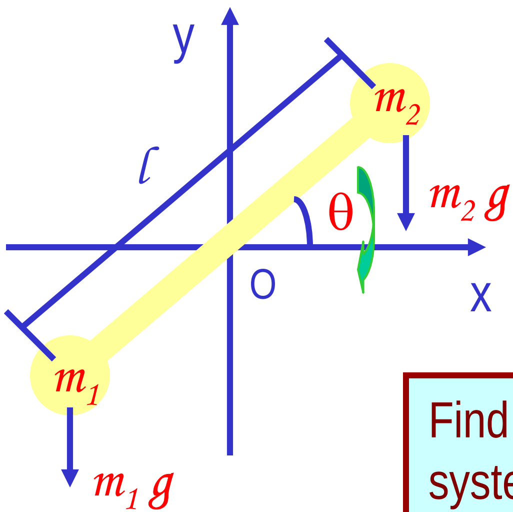
Thus the torque-angular momentum relationship becomes

$$\sum \tau_{ext} = \frac{dL_z}{dt} = I\alpha$$

Thus the net external torque acting on a rigid body rotating about a fixed axis is equal to the moment of inertia about that axis multiplied by the object's angular acceleration with respect to that axis.

Example for Rigid Body Angular Momentum

A rigid rod of mass $\mathcal{M}$ and length ℓ is pivoted without friction at its center. Two particles of mass m_1 and m_2 are attached to either end of the rod. The combination rotates on a vertical plane with an angular speed of ω. Find an expression for the magnitude of the angular momentum.



The moment of inertia of this system is

$$I = I_{rod} + I_{m_1} + I_{m_2} = \frac{1}{12}Ml^2 + \frac{1}{4}m_1 l^2 + \frac{1}{4}m_2 l^2$$

$$= \frac{l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)$$

$$L = I\omega = \frac{\omega l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)$$

Find an expression for the magnitude of the angular acceleration of the system when the rod makes an angle θ with the horizon.

If $m_1 = m_2$, no angular momentum because net torque is 0.

If θ=+/−π/2, at equilibrium so no angular momentum.

First compute net external torque

$$\tau_1 = m_1 g \frac{l}{2}\cos\theta \qquad \tau_2 = -m_2 g \frac{l}{2}\cos\theta$$

$$\tau_{ext} = \tau_1 + \tau_2 = \frac{gl\cos\theta(m_1 - m_2)}{2}$$

Thus α becomes

$$\alpha = \frac{\sum \tau_{ext}}{I} = \frac{\frac{1}{2}(m_1 - m_1)gl\cos\theta}{\frac{l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)} = \frac{2(m_1 - m_1)\cos\theta}{\left(\frac{1}{3}M + m_1 + m_2\right)} g/l$$

Conservation of Angular Momentum

Remember under what condition the linear momentum is conserved?

Linear momentum is conserved when the net external force is 0.

$$\sum \vec{F} = 0 = \frac{d\vec{p}}{dt}$$

$$\vec{p} = const$$

By the same token, the angular momentum of a system is constant in both magnitude and direction, if the resultant external torque acting on the system is 0.

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt} = 0$$

$$\vec{L} = const$$

What does this mean?

Angular momentum of the system before and after a certain change is the same.

$$\vec{L}_i = \vec{L}_f = \text{constant}$$

Three important conservation laws for isolated system that does not get affected by external forces

$K_i + U_i = K_f + U_f$	Mechanical Energy
$\vec{p}_i = \vec{p}_f$	Linear Momentum
$\vec{L}_i = \vec{L}_f$	Angular Momentum

Example for Angular Momentum Conservation

A star rotates with a period of 30 days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0×10^4km, collapses into a neutron star of radius 3.0km. Determine the period of rotation of the neutron star.

What is your guess about the answer?

The period will be significantly shorter, because its radius got smaller.

Let's make some assumptions:

1. There is no external torque acting on it
2. The shape remains spherical
3. Its mass remains constant

Using angular momentum conservation

$$L_i = L_f$$

$$I_i \omega_i = I_f \omega_f$$

The angular speed of the star with the period T is

$$\omega = \frac{2\pi}{T}$$

Thus

$$\omega_f = \frac{I_i \omega_i}{I_f} = \frac{m r_i^2}{m r_f^2} \frac{2\pi}{T_i}$$

$$T_f = \frac{2\pi}{\omega_f} = \left(\frac{r_f^2}{r_i^2}\right) T_i = \left(\frac{3.0}{1.0 \times 10^4}\right)^2 \times 30\ days \quad = 2.7 \times 10^{-6}\ days \quad = 0.23\ s$$

Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Quantities	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = mr^2$
Length of motion	Distance L	Angle θ (Radian)
Speed	$v = \frac{\Delta r}{\Delta t}$	$\omega = \frac{\Delta \theta}{\Delta t}$
Acceleration	$a = \frac{\Delta v}{\Delta t}$	$\alpha = \frac{\Delta \omega}{\Delta t}$
Force	Force $\vec{F} = m\vec{a}$	Torque $\vec{\tau} = I\vec{\alpha}$
Work	Work $W = \vec{F} \cdot \vec{d}$	Work $W = \tau\theta$
Power	$P = \vec{F} \cdot \vec{v}$	$P = \tau\omega$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{\omega}$
Kinetic Energy	Kinetic $K = \frac{1}{2}mv^2$	Rotational $K_R = \frac{1}{2}I\omega^2$

Conditions for Equilibrium

What do you think the term "An object is at its equilibrium" means?

The object is either at rest (Static Equilibrium) or its center of mass is moving with a constant velocity (Dynamic Equilibrium).

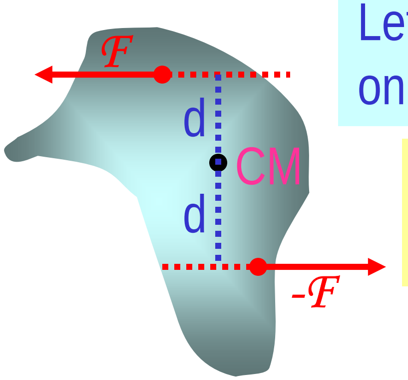
When do you think an object is at its equilibrium?

Translational Equilibrium: Equilibrium in linear motion $\sum \vec{F} = 0$

Is this it?

The above condition is sufficient for a point-like particle to be at its translational equilibrium. However for object with size this is not sufficient. One more condition is needed. What is it?

Let's consider two forces equal magnitude but in opposite direction acting on a rigid object as shown in the figure. What do you think will happen?



The object will rotate about the CM. The net torque acting on the object about any axis must be 0. $\sum \vec{\tau} = 0$

For an object to be at its *static equilibrium*, the object should not have linear or angular speed. $v_{CM} = 0 \quad \omega = 0$

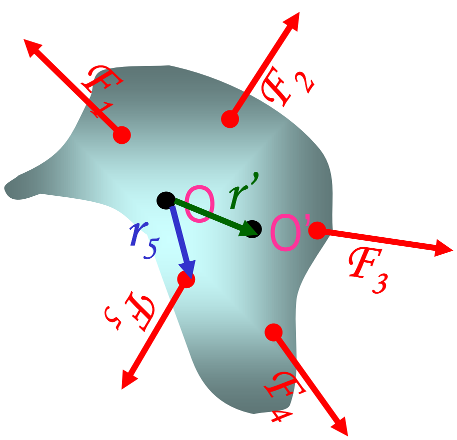
More on Conditions for Equilibrium

To simplify the problem, we will only deal with forces acting on x-y plane, giving torque only along z-axis. What do you think the conditions for equilibrium be in this case?

The six possible equations from the two vector equations turns to three equations.

$$\sum \vec{F} = 0 \Rightarrow \sum F_x = 0, \quad \sum F_y = 0 \quad \textbf{AND} \quad \sum \vec{\tau} = 0 \Rightarrow \sum \tau_z = 0$$

What happens if there are many forces exerting on the object?



If an object is at its translational static equilibrium, and if the net torque acting on the object is 0 about one axis, the net torque must be 0 about any arbitrary axis.

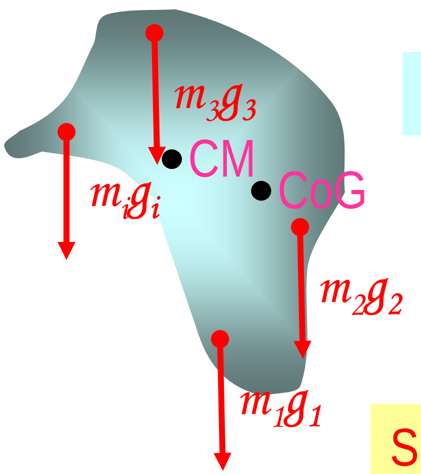
Why is this true?

Because the object is ***not moving***, no matter what the rotational axis is, there should not be in motion. It is simply a matter of mathematical calculation.

Center of Gravity Revisited

When is the center of gravity of a rigid body the same as the center of mass?

Under the uniform gravitational field throughout the body of the object.



Let's consider an arbitrary shaped object

The center of mass of this object is

$$x_{CM} = \frac{\sum m_i x_i}{\sum m_i} = \frac{\sum m_i x_i}{M}$$

$$y_{CM} = \frac{\sum m_i y_i}{\sum m_i} = \frac{\sum m_i y_i}{M}$$

Let's now examine the case with gravitational acceleration on each point is g_i

Since the CoG is the point as if all the gravitational force is exerted on, the torque due to this force becomes

$$(m_1 g_1 + m_2 g_2 + \cdots) x_{CoG} = m_1 g_1 x_1 + m_2 g_2 x_2 + \cdots$$

Generalized expression for different g throughout the body

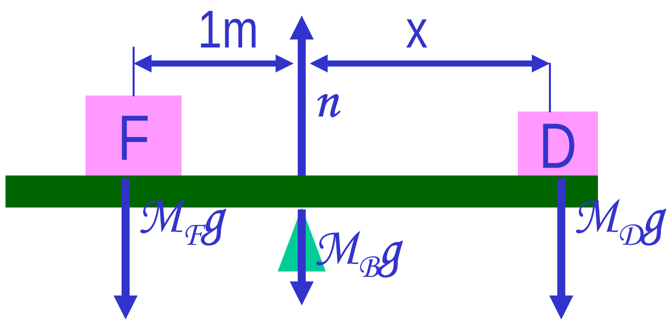
If g is uniform throughout the body

$$(m_1 + m_2 + \cdots) g x_{CoG} = (m_1 x_1 + m_2 x_2 + \cdots) g$$

$$x_{CoG} = \frac{\sum m_i x_i}{\sum m_i} = x_{CM}$$

Example for Mechanical Equilibrium

A uniform 40.0 N board supports the father and the daughter each weighing 800 N and 350 N, respectively. If the support (or fulcrum) is under the center of gravity of the board and the father is 1.00 m from CoG, what is the magnitude of normal force n exerted on the board by the support?



Since there is no linear motion, this system is in its translational equilibrium

$$\sum F_x = 0$$

$$\sum F_y = n - M_B g - M_F g - M_D g = 0$$

Therefore the magnitude of the normal force $n = 40.0 + 800 + 350 = 1190N$

Determine where the child should sit to balance the system.

The net torque about the fulcrum by the three forces are

$$\tau = M_B g \cdot 0 + n \cdot 0 + M_F g \cdot 1.00 - M_D g \cdot x = 0$$

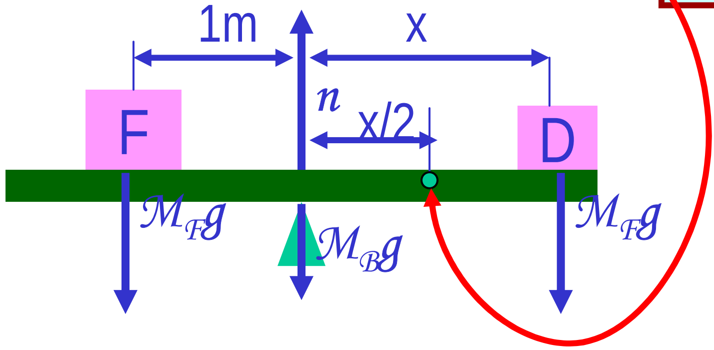
Therefore to balance the system the daughter must sit

$$x = \frac{M_F g}{M_D g} \cdot 1.00m = \frac{800}{350} \cdot 1.00m = 2.29m$$

Example for Mech. Equilibrium Cont'd

Rotational axis

Determine the position of the child to balance the system for different position of axis of rotation.



The net torque about the axis of rotation by all the forces are

$$\tau = M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2) - n \cdot x/2 - M_D g \cdot x/2 = 0$$

Since the normal force is $n = M_B g + M_F g + M_D g$

The net torque can be rewritten

$$\tau = M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2)$$
$$- (M_B g + M_F g + M_D g) \cdot x/2 - M_D g \cdot x/2$$
$$= M_F g \cdot 1.00 - M_D g \cdot x = 0$$

Therefore

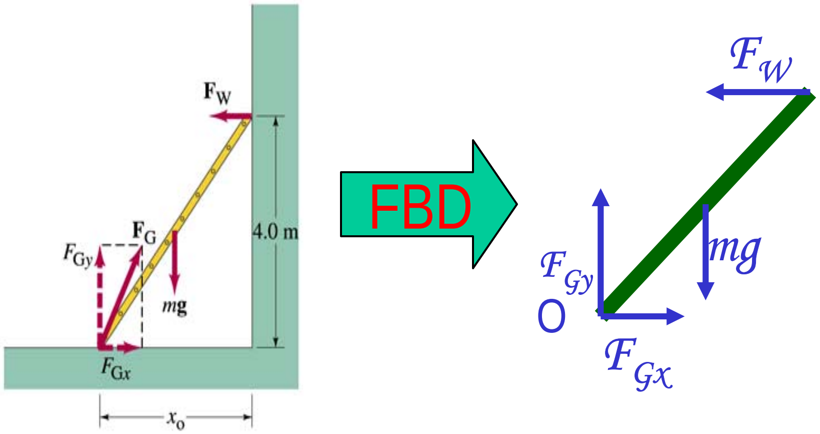
$$x = \frac{M_F g}{M_D g} \cdot 1.00 m = \frac{800}{350} \cdot 1.00 m = 2.29 m$$

What do we learn?

No matter where the rotation axis is, net effect of the torque is identical.

Example 12 – 8

A 5.0 m long ladder leans against a wall at a point 4.0m above the ground. The ladder is uniform and has mass 12.0kg. Assuming the wall is frictionless (but ground is not), determine the forces exerted on the ladder by the ground and the wall.



First the translational equilibrium, using components

$$\sum F_x = F_{Gx} - F_W = 0$$

$$\sum F_y = -mg + F_{Gy} = 0$$

Thus, the y component of the force by the ground is

$$F_{Gy} = mg = 12.0 \times 9.8N = 118N$$

The length x_0 is, from Pythagorian theorem

$$x_0 = \sqrt{5.0^2 - 4.0^2} = 3.0m$$

Example 12 – 8 cont'd

From the rotational equilibrium $\sum \tau_O = -mg\, x_0/2 + F_W 4.0 \;\; = 0$

Thus the force exerted on the ladder by the wall is

$$F_W = \frac{mg\, x_0/2}{4.0} = \frac{118 \cdot 1.5}{4.0} = 44N$$

Tx component of the force by the ground is

$$\sum F_x = F_{Gx} - F_W = 0$$

Solve for F_{Gx}

$$F_{Gx} = F_W = 44N$$

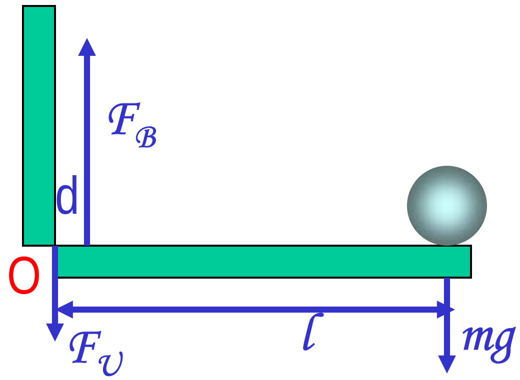
Thus the force exerted on the ladder by the ground is

$$F_G = \sqrt{F_{Gx}^2 + F_{Gy}^2} = \sqrt{44^2 + 118^2} \approx 130N$$

The angle between the ladder and the wall is $\theta = \tan^{-1}\left(\frac{F_{Gy}}{F_{Gx}}\right) = \tan^{-1}\left(\frac{118}{44}\right) = 70^\circ$

Example for Mechanical Equilibrium

A person holds a 50.0N sphere in his hand. The forearm is horizontal. The biceps muscle is attached 3.00 cm from the joint, and the sphere is 35.0cm from the joint. Find the upward force exerted by the biceps on the forearm and the downward force exerted by the upper arm on the forearm and acting at the joint. Neglect the weight of forearm.



Since the system is in equilibrium, from the translational equilibrium condition

$$\sum F_x = 0$$

$$\sum F_y = F_B - F_U - mg = 0$$

From the rotational equilibrium condition

$$\sum \tau = F_U \cdot 0 + F_B \cdot d - mg \cdot l = 0$$

Thus, the force exerted by the biceps muscle is

$$F_B \cdot d = mg \cdot l$$

$$F_B = \frac{mg \cdot l}{d} = \frac{50.0 \times 35.0}{3.00} = 583N$$

Force exerted by the upper arm is

$$F_U = F_B - mg = 583 - 50.0 = 533\,N$$

How do we solve equilibrium problems?

1. Identify all the forces and their directions and locations
2. Draw a free-body diagram with forces indicated on it
3. Write down force equation for each x and y component with proper signs
4. Select a rotational axis for torque calculations → Selecting the axis such that the torque of one of the unknown forces become 0 makes the problem easier to solve
5. Write down torque equation with proper signs
6. Solve the equations for unknown quantities

Elastic Properties of Solids

We have been assuming that the objects do not change their shapes when external forces are exerting on it. It this realistic?

No. In reality, the objects get deformed as external forces act on it, though the internal forces resist the deformation as it takes place.

Deformation of solids can be understood in terms of Stress and Strain

Stress: A quantity proportional to the force causing deformation.
Strain: Measure of degree of deformation

It is empirically known that for small stresses, strain is proportional to stress

The constants of proportionality are called Elastic Modulus

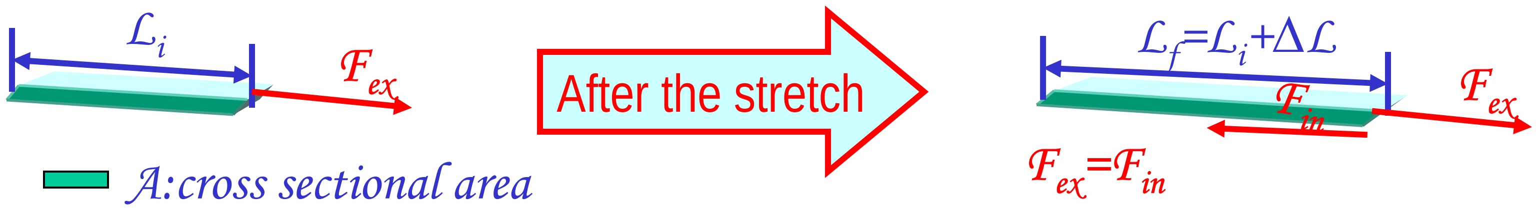
$$\text{Elastic Modulus} \equiv \frac{\text{stress}}{\text{strain}}$$

Three types of Elastic Modulus

1. **Young's modulus**: Measure of the elasticity in length
2. **Shear modulus**: Measure of the elasticity in plane
3. **Bulk modulus**: Measure of the elasticity in volume

Young's Modulus

Let's consider a long bar with cross sectional area A and initial length L_i.



Tensile stress $\text{Tensile Stress} \equiv \frac{F_{ex}}{A}$ Tensile strain $\text{Tensile Strain} \equiv \frac{\Delta L}{L_i}$

Young's Modulus is defined as

$$Y \equiv \frac{\text{Tensile Stress}}{\text{Tensile Strain}} = \frac{F_{ex}/A}{\Delta L / L_i}$$

Used to characterize a rod or wire stressed under tension or compression

What is the unit of Young's Modulus?

Force per unit area

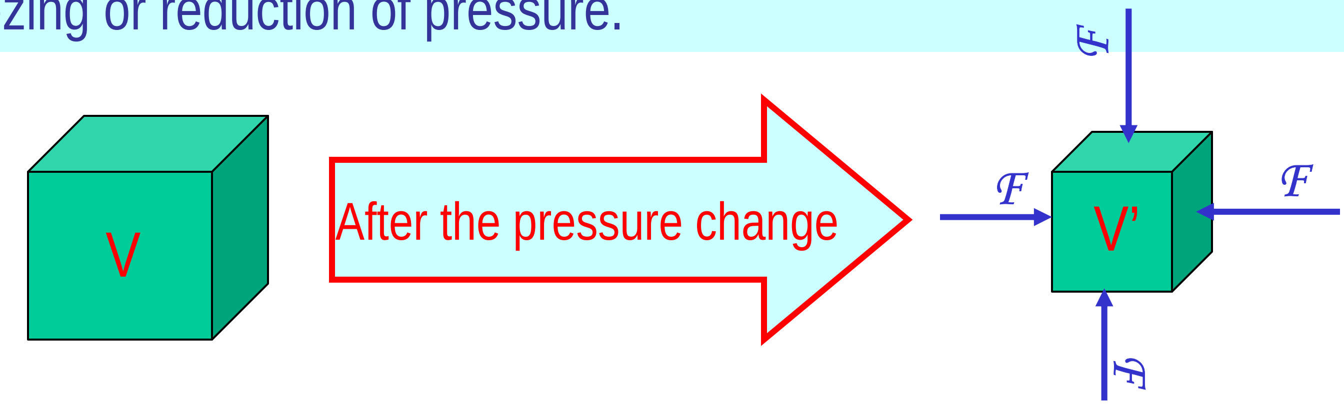
Experimental Observations

1. For fixed external force, the change in length is proportional to the original length
2. The necessary force to produce a given strain is proportional to the cross sectional area

Elastic limit: Maximum stress that can be applied to the substance before it becomes permanently deformed

Bulk Modulus

Bulk Modulus characterizes the response of a substance to uniform squeezing or reduction of pressure.



Volume stress =pressure

$$\text{Pressure} \equiv \frac{\text{Normal Force}}{\text{Surface Area the force applies}} = \frac{F}{A}$$

If the pressure on an object changes by $\Delta P = \Delta F / A$, the object will undergo a volume change ΔV.

Bulk Modulus is defined as

$$B \equiv \frac{\text{Volume Stress}}{\text{Volume Strain}} = -\frac{\Delta F / A}{\Delta V / V_i} = -\frac{\Delta P}{\Delta V / V_i}$$

Because the change of volume is reverse to change of pressure.

Compressibility is the reciprocal of Bulk Modulus

Example for Solid's Elastic Property

A solid brass sphere is initially under normal atmospheric pressure of $1.0\times10^5 N/m^2$. The sphere is lowered into the ocean to a depth at which the pressures is $2.0\times10^7 N/m^2$. The volume of the sphere in air is $0.5m^3$. By how much its volume change once the sphere is submerged?

Since bulk modulus is $$B = -\frac{\Delta P}{\Delta V / V_i}$$

The amount of volume change is $$\Delta V = -\frac{\Delta P V_i}{B}$$

From table 12.1, bulk modulus of brass is 6.1×10^{10} N/m^2

The pressure change ΔP is $$\Delta P = P_f - P_i = 2.0\times10^7 - 1.0\times10^5 \approx 2.0\times10^7$$

Therefore the resulting volume change ΔV is $$\Delta V = V_f - V_i = -\frac{2.0\times10^7 \times 0.5}{6.1\times10^{10}} = -1.6\times10^{-4} m^3$$

The volume has decreased.