PHYS 1443 – Section 001
Lecture #8

Monday, June 11, 2007

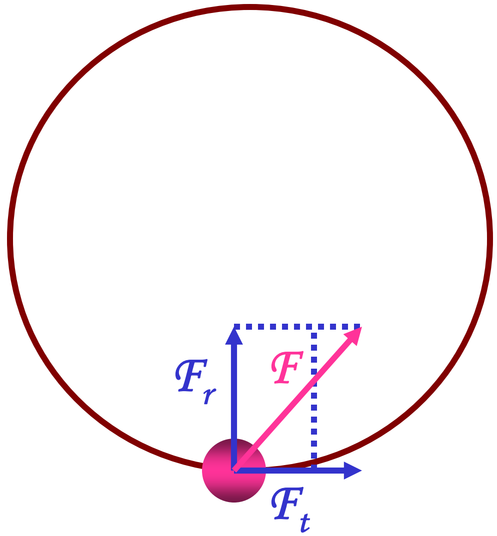
Dr. Jaehoon Yu

- Forces in Non-uniform Circular Motion
- Resistive Forces and Terminal Velocity
- Newton's Law of Universal Gravitation
- Kepler's Laws
- Motion in Accelerated Frames
- Work done by a constant force
- Scalar Product of Vectors

Announcements

- Quiz result
  - Average: 44.7/80
    - Equivalent to 56/100
    - Quiz 1: 71/100
  - Top score:80/80
- Mid-term exam
  - 8:00 – 10am, Thursday, June 14, in class
  - CH 1 – 8 or 9?

Forces in Non-uniform Circular Motion



The object has both tangential and radial accelerations.

What does this statement mean?

The object is moving under both tangential and radial forces.

$$\vec{F} = \vec{F}_r + \vec{F}_t$$

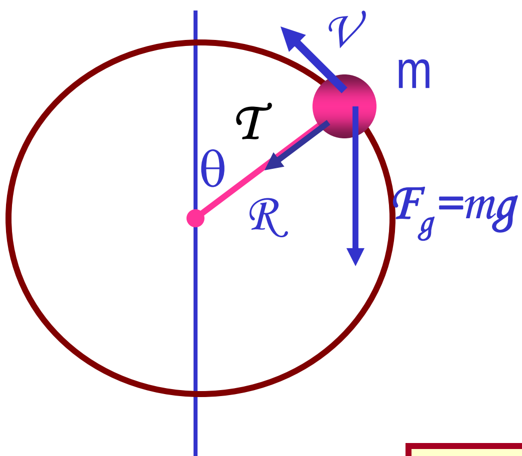
These forces cause not only the velocity but also the speed of the ball to change. The object undergoes a curved motion under the absence of constraints, such as a string.

What is the magnitude of the net acceleration?

$$a = \sqrt{a_r^2 + a_t^2}$$

Example for Non-Uniform Circular Motion

A ball of mass m is attached to the end of a cord of length R. The ball is moving in a vertical circle. Determine the tension of the cord at any instant when the speed of the ball is v and the cord makes an angle θ with vertical.



What are the forces involved in this motion?

- The gravitational force F_g
- The radial force, T, providing tension.

tangential comp.

$$\sum F_t = mg\sin\theta = ma_t$$

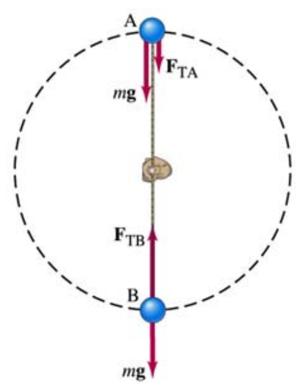
$$a_t = g\sin\theta$$

Radial comp.

$$\sum F_r = T + mg\cos\theta = ma_r = m\frac{v^2}{R}$$

$$T = m\left(\frac{v^2}{R} - g\cos\theta\right)$$

At what angles the tension becomes maximum and minimum. What are the tensions?



Motion in Resistive Forces

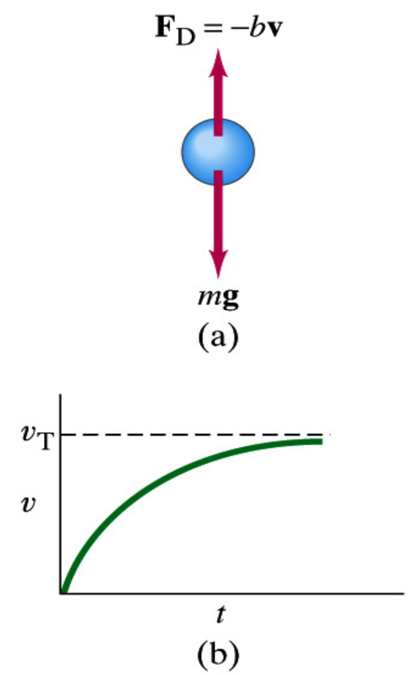
Medium can exert resistive forces on an object moving through it due to viscosity or other types frictional properties of the medium.

Some examples?

Air resistance, viscous force of liquid, etc

These forces are exerted on moving objects in opposite direction of the movement.

These forces are proportional to such factors as speed. They almost always increase with increasing speed.

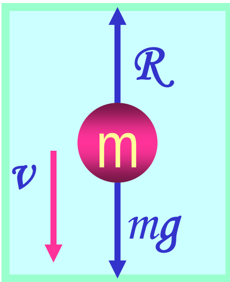


Two different cases of proportionality:

1. Forces linearly proportional to speed: Slowly moving or very small objects
2. Forces proportional to square of speed: Large objects w/ reasonable speed

Resistive Force Proportional to Speed

Since the resistive force is proportional to speed, we can write R=bv.



Let's consider that a ball of mass m is falling through a liquid.

$$\sum \vec{F} = \vec{F}_g + \vec{R} = m\vec{a} \quad \sum F_x = 0$$

In other words

$$\sum F_y = mg - bv = ma = m\frac{dv}{dt} \qquad \frac{dv}{dt} = g - \frac{b}{m}v \qquad \frac{dv}{dt} = g - \frac{b}{m}v = g\text{, when } v = 0$$

The above equation also tells us that as time goes on the speed increases and the acceleration decreases, eventually reaching 0.

What does this mean?

An object moving in a viscous medium will obtain speed to a certain speed (terminal speed) and then maintain the same speed without any more acceleration.

What is the terminal speed in above case?

$$\frac{dv}{dt} = g - \frac{b}{m}v = 0; \; v_t = \frac{mg}{b}$$

How do the speed and acceleration depend on time?

$$v = \frac{mg}{b}\left(1 - e^{-bt/m}\right); \quad v = 0 \text{ when } t = 0;$$

$$a = \frac{dv}{dt} = \frac{mg}{b}\frac{b}{m}e^{-bt/m} = ge^{-t/\tau}; \; a = g \text{ when } t = 0;$$

$$\frac{dv}{dt} = \frac{mg}{b}\frac{b}{m}e^{-t/\tau} = \frac{mg}{b}\frac{b}{m}\left(1 - 1 + e^{-t/\tau}\right) = g - \frac{b}{m}v$$

The time needed to reach 63.2% of the terminal speed is defined as the time constant, τ=m/b.

Newton's Law of Universal Gravitation

People have been very curious about stars in the sky, making observations for a long time. The data people collected, however, have not been explained until Newton has discovered the law of gravitation.

Every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

How would you write this law mathematically?

$$F_g \propto \frac{m_1 m_2}{r_{12}^2}$$

With G

$$F_g = G\frac{m_1 m_2}{r_{12}^2}$$

G is the universal gravitational constant, and its value is

$$G = 6.673 \times 10^{-11}$$

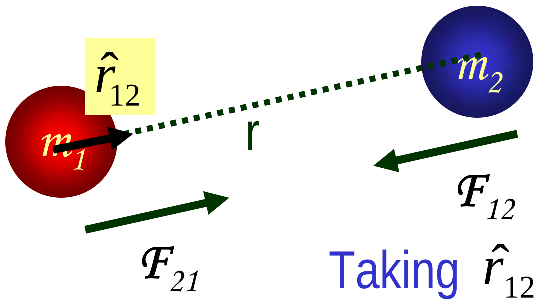
Unit? $N \cdot m^2 / kg^2$

This constant is not given by the theory but must be measured by experiments.

This form of forces is known as the inverse-square law, because the magnitude of the force is inversely proportional to the square of the distances between the objects.

More on Law of Universal Gravitation

Consider two particles exerting gravitational forces to each other.



Two objects exert gravitational force on each other following Newton's 3rd law.

Taking $\hat{r}_{12}$ as the unit vector, we can write the force m_2 experiences as

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r^2}\hat{r}_{12}$$

What do you think the negative sign means?

It means that the force exerted on the particle 2 by particle 1 is an attractive force, pulling #2 toward #1.

Gravitational force is a field force: Forces act on object without a physical contact between the objects at all times, independent of medium between them.

The gravitational force exerted by a finite size, spherically symmetric mass distribution on an object outside of it is the same as when the entire mass of the distributions is concentrated at the center of the object.

What do you think the gravitational force on the surface of the earth looks?

$$F_g = G\frac{M_E m}{R_E^2} = mg$$

Example for Gravitation

Using the fact that g=9.80m/s^2 on the Earth's surface, find the average density of the Earth.

Since the gravitational acceleration is

$$F_g = G\frac{M_E m}{R_E^2} = mg$$

Solving for g

$$g = G\frac{M_E}{R_E^{\ 2}} = 6.67\times 10^{-11}\frac{M_E}{R_E^{\ 2}}$$

Solving for M_E

$$M_E = \frac{R_E^{\ 2} g}{G}$$

Therefore the density of the Earth is

$$\rho = \frac{M_E}{V_E} = \frac{\frac{R_E^{\ 2} g}{G}}{\frac{4\pi}{3}R_E^{\ 3}} = \frac{3g}{4\pi G R_E}$$

$$= \frac{3\times 9.80}{4\pi\times 6.67\times 10^{-11}\times 6.37\times 10^{6}} = 5.50\times 10^{3} kg/m^3$$

Free Fall Acceleration & Gravitational Force

Weight of an object with mass *m* is *mg*. Using the force exerting on a particle of mass *m* on the surface of the Earth, one can obtain

$$mg = G\frac{M_E m}{R_E^2}$$

$$g = G\frac{M_E}{R_E^2}$$

What would the gravitational acceleration be if the object is at an altitude *h* above the surface of the Earth?

$$F_g = mg' = G\frac{M_E m}{r^2} = G\frac{M_E m}{(R_E + h)^2}$$

$$g' = G\frac{M_E}{(R_E + h)^2}$$

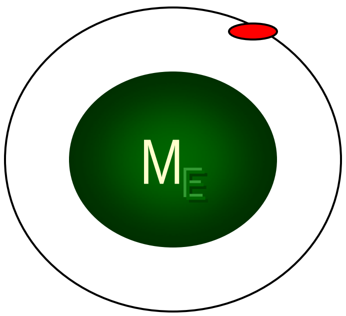
Distance from the center of the Earth to the object at the altitude h.

What do these tell us about the gravitational acceleration?

- The gravitational acceleration is independent of the mass of the object
- The gravitational acceleration decreases as the altitude increases
- If the distance from the surface of the Earth gets infinitely large, the weight of the object approaches 0.

Example for Gravitational Force

The international space station is designed to operate at an altitude of 350km. When completed, it will have a weight (measured on the surface of the Earth) of $4.22\text{x}10^6$N. What is its weight when in its orbit?



The total weight of the station on the surface of the Earth is

$$F_{GE} = mg = G\frac{M_E m}{R_E^{\ 2}} = 4.22\times 10^6 N$$

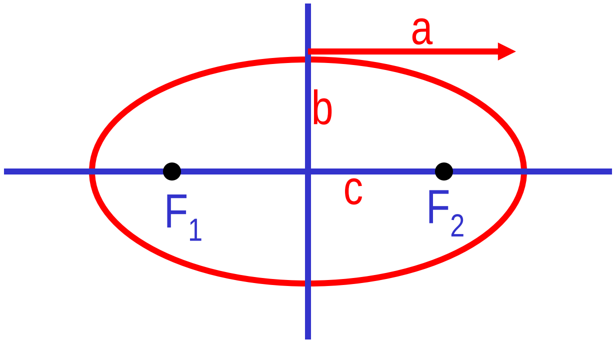
Since the orbit is at 350km above the surface of the Earth, the gravitational force at that height is

$$F_O = mg' = G\frac{M_E m}{(R_E + h)^2} = \frac{R_E^2}{(R_E + h)^2}F_{GE}$$

Therefore the weight in the orbit is

$$F_O = \frac{R_E^2}{(R_E + h)^2}F_{GE} = \frac{(6.37\times 10^6)^2}{(6.37\times 10^6 + 3.50\times 10^5)^2}\times 4.22\times 10^6 = 3.80\times 10^6 N$$

Kepler's Laws & Ellipse



Ellipses have two different axis, major (long) and minor (short) axis, and two focal points, F_1 & F_2

a is the length of the semi-major axis

b is the length of the semi-minor axis

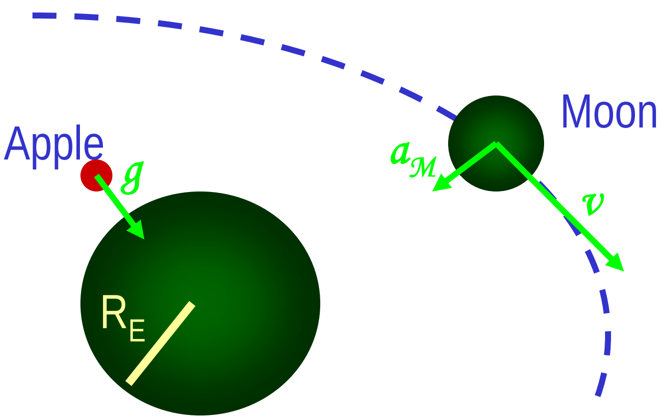
Kepler lived in Germany and discovered the law's governing planets' movements some 70 years before Newton, by analyzing data.

1. All planets move in elliptical orbits with the Sun at one focal point.
2. The radius vector drawn from the Sun to a planet sweeps out equal area in equal time intervals. (*Angular momentum conservation*)
3. The square of the orbital period of any planet is proportional to the cube of the semi-major axis of the elliptical orbit.

Newton's laws explain the cause of the above laws. Kepler's third law is a direct consequence of the law of gravitation being inverse square law.

The Law of Gravity and Motions of Planets

- Newton assumed that the law of gravitation applies the same whether it is the apple on the surface of the Moon or on the surface of the Earth.
- The interacting bodies are assumed to be point like particles.



Newton predicted that the ratio of the Moon's acceleration $a_{\mathcal{M}}$ to the apple's acceleration g would be

$$\frac{a_M}{g} = \frac{(1/r_M)^2}{(1/R_E)^2} = \left(\frac{R_E}{r_M}\right)^2 = \left(\frac{6.37\times10^6}{3.84\times10^8}\right)^2 = 2.75\times10^{-4}$$

Therefore the centripetal acceleration of the Moon, $a_{\mathcal{M}}$, is

$$a_M = 2.75\times10^{-4}\times9.80 = 2.70\times10^{-3}\,m/s^2$$

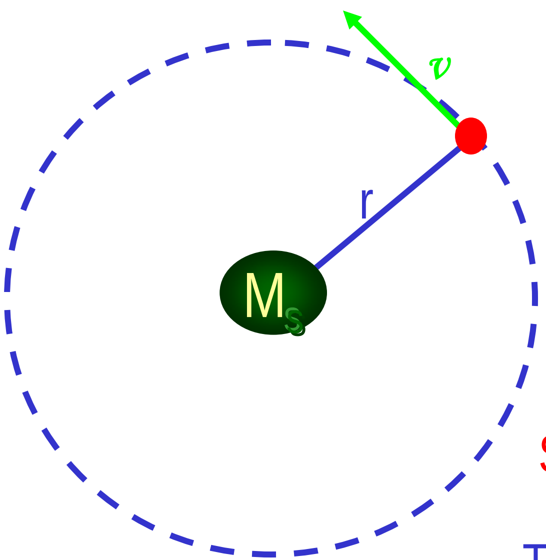
Newton also calculated the Moon's orbital acceleration $a_{\mathcal{M}}$ from the knowledge of its distance from the Earth and its orbital period, T=27.32 days=2.36×10^6s

$$a_M = \frac{v^2}{r_M} = \frac{(2\pi r_M/T)^2}{r_M} = \frac{4\pi^2 r_M}{T^2} = \frac{4\pi^2\times3.84\times10^8}{(2.36\times10^6)^2} = 2.72\times10^{-3}\,m/s^2 \approx \frac{9.80}{(60)^2}$$

This means that the distance to the Moon is about 60 times that of the Earth's radius, and its acceleration is reduced by the square of the ratio. This proves that the inverse square law is valid.

Kepler's Third Law

It is crucial to show that Keper's third law can be predicted from the inverse square law for circular orbits.



Since the gravitational force exerted by the Sun is radially directed toward the Sun to keep the planet on a near circular path, we can apply Newton's second law

$$\frac{GM_sM_P}{r^2} = \frac{M_pv^2}{r}$$

Since the orbital speed, v, of the planet with period T is $v = \frac{2\pi r}{T}$

The above can be written $\frac{GM_sM_P}{r^2} = \frac{M_p(2\pi r/T)^2}{r}$

Solving for T one can obtain $T^2 = \left(\frac{4\pi^2}{GM_s}\right)r^3 = K_s r^3$ and $K_s = \left(\frac{4\pi^2}{GM_s}\right) = 2.97 \times 10^{-19}\, s^2/m^3$

This is Kepler's third law. It's also valid for the ellipse with r as the length of the semi-major axis. The constant K_s is independent of mass of the planet.

Example of Kepler's Third Law

Calculate the mass of the Sun using the fact that the period of the Earth's orbit around the Sun is 3.16×10^7 s, and its distance from the Sun is 1.496×10^{11} m.

Using Kepler's third law.

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3 = K_s r^3$$

The mass of the Sun, M_s, is

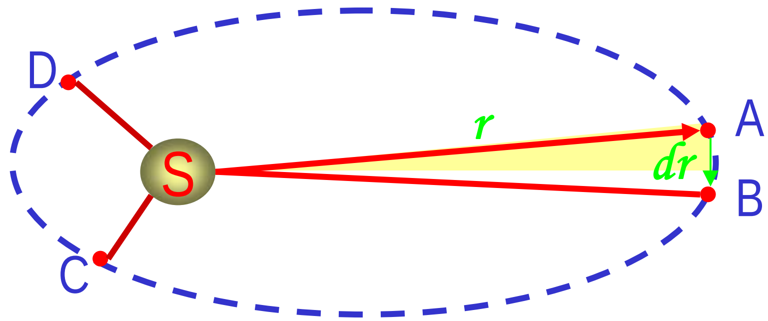
$$M_s = \left(\frac{4\pi^2}{GT^2}\right) r^3$$

$$= \left(\frac{4\pi^2}{6.67 \times 10^{-11} \times \left(3.16 \times 10^7\right)^2}\right) \times \left(1.496 \times 10^{11}\right)^3$$

$$= 1.99 \times 10^{30}\, kg$$

Kepler's Second Law and Angular Momentum Conservation

Consider a planet of mass M_p moving around the Sun in an elliptical orbit.



Since the gravitational force acting on the planet is always toward radial direction, it is a *central force*

Therefore the torque acting on the planet by this force is always 0.

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times F\hat{r} = 0$$

Since torque is the time rate change of angular momentum *L*, the angular momentum is constant.

$$\vec{\tau} = \frac{d\vec{L}}{dt} = 0 \qquad \vec{L} = const$$

Because the gravitational force exerted on a planet by the Sun results in no torque, the angular momentum L of the planet is constant.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times M_p\vec{v} = M_p\vec{r} \times \vec{v} = const$$

Since the area swept by the motion of the planet is

$$dA = \frac{1}{2}\left|\vec{r} \times d\vec{r}\right| = \frac{1}{2}\left|\vec{r} \times \vec{v}dt\right| = \frac{L}{2M_p}dt \Rightarrow \frac{dA}{dt} = \frac{L}{2M_p} = const$$

This is Keper's second law which states that the radius vector from the Sun to a planet sweeps out equal areas in equal time intervals.