PHYS 1441 – Section 001
Lecture # 15

Wednesday, June 25, 2008

Dr. Jaehoon Yu

- Moment of Inertia
- Work, Power and Energy in Rotation
- Rotational Kinetic Energy

Announcements

- Quiz Tomorrow, June 26
  - Beginning of the class
  - Covers CH8 – 9
- Final exam
  - 8 – 10am, Monday, June 30, in SH103
  - Comprehensive exam: Covers CH 1 – 9 + Appendices A – E
    - There will be a review session by Dr. Satyanand in class tomorrow, after the quiz
  - Please take full advantage of this review

Moment of Inertia

Rotational Inertia: Measure of resistance of an object to changes in its rotational motion. Equivalent to mass in linear motion.

For a group of objects

$$I \equiv \sum_i m_i r_i^2$$

For a rigid body

$$I \equiv \int r^2 dm$$

What are the dimension and unit of Moment of Inertia?

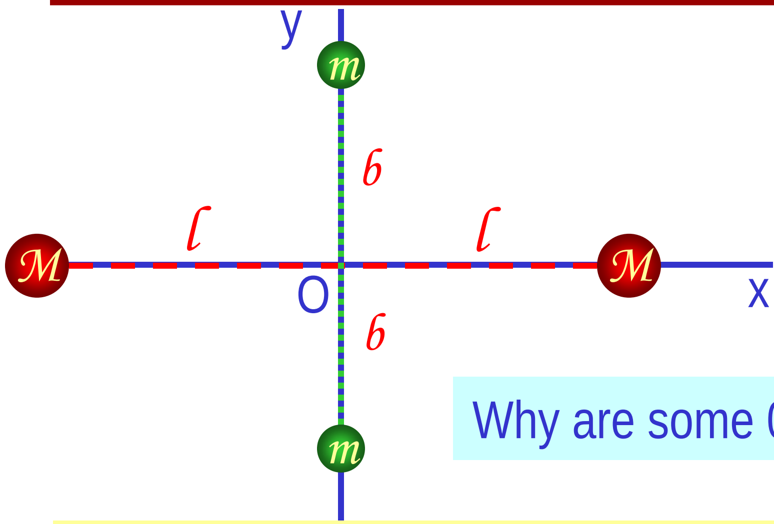
$$[ML^2] \quad kg \cdot m^2$$

Determining Moment of Inertia is extremely important for computing equilibrium of a rigid body, such as a building.

Dependent on the axis of rotation!!!

Example for Moment of Inertia

In a system of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at angular speed ω.



Since the rotation is about y axis, the moment of inertia about y axis, I_y, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m\cdot 0^2 + m\cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2\right)\omega^2 = Ml^2\omega^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2\left(Ml^2 + mb^2\right)$$

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2 + 2mb^2\right)\omega^2 = \left(Ml^2 + mb^2\right)\omega^2$$

Check out Table 9.1 for moment of inertia for various shaped objects

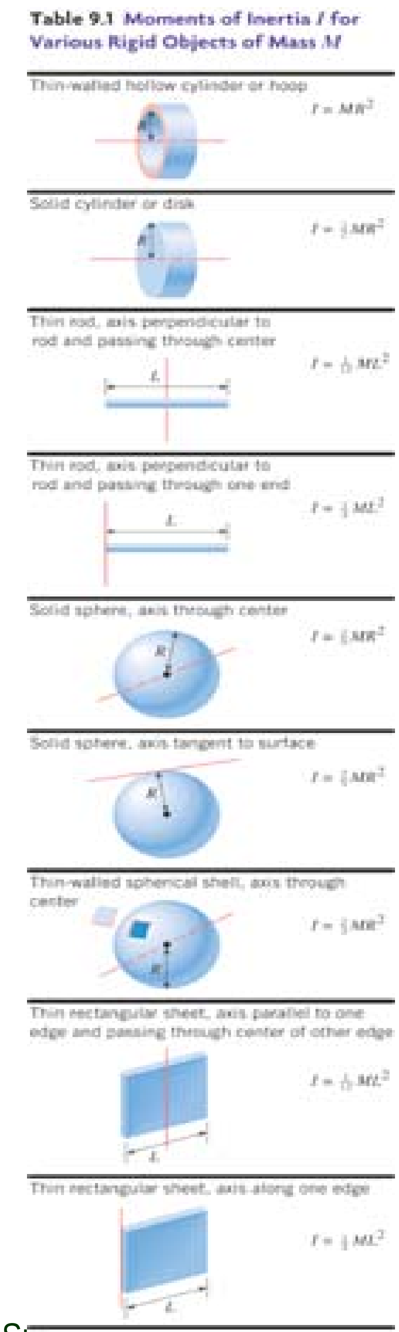
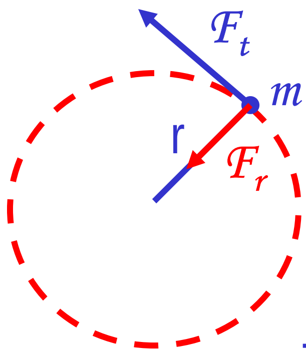


Table 9.1 Moments of Inertia I for Various Rigid Objects of Mass M

Thin-walled hollow cylinder or hoop	$I = MR^2$
Solid cylinder or disk	$I = \frac{1}{2}MR^2$
Thin rod, axis perpendicular to rod and passing through center	$I = \frac{1}{12}ML^2$
Thin rod, axis perpendicular to rod and passing through one end	$I = \frac{1}{3}ML^2$
Solid sphere, axis through center	$I = \frac{2}{5}MR^2$
Solid sphere, axis tangent to surface	$I = \frac{7}{5}MR^2$
Thin-walled spherical shell, axis through center	$I = \frac{2}{3}MR^2$
Thin rectangular sheet, axis parallel to one edge and passing through center of other edge	$I = \frac{1}{12}ML^2$
Thin rectangular sheet, axis along one edge	$I = \frac{1}{3}ML^2$

Torque & Angular Acceleration



Let's consider a point object with mass m rotating on a circle.

What forces do you see in this motion?

The tangential force $\mathcal{F}_t$ and the radial force $\mathcal{F}_r$

The tangential force $\mathcal{F}_t$ is $F_t = ma_t = mr\alpha$

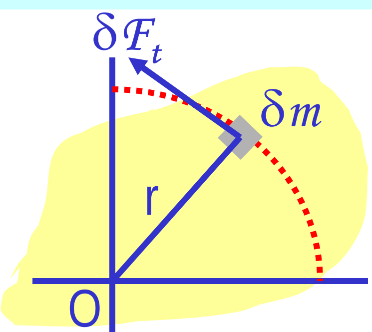
The torque due to tangential force $\mathcal{F}_t$ is $\tau = F_t r = ma_t r = mr^2\alpha = I\alpha$

What do you see from the above relationship? $\tau = I\alpha$

What does this mean? Torque acting on a particle is proportional to the angular acceleration.

What law do you see from this relationship? Analogs to Newton's 2nd law of motion in rotation.

How about a rigid object?



The external tangential force $\delta\mathcal{F}_t$ is $\delta F_t = \delta m a_t = \delta m r\alpha$

The torque due to tangential force $\mathcal{F}_t$ is $\delta\tau = \delta F_t r = \left(r^2\delta m\right)\alpha$

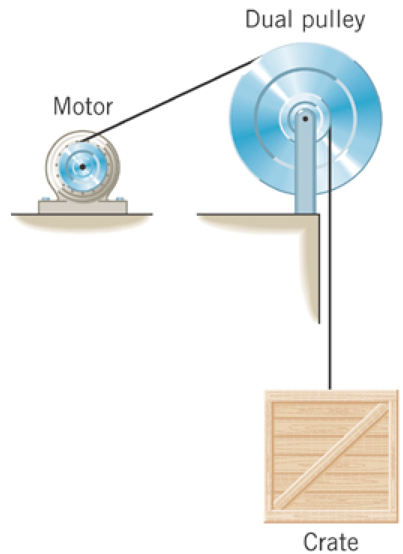
The total torque is $\sum\delta\tau = \alpha\sum r^2\delta m = I\alpha$

What is the contribution due to radial force and why?

Contribution from radial force is 0, because its line of action passes through the pivoting point, making the moment arm 0.

Ex. 12 Hosting a Crate

The combined moment of inertia of the dual pulley is 50.0 kg·m². The crate weighs 4420 N. A tension of 2150 N is maintained in the cable attached to the motor. Find the angular acceleration of the dual pulley.



(a)

$$\sum F_y = T_2' - mg = ma_y$$

$$T_2' = mg + ma_y$$

$$\sum \tau = T_1\ell_1 - T_2'\ell_2 =$$

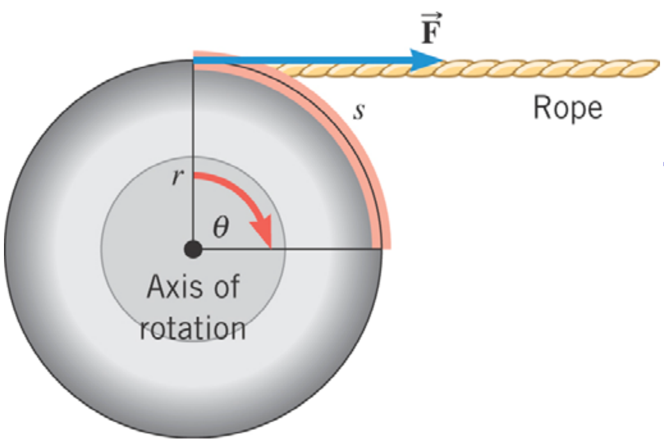
$$T_1\ell_1 - \left(mg + ma_y\right)\ell_2 = I\alpha$$

since $a_y = \ell_2\alpha$ ⇒ $T_1\ell_1 - \left(mg + m\ell_2\alpha\right)\ell_2 = I\alpha$

Solve for α ⇒

$$\alpha = \frac{T_1\ell_1 - mg\ell_2}{I + m\ell_2^2} = \frac{(2150\ \mathrm{N})(0.600\ \mathrm{m}) - (451\ \mathrm{kg})(9.80\ \mathrm{m/s^2})(0.200\ \mathrm{m})}{46.0\ \mathrm{kg}\cdot\mathrm{m}^2 + (451\ \mathrm{kg})(0.200\ \mathrm{m})^2} = 6.3\ \mathrm{rad/s^2}$$

Work, Power, and Energy in Rotation



Let's consider the motion of a rigid body with a single external force $\mathcal{F}$ exerting tangentially, moving the object by s.

The rotational work done by the force $\mathcal{F}$ as the object rotates through the distance s=rθ is

$$W = Fs = Fr\theta$$

Since the magnitude of torque is r$\mathcal{F}$,

$$W = Fr\theta = \tau\theta$$

What is the unit of the rotational work? J (Joules)

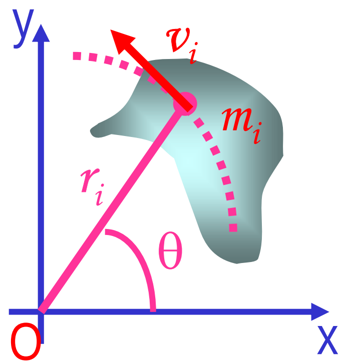
The rate of work, or power, of the constant torque τ becomes

$$P = \frac{\Delta W}{\Delta t} = \tau \frac{\Delta \theta}{\Delta t} = \tau\omega$$

How was the power defined in linear motion?

What is the unit of the rotational power? J/s or W (watts)

Rotational Kinetic Energy



What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, m_i, moving at a tangential speed, v_i, is

$$K_i = \frac{1}{2} m_i v_{Ti}^2 = \frac{1}{2} m_i r_i^2 \omega^2$$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

$$KE_R = \sum_i K_i = \frac{1}{2}\sum_i m_i r_i^2 \omega^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

Since moment of Inertia, I, is defined as

$$I = \sum_i m_i r_i^2$$

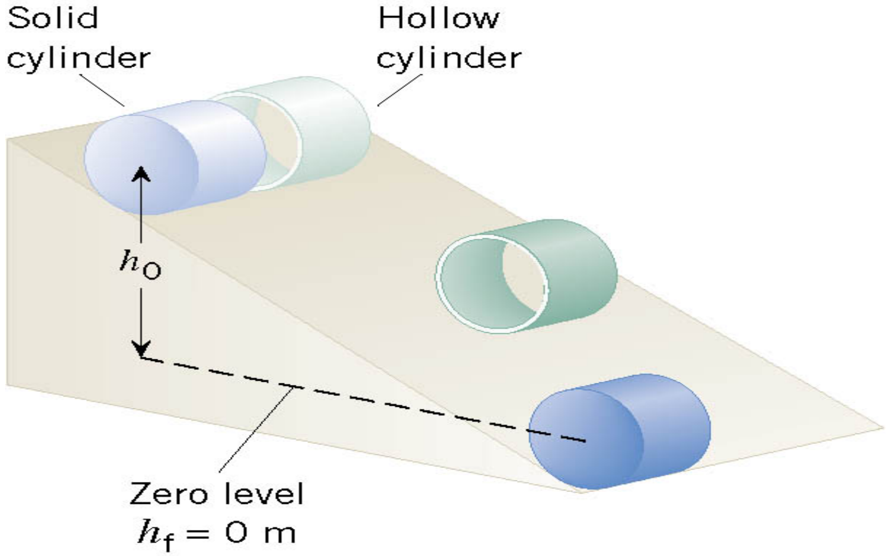
The above expression is simplified as

$$KE_R = \frac{1}{2} I \omega^2$$

Unit? J

Ex. 13 Rolling Cylinders

A thin-walled hollow cylinder (mass = m_h, radius = r_h) and a solid cylinder (mass = m_s, radius = r_s) start from rest at the top of an incline. Determine which cylinder has the greatest translational speed upon reaching the bottom.



Total Mechanical Energy = KE+ KE_R+PE

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 + mgh$$

From Energy Conservation $$\tfrac{1}{2}mv_f^2 + \tfrac{1}{2}I\omega_f^2 + mgh_f = \tfrac{1}{2}mv_i^2 + \tfrac{1}{2}I\omega_i^2 + mgh_0$$

$$\tfrac{1}{2}mv_f^2 + \tfrac{1}{2}I\omega_f^2 = mgh_i$$ since $\omega_f = \frac{v_f}{r}$ $$\tfrac{1}{2}mv_f^2 + \tfrac{1}{2}I\frac{v_f^2}{r^2} = mgh_0$$

Solve for v_f $$v_f = \sqrt{\frac{2mgh_o}{m + I/r^2}}$$

What does this tell you?

The cylinder with the smaller moment of inertia will have a greater final translational speed.

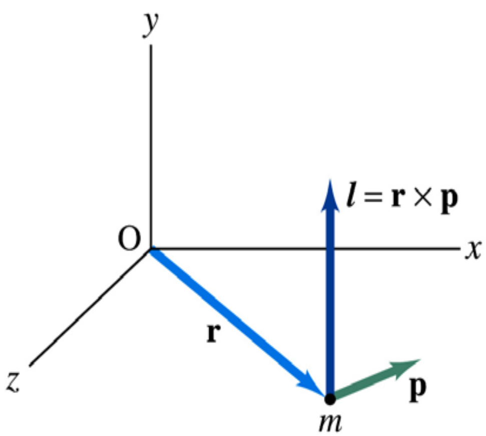
The final speeds of the cylinders are

$$v_f^h = \sqrt{\frac{2mgh_o}{m + I_h/r^2}} = \sqrt{\frac{2mgh_o}{m + mr_h^2/r_h^2}} = \sqrt{\frac{2mgh_o}{2m}} = \sqrt{gh_o}$$

$$v_f^s = \sqrt{\frac{2mgh_o}{m + I_s/r^2}} = \sqrt{\frac{2mgh_o}{m + \frac{1}{2}mr_s^2/r_s^2}} = \sqrt{\frac{2mgh_o}{\frac{3}{2}m}} = \sqrt{\frac{4}{3}gh_o} = \sqrt{\frac{4}{3}}v_f^h = 1.15v_f^h$$

Angular Momentum of a Particle

If you grab onto a pole while running, your body will rotate about the pole, gaining angular momentum. We've used the linear momentum to solve physical problems with linear motions, the angular momentum will do the same for rotational motions.



Let's consider a point-like object (particle) with mass m located at the vector location r and moving with linear velocity v

The angular momentum L of this particle relative to the origin O is

$$\left|\vec{L}\right| \equiv \left|\vec{r}\right|\left|\vec{p}\right|\sin\phi$$

What is the unit and dimension of angular momentum? $kg \cdot m^2/s \quad [ML^2T^{-1}]$

Note that L depends on origin O. Why? Because r changes

What else do you learn? The direction of L is +z

Since p is mv, the magnitude of L becomes $L = mvr = mr^2\varpi = I\varpi$

What do you learn from this?

If the direction of linear velocity points to the origin of rotation, the particle does not have any angular momentum.

If the linear velocity is perpendicular to position vector, the particle moves exactly the same way as a point on a rim.

Conservation of Angular Momentum

Remember under what condition the linear momentum is conserved?

Linear momentum is conserved when the net external force is 0.

$$\sum \vec{F} = 0 = \frac{\Delta \vec{p}}{\Delta t}$$

$$\vec{p} = const$$

By the same token, the angular momentum of a system is constant in both magnitude and direction, if the resultant external torque acting on the system is 0.

$$\sum \vec{\tau}_{ext} = \frac{\Delta \vec{L}}{\Delta t} = 0$$

$$\vec{L} = const$$

What does this mean?

Angular momentum of the system before and after a certain change is the same.

$$\vec{L}_i = \vec{L}_f = \text{constant}$$

Three important conservation laws for isolated system that does not get affected by external forces

$K_i + U_i = K_f + U_f$	Mechanical Energy
$\vec{p}_i = \vec{p}_f$	Linear Momentum
$\vec{L}_i = \vec{L}_f$	Angular Momentum

Example for Angular Momentum Conservation

A star rotates with a period of 30 days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0×10^4 km, collapses into a neutron star of radius 3.0km. Determine the period of rotation of the neutron star.

What is your guess about the answer?

The period will be significantly shorter, because its radius got smaller.

Let's make some assumptions:

1. There is no external torque acting on it
2. The shape remains spherical
3. Its mass remains constant

Using angular momentum conservation

$$L_i = L_f$$

$$I_i \omega_i = I_f \omega_f$$

The angular speed of the star with the period T is

$$\omega = \frac{2\pi}{T}$$

Thus

$$\omega_f = \frac{I_i \omega_i}{I_f} = \frac{m r_i^2}{m r_f^2} \frac{2\pi}{T_i}$$

$$T_f = \frac{2\pi}{\omega_f} = \left(\frac{r_f^2}{r_i^2} \right) T_i = \left(\frac{3.0}{1.0 \times 10^4} \right)^2 \times 30\ days = 2.7 \times 10^{-6}\ days = 0.23\ s$$

Ex. 14 A Spinning Skater

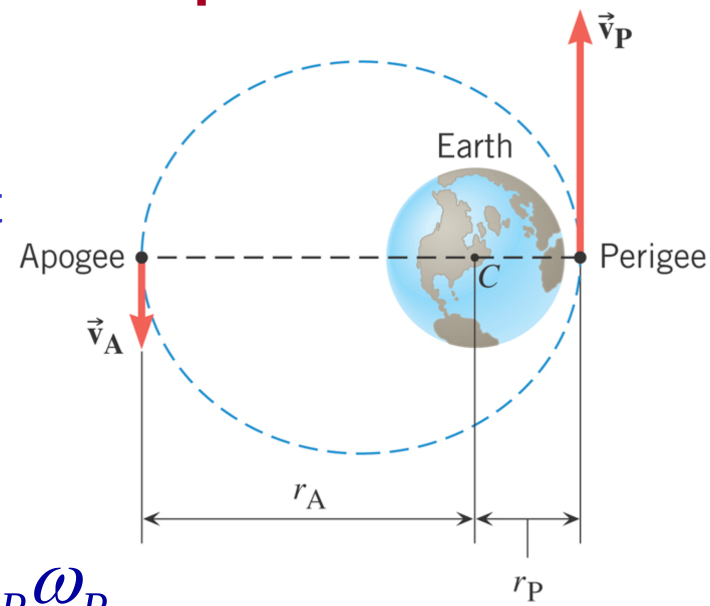
An ice skater is spinning with both arms and a leg outstretched. She pulls her arms and leg inward and her spinning motion changes dramatically. Use the principle of conservation of angular momentum to explain how and why her spinning motion changes.

(a)

The system of the ice skater does not have any net external torque applied to her. Therefore the angular momentum is conserved for her system. By pulling her arm inward, she reduces the moment of inertia (Σmr^2) and thus in order to keep the angular momentum the same, her angular speed has to increase.

Ex. 15 A Satellite in an Elliptical Orbit

A satellite is placed in an elliptical orbit about the earth. Its point of closest approach is $8.37\text{x}10^6$m from the center of the earth, and its point of greatest distance is $25.1\text{x}10^6$m from the center of the earth. The speed of the satellite at the perigee is 8450 m/s. Find the speed at the apogee.



Angular momentum is $L = I\omega$

From angular momentum conservation $I_A\omega_A = I_P\omega_P$

since $I = mr^2$ and $\omega = v/r$ ⇒ $\cancel{m}r_A^2 \frac{v_A}{r_A} = \cancel{m}r_P^2 \frac{v_P}{r_P}$ ⇒ $r_A v_A = r_P v_P$

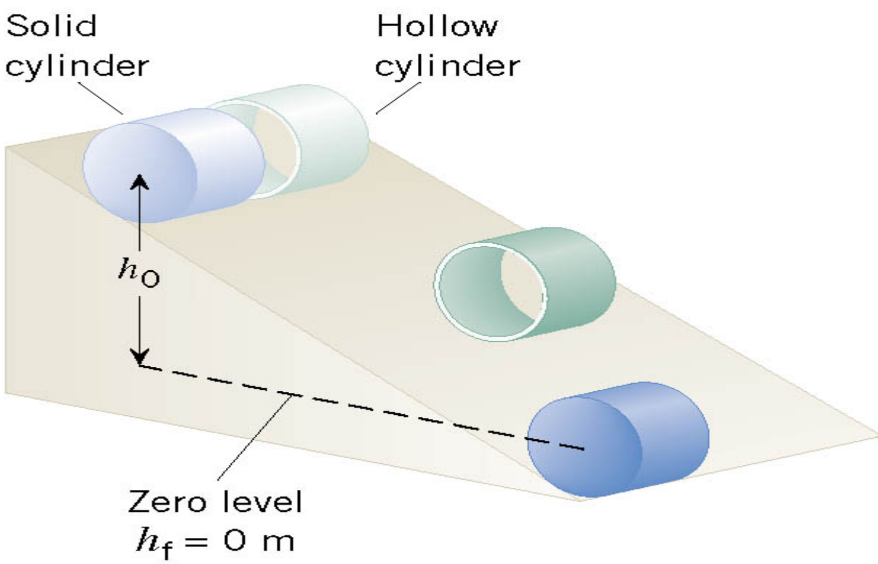
Solve for v_A ⇒ $$v_A = \frac{r_P v_P}{r_A} = \frac{(8.37\times10^6\ \text{m})(8450\,\text{m/s})}{25.1\times10^6\ \text{m}} = 2820\,\text{m/s}$$

Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Quantities	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = mr^2$
Length of motion	Distance L	Angle θ (Radian)
Speed	$v = \frac{\Delta r}{\Delta t}$	$\omega = \frac{\Delta \theta}{\Delta t}$
Acceleration	$a = \frac{\Delta v}{\Delta t}$	$\alpha = \frac{\Delta \omega}{\Delta t}$
Force	Force $\vec{F} = m\vec{a}$	Torque $\vec{\tau} = I\vec{\alpha}$
Work	Work $W = \vec{F} \cdot \vec{d}$	Work $W = \tau\theta$
Power	$P = \vec{F} \cdot \vec{v}$	$P = \tau\omega$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{\omega}$
Kinetic Energy	Kinetic $K = \frac{1}{2}mv^2$	Rotational $K_R = \frac{1}{2}I\omega^2$

A thought problem



- Consider two cylinders – one hollow (mass m_h and radius r_h) and the other solid (mass m_s and radius r_s) – on top of an inclined surface of height h_0 as shown in the figure. Show mathematically how their final speeds at the bottom of the hill compare in the following cases:

1. Totally frictionless surface
2. With some friction but no energy loss due to the friction
3. With energy loss due to kinetic friction

- Moment of Inertia
  - Hollow cylinder: $I_h = mr_h^2$
  - Solid Cylinder: $I_s = \frac{1}{2}mr_s^2$