PHYS 1443 – Section 001
Lecture #15

Thursday, June 30, 2011

Dr. Jaehoon Yu

- Relationship Between Angular and Linear quantities
- Torque and Vector Product
- Moment of Inertia
- Calculation of Moment of Inertia
- Torque and Angular Acceleration

Today's homework is homework #8, due 10pm, Monday, July4!!

Reminder: Extra-Credit Special Project

- Derive the formula for the final velocity of two objects which underwent an elastic collision as a function of known quantities m_1, m_2, v_{01} and v_{02} in page 8 of lecture note on Tuesday, June 28, in a far greater detail than in the note.
  - 20 points extra credit
- Show mathematically what happens to the final velocities if $m_1=m_2$ and explain in detail in words the resulting motion.
  - 5 point extra credit
- NO Credit will be given if the process is too close to the note!
- Due: Start of the class Tuesday, July 5

Extra Credit: 2-D Collisions

- Proton #1 with a speed 5.0×10^6 m/s collides elastically with proton #2 initially at rest. After the collision, proton #1 moves at an angle of 37^o to the horizontal axis and proton #2 deflects at an angle ϕ to the same axis. Find the final speeds of the two protons and the scattering angle of proton #2, ϕ. This must be done in much more detail than the book or on page 13 of lecture note on Tuesday, June 28.
- 10 points
- Due beginning of the class Wednesday, July 6

Rotational Kinematics

The first type of motion we have learned in linear kinematics was under the constant acceleration. We will learn about the rotational motion under constant angular acceleration, because these are the simplest motions in both cases.

Just like the case in linear motion, one can obtain

Angular velocity under constant angular acceleration:

$$\omega_f = \omega_0 + \alpha t$$

Linear kinematics $v = v_o + at$

Angular displacement under constant angular acceleration:

$$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

Linear kinematics $x_f = x_0 + v_o t + \frac{1}{2} a t^2$

One can also obtain

$$\omega_f^2 = \omega_0^2 + 2\alpha\left(\theta_f - \theta_0\right)$$

Linear kinematics $v_f^2 = v_o^2 + 2a\left(x_f - x_i\right)$

Ex. 10 – 4: Rotational Kinematics

A wheel rotates with a constant angular acceleration of 3.50 rad/s^2. If the angular speed of the wheel is 2.00 rad/s at t_i=0, a) through what angle does the wheel rotate in 2.00s?

Using the angular displacement formula in the previous slide, one gets

$$\theta_f - \theta_i = \omega t + \frac{1}{2}\alpha t^2$$

$$= 2.00 \times 2.00 + \frac{1}{2} 3.50 \times (2.00)^2 = 11.0 rad$$

$$= \frac{11.0}{2\pi} rev. = 1.75 rev.$$

Example for Rotational Kinematics cnt'd

What is the angular speed at t=2.00s?

Using the angular speed and acceleration relationship

$$\omega_f = \omega_i + \alpha t = 2.00 + 3.50 \times 2.00 = 9.00 rad/s$$

Find the angle through which the wheel rotates between t=2.00s and t=3.00s.

Using the angular kinematic formula $\theta_f - \theta_i = \omega t + \frac{1}{2}\alpha t^2$

At t=2.00s $\theta_{t=2} = 2.00 \times 2.00 + \frac{1}{2} 3.50 \times 2.00 = 11.0 rad$

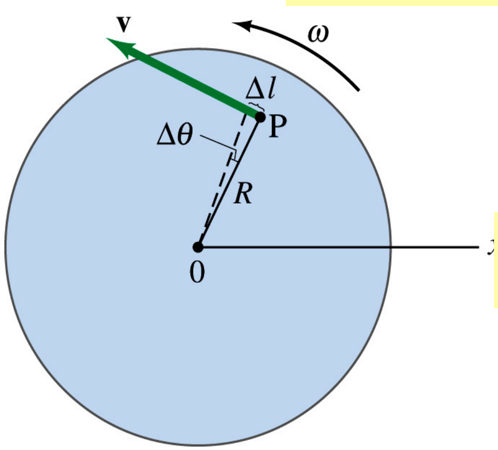
At t=3.00s $\theta_{t=3} = 2.00 \times 3.00 + \frac{1}{2} 3.50 \times (3.00)^2 = 21.8 rad$

Angular displacement $\Delta\theta = \theta_3 - \theta_2 = 10.8 rad = \frac{10.8}{2\pi} rev. = 1.72 rev.$

Relationship Between Angular and Linear Quantities

What do we know about a rigid object that rotates about a fixed axis of rotation?

Every particle (or masslet) in the object moves on a circle centered at the same axis of rotation.



When a point rotates, it has both the linear and angular components in its motion.

What is the linear component of the motion in the figure?

Linear velocity along the tangential direction.

The direction of ω follows the right-hand rule.

How do we related this linear component of the motion with angular component?

The arc-length is $\boxed{l = r\theta}$ So the tangential speed v is $v = \frac{dl}{dt} = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} = r\omega$

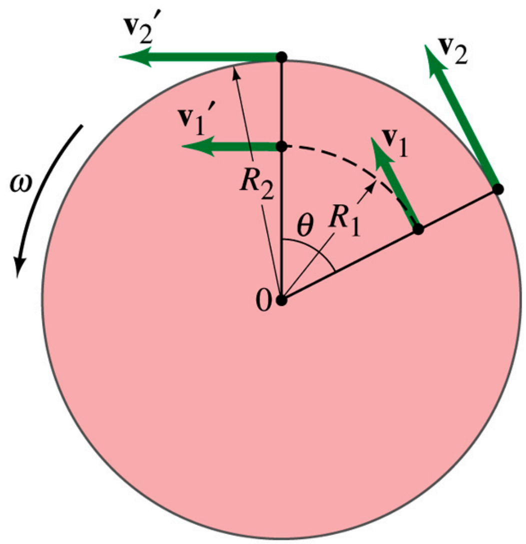
What does this relationship tell you about the tangential speed of the points in the object and their angular speed?:

Although every particle in the object has the same angular speed, its tangential speed differs and is proportional to its distance from the axis of rotation.

The farther away the particle is from the center of rotation, the higher the tangential speed.

Is the lion faster than the horse?

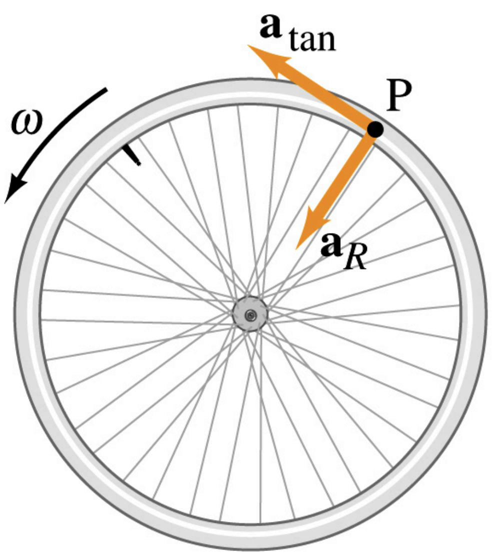
A rotating carousel has one child sitting on a horse near the outer edge and another child on a lion halfway out from the center. (a) Which child has the greater linear speed? (b) Which child has the greater angular speed?



(a) Linear speed is the distance traveled divided by the time interval. So the child sitting at the outer edge travels more distance within the given time than the child sitting closer to the center. Thus, the horse is faster than the lion.

(b) Angular speed is the angle traveled divided by the time interval. The angle both the children travel in the given time interval is the same. Thus, both the horse and the lion have the same angular speed.

How about the acceleration?



How many different linear acceleration components do you see in a circular motion and what are they? Two

Tangential, a_t, and the radial acceleration, a_r.

Since the tangential speed v is $v = r\omega$

The magnitude of tangential acceleration a_t is $a_t = \frac{dv}{dt} = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} = r\alpha$

What does this relationship tell you?

Although every particle in the object has the same angular acceleration, its tangential acceleration differs proportional to its distance from the axis of rotation.

The radial or centripetal acceleration a_r is $a_r = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$

What does this tell you?

The father away the particle is from the rotation axis, the more radial acceleration it receives. In other words, it receives more centripetal force.

Total linear acceleration is $a = \sqrt{a_t^2 + a_r^2} = \sqrt{(r\alpha)^2 + (r\omega^2)^2} = r\sqrt{\alpha^2 + \omega^4}$

Example

(a) What is the linear speed of a child seated 1.2m from the center of a steadily rotating merry-go-around that makes one complete revolution in 4.0s? (b) What is her total linear acceleration?

First, figure out what the angular speed of the merry-go-around is.

$$\varpi = \frac{1rev}{4.0s} = \frac{2\pi}{4.0s} = 1.6rad/s$$

Using the formula for linear speed

$$v = r\omega = 1.2m \times 1.6rad/s = 1.9m/s$$

Since the angular speed is constant, there is no angular acceleration.

Tangential acceleration is

$$a_t = r\alpha = 1.2m \times 0rad/s^2 = 0m/s^2$$

Radial acceleration is

$$a_r = r\omega^2 = 1.2m \times (1.6rad/s)^2 = 3.1m/s^2$$

Thus the total acceleration is

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{0 + (3.1)^2} = 3.1m/s^2$$

Example for Rotational Motion

Audio information on compact discs are transmitted digitally through the readout system consisting of laser and lenses. The digital information on the disc are stored by the pits and flat areas on the track. Since the speed of readout system is constant, it reads out the same number of pits and flats in the same time interval. In other words, the linear speed is the same no matter which track is played. a) Assuming the linear speed is 1.3 m/s, find the angular speed of the disc in revolutions per minute when the inner most (r=23mm) and outer most tracks (r=58mm) are read.

Using the relationship between angular and tangential speed $v = r\omega$

$$r = 23mm \qquad \omega = \frac{v}{r} = \frac{1.3m/s}{23mm} = \frac{1.3}{23\times10^{-3}} = 56.5rad/s$$

$$= 9.00rev/s = 5.4\times10^{2}\,rev/\min$$

$$r = 58mm \qquad \omega = \frac{1.3m/s}{58mm} = \frac{1.3}{58\times10^{-3}} = 22.4rad/s$$

$$= 2.1\times10^{2}\,rev/\min$$

b) The maximum playing time of a standard music CD is 74 minutes and 33 seconds. How many revolutions does the disk make during that time?

$$\overline{\omega} = \frac{(\omega_i + \omega_f)}{2} = \frac{(540 + 210)rev/\min}{2} = 375 rev/\min$$

$$\theta_f = \theta_i + \overline{\omega} t = 0 + \frac{375}{60} rev/s \times 4473s = 2.8 \times 10^4 rev$$

c) What is the total length of the track past through the readout mechanism?

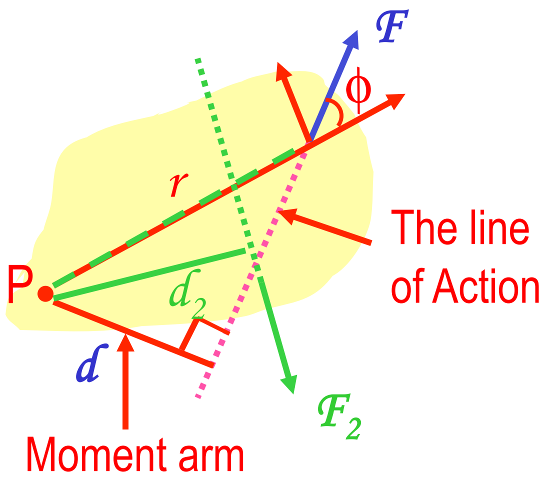
$$l = v_t \Delta t = 1.3 m/s \times 4473 s = 5.8 \times 10^3 m$$

d) What is the angular acceleration of the CD over the 4473s time interval, assuming constant a?

$$\alpha = \frac{(\omega_f - \omega_i)}{\Delta t} = \frac{(22.4 - 56.5) rad/s}{4473 s} = 7.6 \times 10^{-3} rad/s^2$$

Torque

Torque is the tendency of a force to rotate an object about an axis. Torque, τ, is a vector quantity.



Consider an object pivoting about the point P by the force F being exerted at a distance r from P.

The line that extends out of the tail of the force vector is called the line of action.

The perpendicular distance from the pivoting point P to the line of action is called the moment arm.

Magnitude of torque is defined as the product of the force exerted on the object to rotate it and the moment arm.

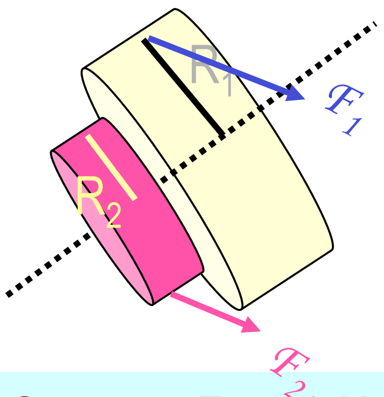
$$\tau \equiv rF\sin\phi = Fd$$

When there are more than one force being exerted on certain points of the object, one can sum up the torque generated by each force vectorially. The convention for sign of the torque is **positive if rotation is in counter-clockwise** and **negative if clockwise**.

$$\sum \tau = \tau_1 + \tau_2$$
$$= F_1 d_1 - F_2 d_2$$

Ex. 10 – 7: Torque

A one piece cylinder is shaped as in the figure with core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the picture. A rope wrapped around the drum whose radius is R_1 exerts force F_1 to the right on the cylinder, and another force exerts F_2 on the core whose radius is R_2 downward on the cylinder. A) What is the net torque acting on the cylinder about the rotation axis?



The torque due to F_1 $\tau_1 = -R_1F_1$ and due to F_2 $\tau_2 = R_2F_2$

So the total torque acting on the system by the forces is

$$\sum\tau = \tau_1 + \tau_2 = -R_1F_1 + R_2F_2$$

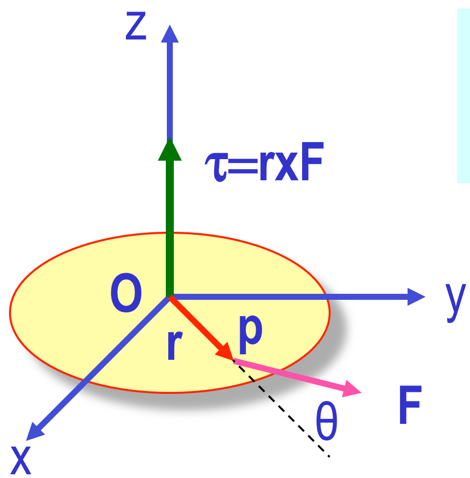
Suppose F_1=5.0 N, R_1=1.0 m, F_2= 15.0 N, and R_2=0.50 m. What is the net torque about the rotation axis and which way does the cylinder rotate from the rest?

Using the above result

$$\sum\tau = -R_1F_1 + R_2F_2$$
$$= -5.0\times1.0 + 15.0\times0.50 = 2.5\ N\cdot m$$

The cylinder rotates in counter-clockwise.

Torque and Vector Product



Let's consider a disk fixed onto the origin O and the force $\mathcal{F}$ exerts on the point p. What happens?

The disk will start rotating counter clockwise about the Z axis

The magnitude of torque given to the disk by the force $\mathcal{F}$ is

$$\tau = Fr\sin\theta$$

But torque is a vector quantity, what is the direction?
How is torque expressed mathematically?

$$\boxed{\vec{\tau} \equiv \vec{r} \times \vec{F}}$$

What is the direction? The direction of the torque follows the right-hand rule!!

The above operation is called the Vector product or Cross product

$$\vec{C} \equiv \vec{A} \times \vec{B}$$

$$\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$$

What is the result of a vector product?

Another vector

What is another vector operation we've learned?

Scalar product

$$C \equiv \vec{A} \cdot \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$$

Result? A scalar

Properties of Vector Product

Vector Product is Non-commutative

What does this mean?

If the order of operation changes the result changes $\vec{A}\times\vec{B}\neq\vec{B}\times\vec{A}$

Following the right-hand rule, the direction changes $\boxed{\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}}$

Vector Product of two parallel vectors is 0.

$$\left|\vec{C}\right|=\left|\vec{A}\times\vec{B}\right|=\left|\vec{A}\right|\left|\vec{B}\right|\sin\theta=\left|\vec{A}\right|\left|\vec{B}\right|\sin 0=0$$

Thus, $\boxed{\vec{A}\times\vec{A}=0}$

If two vectors are perpendicular to each other

$$\left|\vec{A}\times\vec{B}\right|=\left|\vec{A}\right|\left|\vec{B}\right|\sin\theta=\left|\vec{A}\right|\left|\vec{B}\right|\sin 90^\circ=\left|\vec{A}\right|\left|\vec{B}\right|=AB$$

Vector product follows distribution law

$$\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

The derivative of a Vector product with respect to a scalar variable is

$$\frac{d\left(\vec{A}\times\vec{B}\right)}{dt}=\frac{d\vec{A}}{dt}\times\vec{B}+\vec{A}\times\frac{d\vec{B}}{dt}$$

More Properties of Vector Product

The relationship between unit vectors, $\vec{i}$, $\vec{j}$ and $\vec{k}$

$$\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0$$

$$\vec{i}\times\vec{j}=-\vec{j}\times\vec{i}=\vec{k}$$

$$\vec{j}\times\vec{k}=-\vec{k}\times\vec{j}=\vec{i}$$

$$\vec{k}\times\vec{i}=-\vec{i}\times\vec{k}=\vec{j}$$

Vector product of two vectors can be expressed in the following determinant form

$$\vec{A}\times\vec{B}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\vec{i}\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}-\vec{j}\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}+\vec{k}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}$$

$$=\left(A_yB_z-A_zB_y\right)\vec{i}-\left(A_xB_z-A_zB_x\right)\vec{j}+\left(A_xB_y-A_yB_x\right)\vec{k}$$

Moment of Inertia

Rotational Inertia: Measure of resistance of an object to changes in its rotational motion. Equivalent to mass in linear motion.

For a group of particles $I \equiv \sum_i m_i r_i^2$

For a rigid body $I \equiv \int r^2 dm$

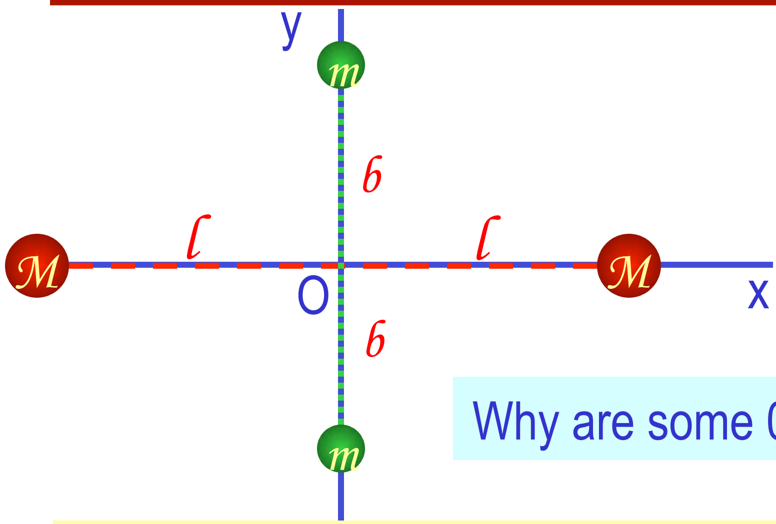
What are the dimension and unit of Moment of Inertia? $\left[ML^2\right]$ $kg \cdot m^2$

Determining Moment of Inertia is extremely important for computing equilibrium of a rigid body, such as a building.

Dependent on the axis of rotation!!!

Example for Moment of Inertia

In a system of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at angular speed ω.



Since the rotation is about y axis, the moment of inertia about y axis, I_y, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m \cdot 0^2 + m \cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2\right)\omega^2 = Ml^2\omega^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2\left(Ml^2 + mb^2\right) \qquad K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2 + 2mb^2\right)\omega^2 = \left(Ml^2 + mb^2\right)\omega^2$$

Calculation of Moments of Inertia

Moments of inertia for large objects can be computed, if we assume the object consists of small volume elements with mass, Δm_i.

The moment of inertia for the large rigid object is $I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$

It is sometimes easier to compute moments of inertia in terms of volume of the elements rather than their mass

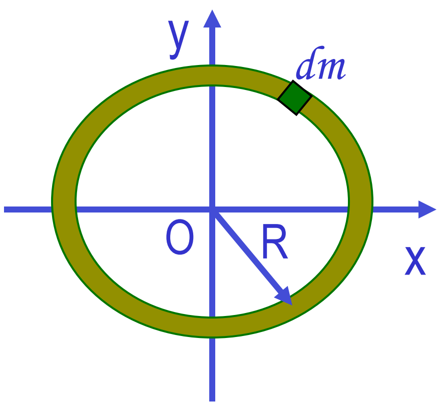
How can we do this?

Using the volume density, ρ, replace dm in the above equation with dV.

$$\rho = \frac{dm}{dV} \Rightarrow dm = \rho dV$$

The moments of inertia becomes $\boxed{I = \int \rho r^2 dV}$

Example: Find the moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.



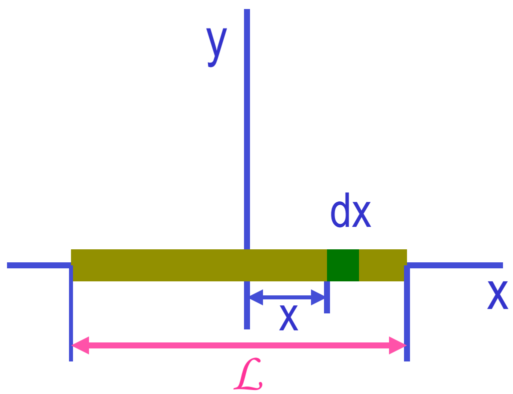
The moment of inertia is $I = \int r^2 dm = R^2 \int dm = MR^2$

What do you notice from this result?

The moment of inertia for this object is the same as that of a point of mass M at the distance R.

Ex.10 – 11 Rigid Body Moment of Inertia

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing through its center of mass.



The line density of the rod is $\lambda = \frac{M}{L}$

so the masslet is $dm = \lambda dx = \frac{M}{L}dx$

The moment of inertia is

$$I = \int r^2\,dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L}dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

What is the moment of inertia when the rotational axis is at one end of the rod.

$$I = \int r^2\,dm = \int_0^L \frac{x^2 M}{L}dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_0^L$$

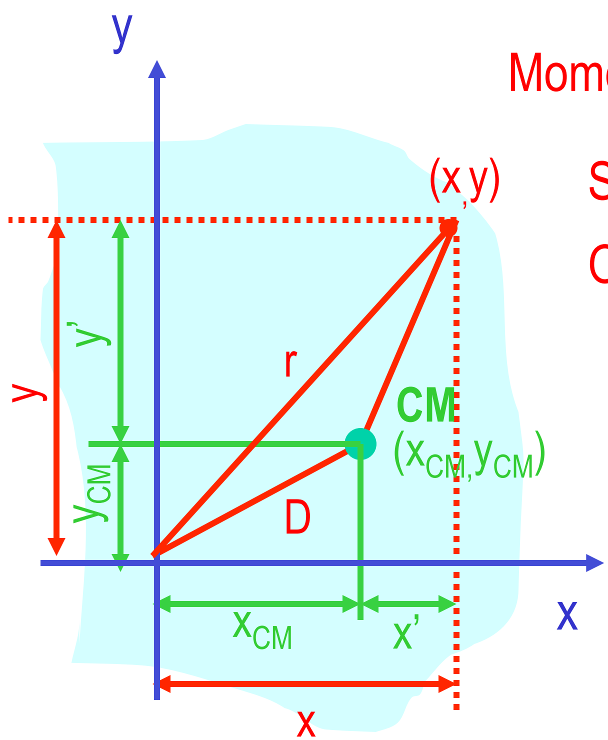
$$= \frac{M}{3L}\left[(L)^3 - 0\right] = \frac{M}{3L}\left(L^3\right) = \frac{ML^2}{3}$$

Will this be the same as the above. Why or why not?

Since the moment of inertia is resistance to motion, it makes perfect sense for it to be harder to move when it is rotating about the axis at one end.

Parallel Axis Theorem

Moments of inertia for highly symmetric object is easy to compute if the rotational axis is the same as the axis of symmetry. However if the axis of rotation does not coincide with axis of symmetry, the calculation can still be done in a simple manner using **parallel-axis theorem**. $I = I_{CM} + MD^2$



Moment of inertia is defined as $I = \int r^2\,dm = \int\left(x^2 + y^2\right)dm \quad (1)$

Since x and y are $x = x_{CM} + x' \quad y = y_{CM} + y'$

One can substitute x and y in Eq. 1 to obtain

$$I = \int\left[\left(x_{CM} + x'\right)^2 + \left(y_{CM} + y'\right)^2\right]dm$$

$$= \left(x_{CM}^2 + y_{CM}^2\right)\int dm + 2x_{CM}\int x'\,dm + 2y_{CM}\int y'\,dm + \int\left(x'^2 + y'^2\right)dm$$

Since the x' and y' are the distance from CM, by definition $\int x'\,dm = 0 \quad \int y'\,dm = 0$

Therefore, the parallel-axis theorem

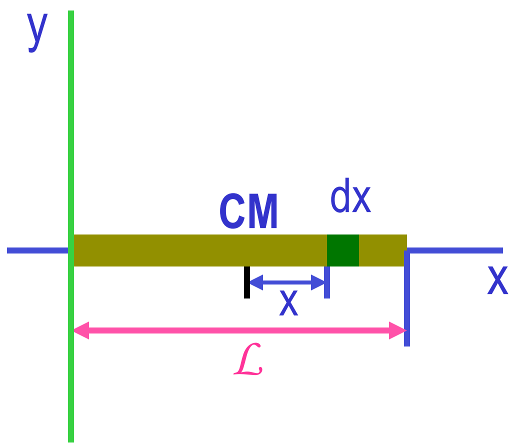
$$I = \left(x_{CM}^2 + y_{CM}^2\right)\int dm + \int\left(x'^2 + y'^2\right)dm = MD^2 + I_{CM}$$

What does this theorem tell you?

Moment of inertia of any object about any arbitrary axis are the same as the sum of moment of inertia for **a rotation about the CM** and **that of the CM about the rotation axis**.

Example for Parallel Axis Theorem

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis that goes through one end of the rod, using parallel-axis theorem.



The line density of the rod is $\lambda = \dfrac{M}{L}$

so the masslet is $dm = \lambda dx = \dfrac{M}{L} dx$

The moment of inertia about the CM

$$I_{CM} = \int r^2\, dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

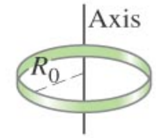
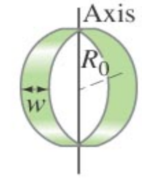
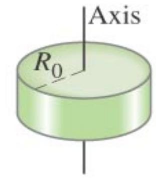
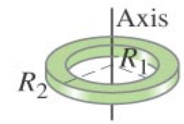
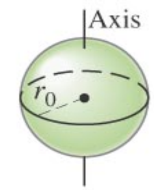
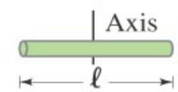
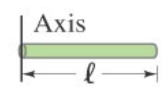
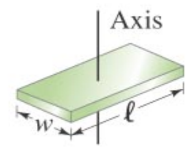
Using the parallel axis theorem

$$I = I_{CM} + D^2 M = \frac{ML^2}{12} + \left(\frac{L}{2}\right)^2 M = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

The result is the same as using the definition of moment of inertia.

Parallel-axis theorem is useful to compute moment of inertia of a rotation of a rigid object with complicated shape about an arbitrary axis

Check out Figure 10 – 20 for moment of inertia for various shaped objects

	Object	Location of axis		Moment of inertia
(a)	**Thin hoop,** radius R_0	Through center		MR_0^2
(b)	**Thin hoop,** radius R_0 width w	Through central diameter		$\frac{1}{2}MR_0^2 + \frac{1}{12}Mw^2$
(c)	**Solid cylinder,** radius R_0	Through center		$\frac{1}{2}MR_0^2$
(d)	**Hollow cylinder,** inner radius R_1 outer radius R_2	Through center		$\frac{1}{2}M(R_1^2 + R_2^2)$
(e)	**Uniform sphere,** radius r_0	Through center		$\frac{2}{5}Mr_0^2$
(f)	**Long uniform rod,** length ℓ	Through center		$\frac{1}{12}M\ell^2$
(g)	**Long uniform rod,** length ℓ	Through end		$\frac{1}{3}M\ell^2$
(h)	**Rectangular thin plate,** length ℓ, width w	Through center		$\frac{1}{12}M(\ell^2 + w^2)$