PHYS 1441 – Section 001
Lecture #16

Thursday, July 9, 2015
Dr. Jaehoon Yu

- Relationship Between Angular and Linear Quantities
- Rolling Motion of a Rigid Body
- Torque
- Moment of Inertia
- Angular Momentum & Its Conservation
- Equilibrium

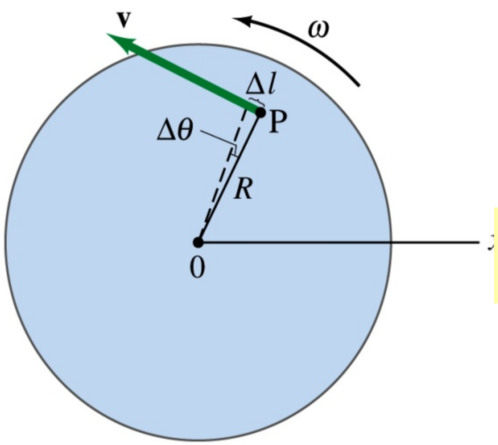
Announcements

- Reading Assignments: CH8.9 and 9.3
- Final exam
  - 10:30am – 12:30pm, Monday, July 13, in this room
  - Comprehensive exam, covers from CH1.1 – CH9.3 plus appendices A1 – A8
  - Bring your calculator but DO NOT input formula into it!
    - Your phones or portable computers are NOT allowed as a replacement!
  - You can prepare a one 8.5x11.5 sheet (front and back) of **handwritten** formulae and values of constants for the exam ➔ no solutions, derivations or definitions!
    - No additional formulae or values of constants will be provided!
- Quiz 4 results
  - Class average: 20.6/40
    - Equivalent to 51.5/100
    - Previous quizzes: 74.7/100, 50.8/100 and 65.4/100
  - Top score: 40/40

Relationship Between Angular and Linear Quantities

What do we know about a rigid object that rotates about a fixed axis of rotation?

Every particle (or masslet) in an object moves in a circle centered at the same axis of rotation with the same angular velocity.



When a point rotates, it has both the linear and angular components in its motion.

What is the linear component of the motion you see?

Linear velocity along the tangential direction.

How do we related this linear component of the motion with angular component?

The direction of ω follows the right-hand rule.

The arc-length is $l = r\theta$ So the tangential speed v is

$$v = \frac{\Delta l}{\Delta t} = \frac{\Delta(r\theta)}{\Delta t} = r\frac{\Delta\theta}{\Delta t} = r\omega$$

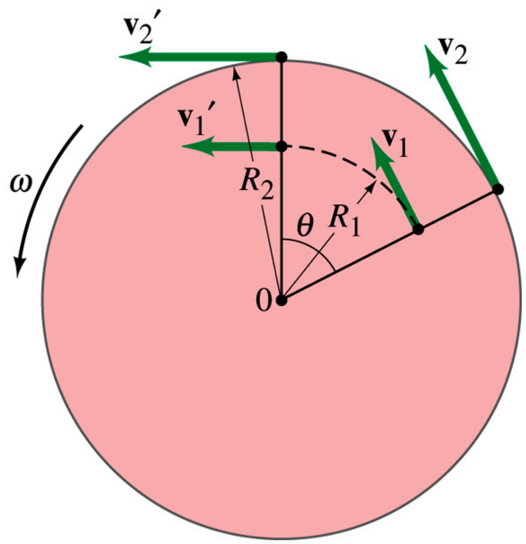
What does this relationship tell you about the tangential speed of the points in the object and their angular speed?

Although every particle in the object has the same angular speed, its tangential speed differs and is proportional to its distance from the axis of rotation.

The farther away the particle is from the center of rotation, the higher the tangential speed.

Is the lion faster than the horse?

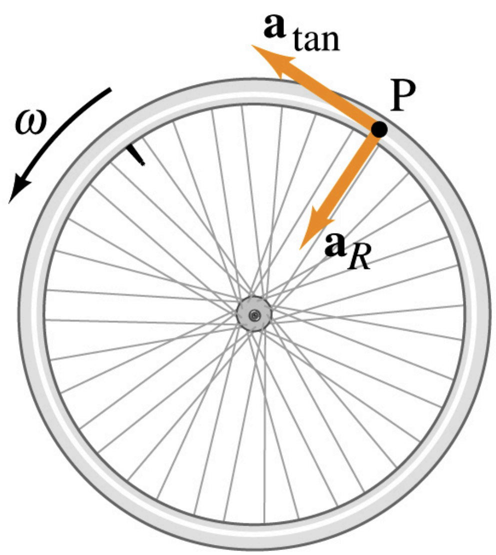
A rotating carousel has one child sitting on the horse near the outer edge and another child on the lion halfway out from the center. (a) Which child has the greater linear speed? (b) Which child has the greater angular speed?



(a) Linear speed is the distance traveled divided by the time interval. So the child sitting at the outer edge travels more distance within the given time than the child sitting closer to the center. Thus, the horse is faster than the lion.

(b) Angular speed is the angle traveled divided by the time interval. The angle both the children travel in the given time interval is the same. Thus, both the horse and the lion have the same angular speed.

How about the acceleration?



How many different linear acceleration components do you see in a circular motion and what are they? Two

Tangential, a_t, and the radial acceleration, a_r.

Since the tangential speed v is $v_t = r\omega$

The magnitude of tangential acceleration a_t is $a_t = \frac{v_{tf} - v_{t0}}{\Delta t} = \frac{r\omega_f - r\omega_0}{\Delta t} = r\frac{\omega_f - \omega_0}{\Delta t} = r\alpha$

What does this relationship tell you?

Although every particle in the object has the same angular acceleration, its tangential acceleration differs proportional to its distance from the axis of rotation.

The radial or centripetal acceleration a_r is $a_r = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$

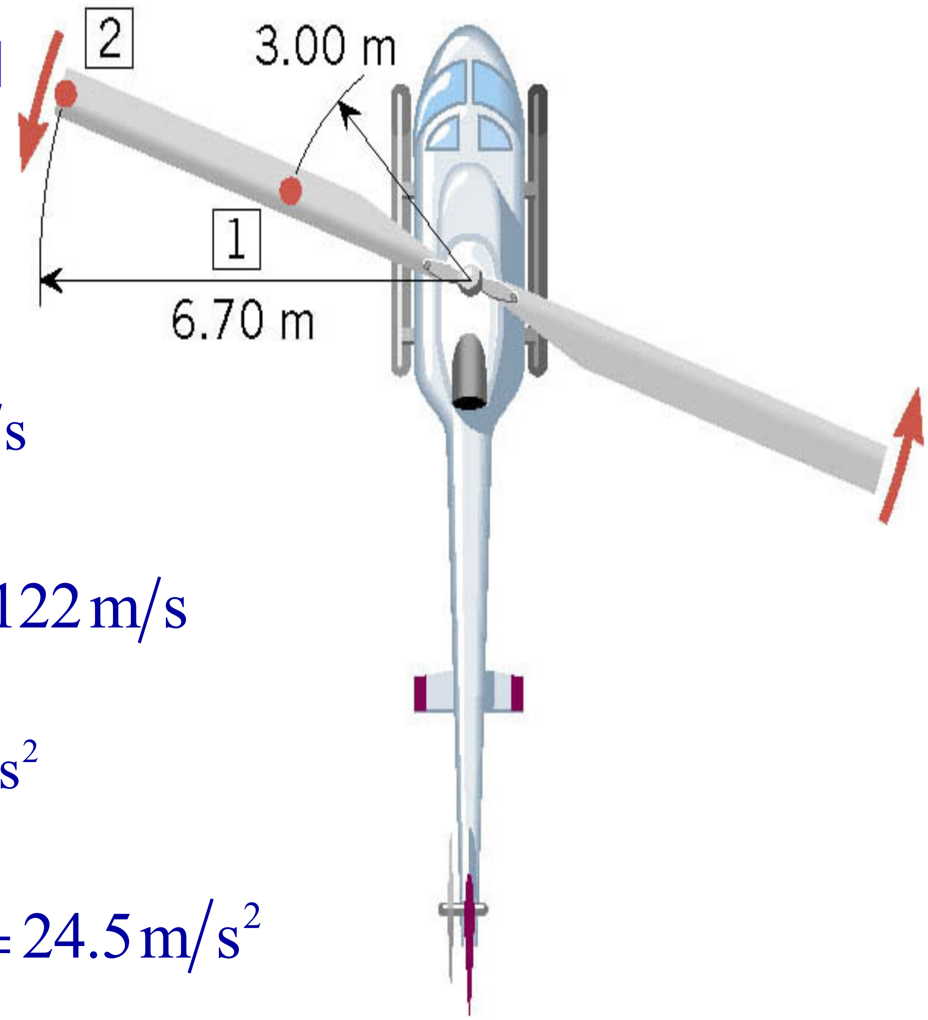
What does this tell you?

The father away the particle is from the rotation axis, the more radial acceleration it receives. In other words, it receives more centripetal force.

Total linear acceleration is $a = \sqrt{a_t^2 + a_r^2} = \sqrt{(r\alpha)^2 + (r\omega^2)^2} = r\sqrt{\alpha^2 + \omega^4}$

Ex. A Helicopter Blade

A helicopter blade has an angular velocity of +6.50 rev/s and an angular acceleration of +1.30 rev/s^2. For point 1 on the blade, find the magnitude of (a) the tangential speed and (b) the tangential acceleration.



$$\omega = \left(6.50\ \frac{\text{rev}}{\text{s}}\right)\left(\frac{2\pi\ \text{rad}}{1\ \text{rev}}\right) = 40.8\ \text{rad/s}$$

$$v_T = r\varpi = (3.00\ \text{m})(40.8\,\text{rad/s}) = 122\,\text{m/s}$$

$$\alpha = \left(1.30\ \frac{\text{rev}}{\text{s}^2}\right)\left(\frac{2\pi\ \text{rad}}{1\ \text{rev}}\right) = 8.17\ \text{rad/s}^2$$

$$a_T = r\alpha = (3.00\ \text{m})(8.17\,\text{rad/s}^2) = 24.5\,\text{m/s}^2$$

Rolling Motion of a Rigid Body

What is a rolling motion?

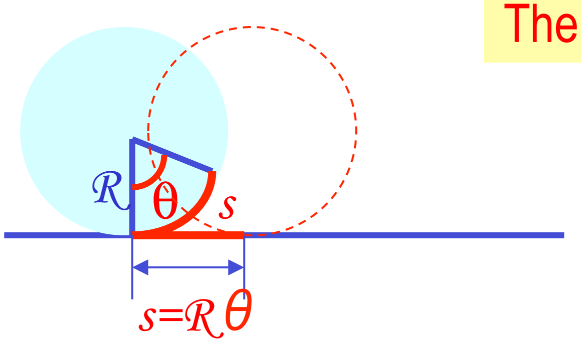
A more generalized case of a motion where the rotational axis moves together with an object

A rotational motion about a moving axis

To simplify the discussion, let's make a few assumptions

1. Limit our discussion on very symmetric objects, such as cylinders, spheres, etc
2. The object rolls on a flat surface

Let's consider a cylinder rolling on a flat surface, without slipping.



Under what condition does this "Pure Rolling" happen?

The total linear distance the CM of the cylinder moved is $s = R\theta$

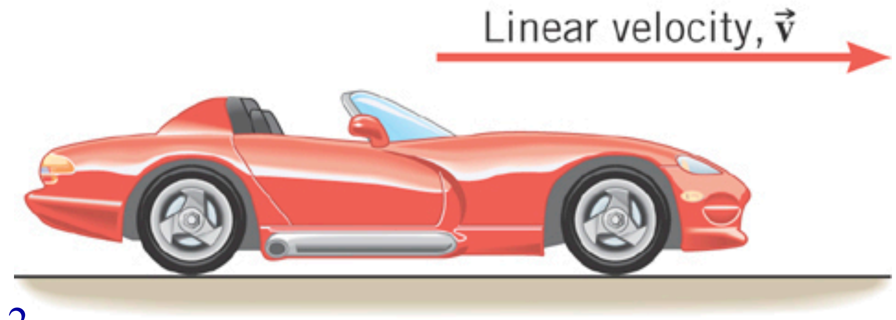
Thus the linear speed of the CM is

$$\bar{v}_{CM} = \frac{\Delta s}{\Delta t} = R\frac{\Delta \theta}{\Delta t} = R\omega$$

The condition for a "Pure Rolling motion"

Ex. An Accelerating Car

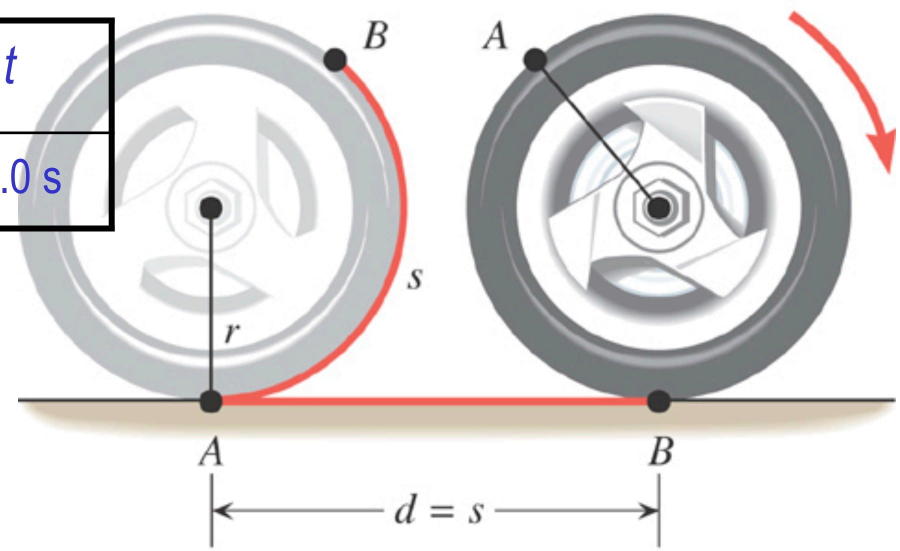
Starting from rest, the car accelerates for 20.0 s with a constant linear acceleration of 0.800 m/s². The radius of the tires is 0.330 m. What is the angle through which each wheel has rotated?



(a)

$$\alpha = \frac{a}{r} = \frac{0.800\,\mathrm{m/s^2}}{0.330\ \mathrm{m}} = 2.42\,\mathrm{rad/s^2}$$

θ	α	ω	ω_o	t
?	-2.42 rad/s²		0 rad/s	20.0 s

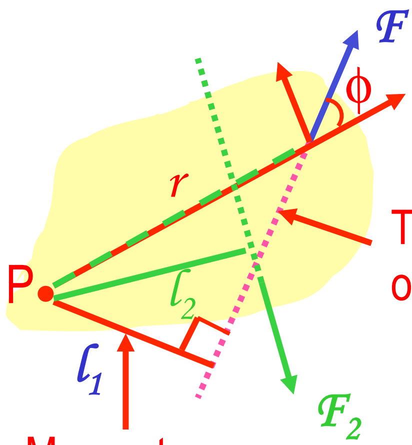


(b)

$$\theta = \omega_o t + \tfrac{1}{2}\alpha t^2$$
$$= \tfrac{1}{2}\left(-2.42\,\mathrm{rad/s^2}\right)^2 (20.0\ \mathrm{s})^2$$
$$= -484\ \mathrm{rad}$$

Torque

Torque is the tendency of a force to rotate an object about an axis. Torque, τ, is a vector quantity.



Consider an object pivoting about the point P by the force F being exerted at a distance r from P.

The line that extends out of the tail of the force vector is called the line of action.

The perpendicular distance from the pivoting point P to the line of action is called the moment arm.

$$\left|\vec{\tau}\right| \equiv (\text{Magnitude of the Force}) \times (\text{Lever Arm}) = (F)(r\sin\phi) = Fl$$

Magnitude of torque is defined as the product of the force exerted on the object to rotate it and the moment arm.

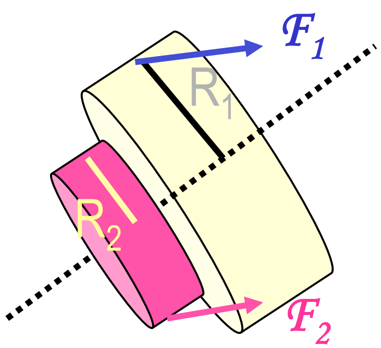
When there are more than one force being exerted on certain points of the object, one can sum up the torque generated by each force vectorially. The convention for sign of the torque is **positive if rotation is in counter-clockwise** and **negative if clockwise**.

$$\sum \tau = \tau_1 + \tau_2 = F_1 l_1 - F_2 l_2$$

Unit? $N \cdot m$

Example for Torque

A one piece cylinder is shaped as in the figure with core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the picture. A rope wrapped around the drum whose radius is R_1 exerts force F_1 to the right on the cylinder, and another force exerts F_2 on the core whose radius is R_2 downward on the cylinder. A) What is the net torque acting on the cylinder about the rotation axis?



The torque due to F_1 $\tau_1 = -R_1 F_1$ and due to F_2 $\tau_2 = R_2 F_2$

So the total torque acting on the system by the forces is

$$\sum \tau = \tau_1 + \tau_2 = -R_1 F_1 + R_2 F_2$$

Suppose F_1=5.0 N, R_1=1.0 m, F_2= 15.0 N, and R_2=0.50 m. What is the net torque about the rotation axis and which way does the cylinder rotate from the rest?

Using the above result

$$\sum \tau = -R_1 F_1 + R_2 F_2$$
$$= -5.0 \times 1.0 + 15.0 \times 0.50 = 2.5 N \bullet m$$

The cylinder rotates in counter-clockwise.

Moment of Inertia

Rotational Inertia: Measure of resistance of an object to changes in its rotational motion. Equivalent to mass in linear motion.

For a group of objects $$I \equiv \sum_i m_i r_i^2$$

For a rigid body $$I \equiv \int r^2 dm$$

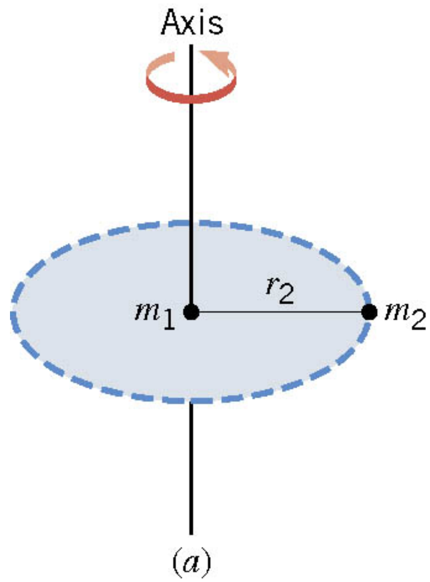
What are the dimension and unit of Moment of Inertia? $\left[ML^2\right]$ $kg \cdot m^2$

Determining Moment of Inertia is extremely important for computing equilibrium of a rigid body, such as a building.

Dependent on the axis of rotation!!!

Ex. The Moment of Inertia Depends on Where the Axis Is.

Two particles each have mass m_1 and m_2 and are fixed at the ends of a thin rigid rod. The length of the rod is *L*. Find the moment of inertia when this object rotates relative to an axis that is perpendicular to the rod at (a) one end and (b) the center.



(a) $I = \sum\left(mr^2\right) = m_1 r_1^2 + m_2 r_2^2$

$m_1 = m_2 = m \qquad r_1 = 0 \quad r_2 = L$

$$I = m(0)^2 + m(L)^2 = mL^2$$

(b) $I = \sum\left(mr^2\right) = m_1 r_1^2 + m_2 r_2^2$

$m_1 = m_2 = m \qquad r_1 = L/2 \quad r_2 = L/2$

$$I = m(L/2)^2 + m(L/2)^2 = \tfrac{1}{2}mL^2$$

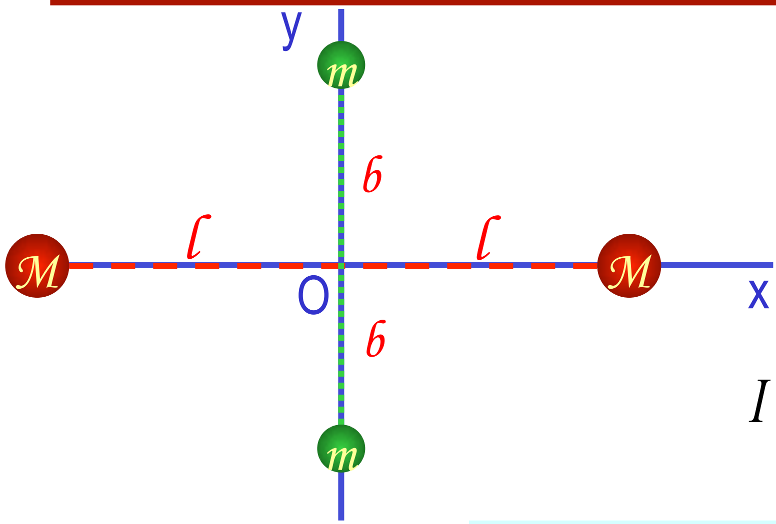
Which case is easier to spin?

Case (b)

Why? Because the moment of inertia is smaller

Example for Moment of Inertia

In a system of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at angular speed ω.



Since the rotation is about y axis, the moment of inertia about y axis, I_y, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m\cdot 0^2 + m\cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about the y axis, and the radii of the spheres are negligible.

Check out Figure 8 – 20 for moment of inertia for various shaped objects

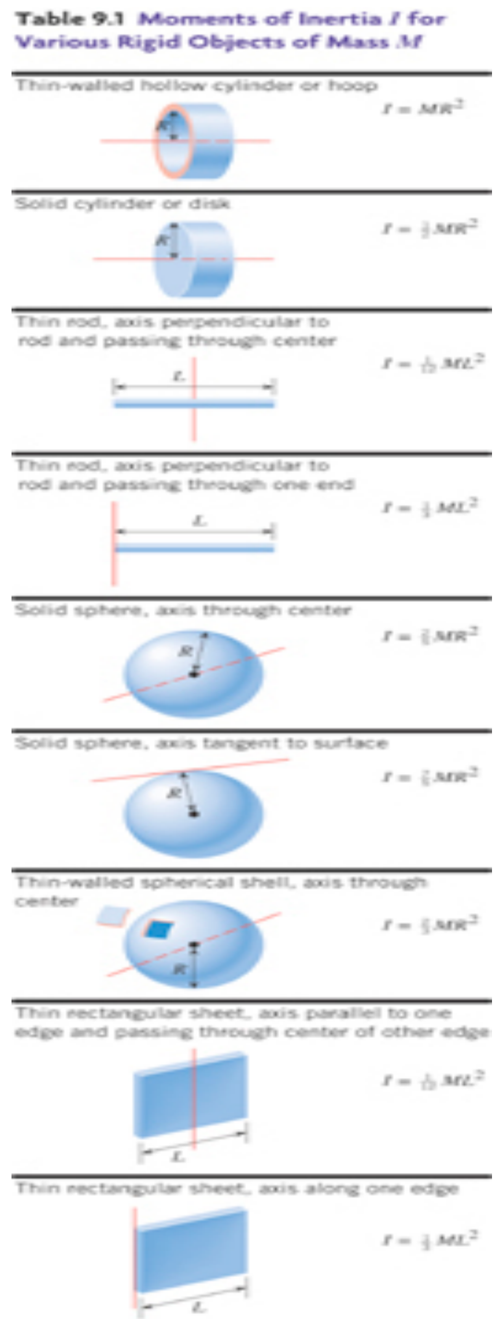
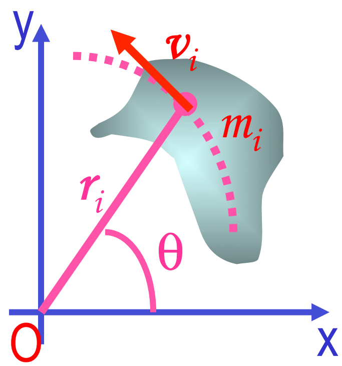


Table 9.1 Moments of Inertia I for Various Rigid Objects of Mass M

Thin-walled hollow cylinder or hoop	$I = MR^2$
Solid cylinder or disk	$I = \frac{1}{2}MR^2$
Thin rod, axis perpendicular to rod and passing through center	$I = \frac{1}{12}ML^2$
Thin rod, axis perpendicular to rod and passing through one end	$I = \frac{1}{3}ML^2$
Solid sphere, axis through center	$I = \frac{2}{5}MR^2$
Solid sphere, axis tangent to surface	$I = \frac{7}{5}MR^2$
Thin-walled spherical shell, axis through center	$I = \frac{2}{3}MR^2$
Thin rectangular sheet, axis parallel to one edge and passing through center of other edge	$I = \frac{1}{12}ML^2$
Thin rectangular sheet, axis along one edge	$I = \frac{1}{3}ML^2$

Rotational Kinetic Energy



What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, m_i, moving at a tangential speed, v_i, is $K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 \omega^2$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

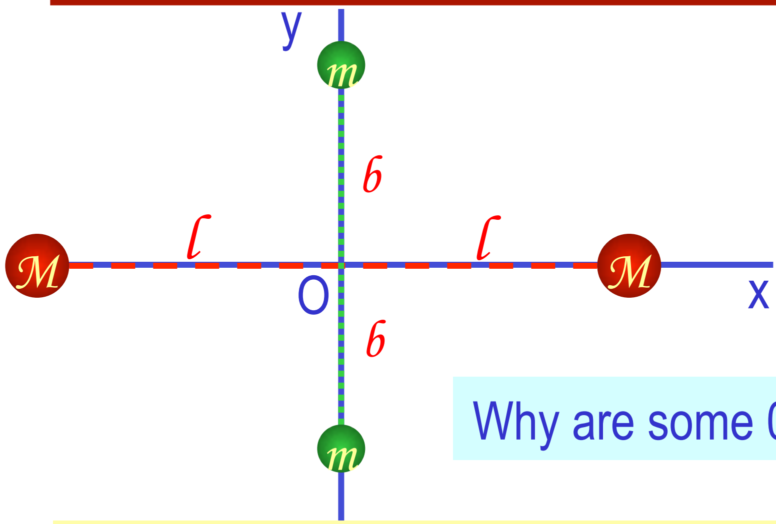
$$K_R = \sum_i K_i = \frac{1}{2} \sum_i m_i r_i^2 \omega^2 = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2$$

Since moment of Inertia, I, is defined as $I = \sum_i m_i r_i^2$

The above expression is simplified as $K_R = \frac{1}{2} I \omega^2$

Example for Moment of Inertia

In a system of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at angular speed ω.



Since the rotation is about y axis, the moment of inertia about y axis, I_y, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m\cdot 0^2 + m\cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

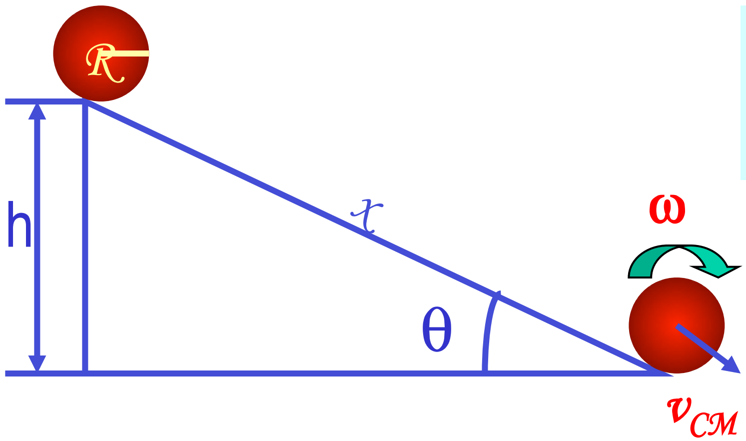
Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(2Ml^2\right)\omega^2 = Ml^2\omega^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2\left(Ml^2 + mb^2\right) \quad K_R = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(2Ml^2 + 2mb^2\right)\omega^2 = \left(Ml^2 + mb^2\right)\omega^2$$

Kinetic Energy of a Rolling Sphere



Let's consider a sphere with radius R rolling down the hill without slipping.

$$K = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M R^2 \omega^2$$

Since $v_{CM} = R\omega$

$$= \frac{1}{2} I_{CM} \left(\frac{v_{CM}}{R} \right)^2 + \frac{1}{2} M v_{CM}^2$$

$$= \frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}^2$$

What is the speed of the CM in terms of known quantities and how do you find this out?

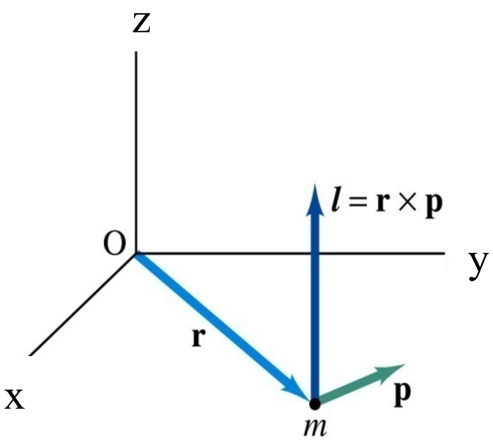
Since the kinetic energy at the bottom of the hill must be equal to the potential energy at the top of the hill

$$K = \frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}^2 = Mgh$$

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM} / MR^2}} = \sqrt{\frac{2gh}{1 + \left(2MR^2/5\right) / MR^2}} = \sqrt{\frac{2gh}{1 + 2/5}} = \sqrt{\frac{10gh}{7}}$$

Angular Momentum of a Particle

If you grab onto a pole while running, your body will rotate about the pole, gaining angular momentum. We've used the linear momentum to solve physical problems with linear motions, the angular momentum will do the same for rotational motions.



Let's consider a point-like object (particle) with mass m located at the vector location r and moving with linear velocity v

The angular momentum L of this particle relative to the origin O is

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

What is the unit and dimension of angular momentum? $kg \cdot m^2/s \quad [ML^2T^{-1}]$

Note that L depends on origin O. Why? Because r changes

What else do you learn? The direction of L is +z.

Since p is mv, the magnitude of L becomes $L = mvr = mr^2\varpi = I\varpi$

What do you learn from this?

If the direction of linear velocity points to the origin of rotation, the particle does not have any angular momentum.

If the linear velocity is perpendicular to position vector, the particle moves exactly the same way as a point on a rim.

Conservation of Angular Momentum

Remember under what condition the linear momentum is conserved?

Linear momentum is conserved when the net external force is 0.

$$\sum \vec{F} = 0 = \frac{\Delta \vec{p}}{\Delta t}$$

$$\vec{p} = const$$

By the same token, the angular momentum of a system is constant in both magnitude and direction, if the resultant external torque acting on the system is 0.

$$\sum \vec{\tau}_{ext} = \frac{\Delta \vec{L}}{\Delta t} = 0$$

$$\boxed{\vec{L} = const}$$

What does this mean?

Angular momentum of the system before and after a certain change is the same.

$$\vec{L}_i = \vec{L}_f = \text{constant}$$

Three important conservation laws for isolated system that does not get affected by external forces

$$K_i + U_i = K_f + U_f$$ **Mechanical Energy**

$$\vec{p}_i = \vec{p}_f$$ **Linear Momentum**

$$\vec{L}_i = \vec{L}_f$$ **Angular Momentum**

Example for Angular Momentum Conservation

A star rotates with a period of 30 days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0×10^4km, collapses into a neutron star of radius 3.0km. Determine the period of rotation of the neutron star.

What is your guess about the answer?

The period will be significantly shorter, because its radius got smaller.

Let's make some assumptions:

1. There is no external torque acting on it
2. The shape remains spherical
3. Its mass remains constant

Using angular momentum conservation

$$L_i = L_f$$

$$I_i\omega_i = I_f\omega_f$$

The angular speed of the star with the period T is

$$\omega = \frac{2\pi}{T}$$

Thus

$$\omega_f = \frac{I_i\omega_i}{I_f} = \frac{mr_i^2}{mr_f^2}\frac{2\pi}{T_i}$$

$$T_f = \frac{2\pi}{\omega_f} = \left(\frac{r_f^2}{r_i^2}\right)T_i = \left(\frac{3.0}{1.0\times10^4}\right)^2\times 30days = 2.7\times10^{-6}days = 0.23s$$

Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Quantities	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = mr^2$
Length of motion	Distance L	Angle θ (Radian)
Speed	$v = \frac{\Delta r}{\Delta t}$	$\omega = \frac{\Delta \theta}{\Delta t}$
Acceleration	$a = \frac{\Delta v}{\Delta t}$	$\alpha = \frac{\Delta \omega}{\Delta t}$
Force	Force $\vec{F} = m\vec{a}$	Torque $\vec{\tau} = I\vec{\alpha}$
Work	Work $W = \vec{F} \cdot \vec{d}$	Work $W = \tau\theta$
Power	$P = \vec{F} \cdot \vec{v}$	$P = \tau\omega$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{\omega}$
Kinetic Energy	Kinetic $K = \frac{1}{2}mv^2$	Rotational $K_R = \frac{1}{2}I\omega^2$

Conditions for Equilibrium

What do you think the term "An object is at its equilibrium" means?

The object is either at rest (<u>Static Equilibrium</u>) or its center of mass is moving at a constant velocity (<u>Dynamic Equilibrium</u>).

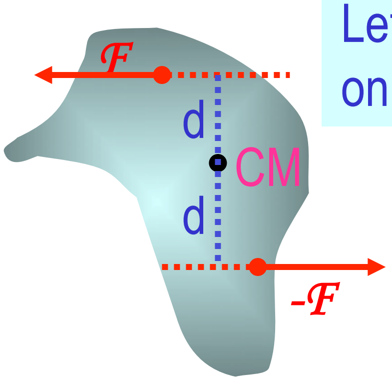
When do you think an object is at its equilibrium?

Translational Equilibrium: Equilibrium in linear motion

$$\sum \vec{F} = 0$$

Is this it?

The above condition is sufficient for a point-like object to be at its translational equilibrium. However for an object with size this is not sufficient. One more condition is needed. What is it?



Let's consider two forces equal in magnitude but in opposite direction acting on a rigid object as shown in the figure. What do you think will happen?

The object will rotate about the CM. Since the net torque acting on the object about a rotational axis is not 0.

$$\sum \vec{\tau} = 0$$

For an object to be at its *static equilibrium*, the object should not have linear or angular speed. $v_{CM} = 0 \quad \omega = 0$

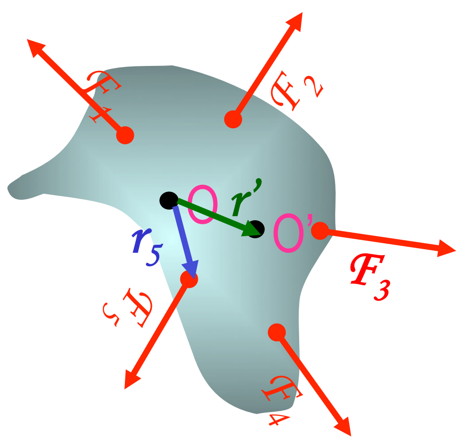
More on Conditions for Equilibrium

To simplify the problem, we will only deal with forces acting on x-y plane, giving torque only along z-axis. What do you think the conditions for equilibrium be in this case?

The six possible equations from the two vector equations turns to three equations.

$$\sum \vec{F} = 0 \Rightarrow \sum F_x = 0, \quad \sum F_y = 0 \qquad \textbf{AND} \qquad \sum \vec{\tau} = 0 \Rightarrow \sum \tau_z = 0$$

What happens if there are many forces exerting on an object?



If an object is at its translational static equilibrium, and if the net torque acting on the object is 0 about one axis, the net torque must be 0 about any arbitrary axis.

Why is this true?

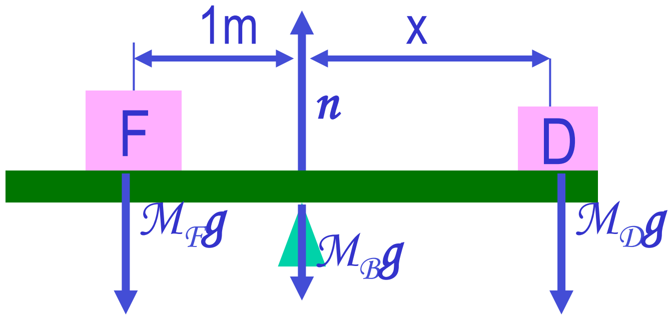
Because the object is ***not moving***, no matter what the rotational axis is, there should not be any motion. It is simply a matter of mathematical manipulation.

How do we solve static equilibrium problems?

1. Select the object to which the equations for equilibrium are to be applied.
2. Identify all the forces and draw a free-body diagram with them indicated on it with their directions and locations properly indicated
3. Choose a convenient set of x and y axes and write down the force equation for each x and y component with correct signs.
4. Apply the equations that specify the balance of forces at equilibrium. Set the net force in the x and y directions equal to 0.
5. Select the most optimal rotational axis for torque calculations ➔ Selecting the axis such that the torque of one or more of the unknown forces become 0 makes the problem much easier to solve.
6. Write down the torque equation with proper signs.
7. Solve the force and torque equations for the desired unknown quantities.

Example for Mechanical Equilibrium

A uniform 40.0 N board supports the father and the daughter each weighing 800 N and 350 N, respectively, and is not moving. If the support (or fulcrum) is under the center of gravity of the board, and the father is 1.00 m from the center of gravity (CoG), what is the magnitude of the normal force n exerted on the board by the support?



Since there is no linear motion, this system is in its translational equilibrium

$$\sum F_x = 0$$

$$\sum F_y = n - M_B g - M_F g - M_D g = 0$$

Therefore the magnitude of the normal force $n = 40.0 + 800 + 350 = 1190N$

Determine where the child should sit to balance the system.

The net torque about the fulcrum by the three forces are

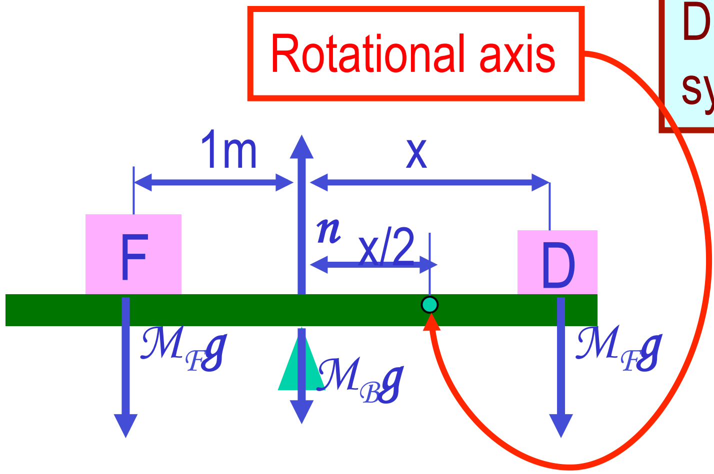
$$\tau = M_B g \cdot 0 + n \cdot 0 + M_F g \cdot 1.00 - M_D g \cdot x = 0$$

Therefore to balance the system the daughter must sit

$$x = \frac{M_F g}{M_D g} \cdot 1.00m = \frac{800}{350} \cdot 1.00m = 2.29m$$

Example for Mech. Equilibrium Cont'd

Determine the position of the child to balance the system for different position of axis of rotation.



The net torque about the axis of rotation by all the forces are

$$\tau = M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2) - n \cdot x/2 - M_D g \cdot x/2 = 0$$

Since the normal force is $n = M_B g + M_F g + M_D g$

The net torque can be rewritten

$$\tau = M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2) - (M_B g + M_F g + M_D g) \cdot x/2 - M_D g \cdot x/2$$
$$= M_F g \cdot 1.00 - M_D g \cdot x = 0$$

Therefore

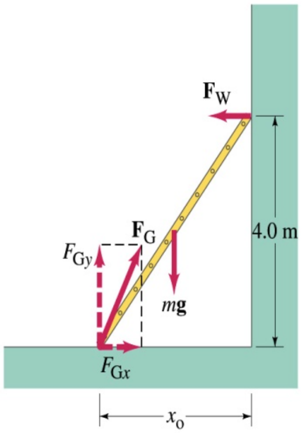
$$x = \frac{M_F g}{M_D g} \cdot 1.00m = \frac{800}{350} \cdot 1.00m = 2.29m$$

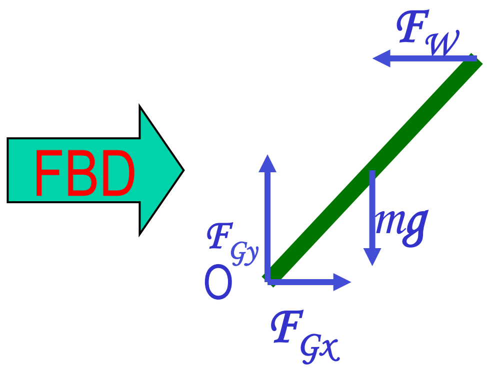
What do we learn?

No matter where the rotation axis is, net effect of the torque is identical.

Example 9 – 7

A 5.0 m long ladder leans against a wall at a point 4.0m above the ground. The ladder is uniform and has mass 12.0kg. Assuming the wall is frictionless (but ground is not), determine the forces exerted on the ladder by the ground and the wall.





First the translational equilibrium, using components

$$\sum F_x = F_{Gx} - F_W = 0$$

$$\sum F_y = -mg + F_{Gy} = 0$$

Thus, the y component of the force by the ground is

$$F_{Gy} = mg = 12.0 \times 9.8N = 118N$$

The length x_0 is, from Pythagorian theorem

$$x_0 = \sqrt{5.0^2 - 4.0^2} = 3.0m$$

Example 9 – 7 cont'd

From the rotational equilibrium $\sum \tau_O = -mg\,x_0/2 + F_W\,4.0 \;= 0$

Thus the force exerted on the ladder by the wall is

$$F_W = \frac{mg\,x_0/2}{4.0} = \frac{118 \cdot 1.5}{4.0} = 44N$$

The x component of the force by the ground is

$$\sum F_x = F_{Gx} - F_W = 0$$

Solve for F_{Gx}

$$F_{Gx} = F_W = 44N$$

Thus the force exerted on the ladder by the ground is

$$F_G = \sqrt{F_{Gx}^2 + F_{Gy}^2} = \sqrt{44^2 + 118^2} \approx 130N$$

The angle between the ground force to the floor $\theta = \tan^{-1}\left(\frac{F_{Gy}}{F_{Gx}}\right) = \tan^{-1}\left(\frac{118}{44}\right) = 70^{\circ}$

Congratulations!!!!

You all are impressive and have done very well!!!

I certainly had a lot of fun with ya'll and am truly proud of you!

Good luck with your exam!!!

Have a safe summer!!