PHYS 1444 – Section 003
Lecture #5

Thursday Sep. 6, 2012

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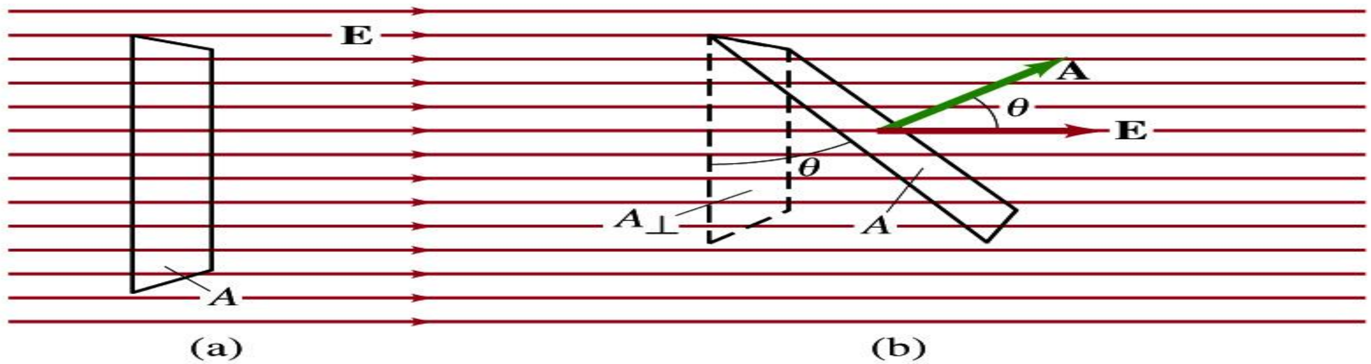
- Chapter 22:
  - Electric Flux
  - Gauss' Law
  - Gauss' Law with many charges
  - What is Gauss' Law good for?

CH 23 Electrical Potential

HW on ch 22 due weds at 11pm

Electric Flux



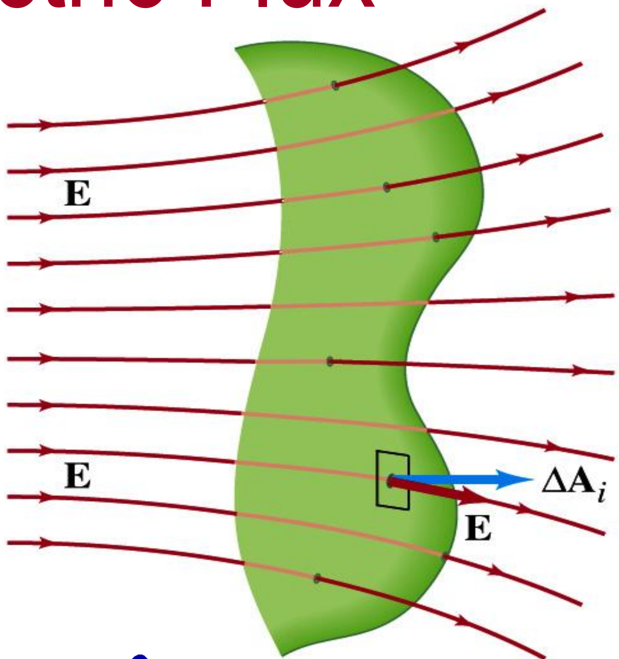
- Let's imagine a surface of area A through which a uniform electric field E passes
- The electric flux is defined as
  - Φ_E=EA, if the field is perpendicular to the surface
  - Φ_E=EAcosθ, if the field makes an angle θ with the surface
- So the electric flux is defined as $\Phi_E = \vec{E} \cdot \vec{A}$.
- How would you define the electric flux in words?
  - Total number of field lines passing through the unit area perpendicular to the field. $N_E \propto EA_{\perp} = \Phi_E$

Generalization of the Electric Flux

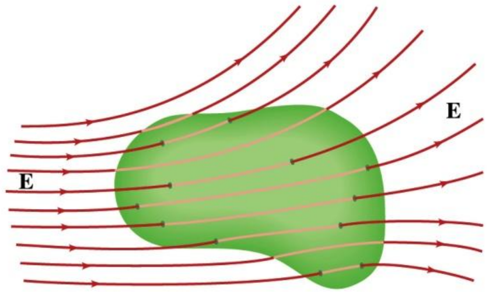
- Let's consider a surface of area A that has an irregular shape, and furthermore, that the field is not uniform.
- The surface can be divided up into infinitesimally small areas of $\Delta \mathbf{A}_i$ that can be considered flat.
- And the electric field through this area can be considered uniform since the area is very small.
- Then the electric flux through the entire surface is

$$\Phi_E \approx \sum_{i=1}^{n} \vec{E}_i \cdot \Delta \vec{A}_i$$

- In the limit where $\Delta \mathbf{A}_i \rightarrow 0$, the discrete summation becomes an integral.

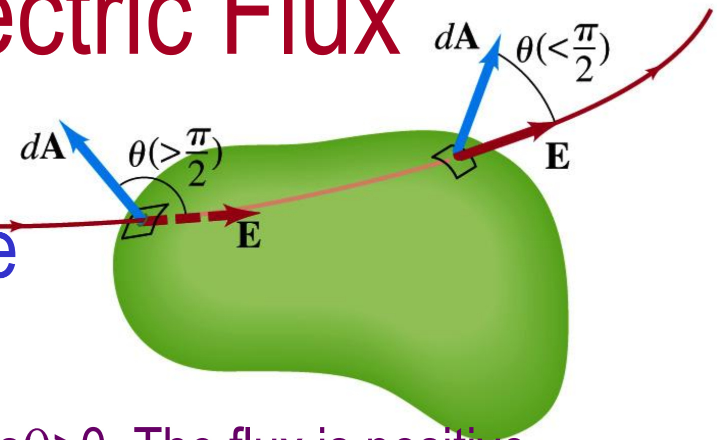


$$\Phi_E = \int \vec{E}_i \cdot d\vec{A}$$ open surface

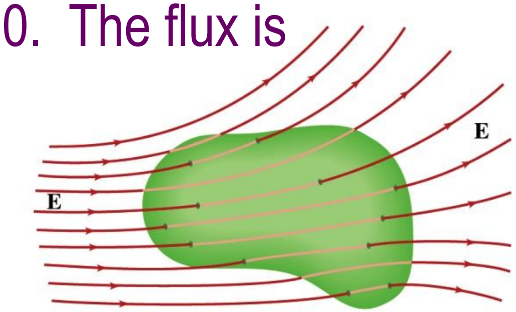


$$\Phi_E = \oint E_i \cdot dA$$ closed surface

Generalization of the Electric Flux



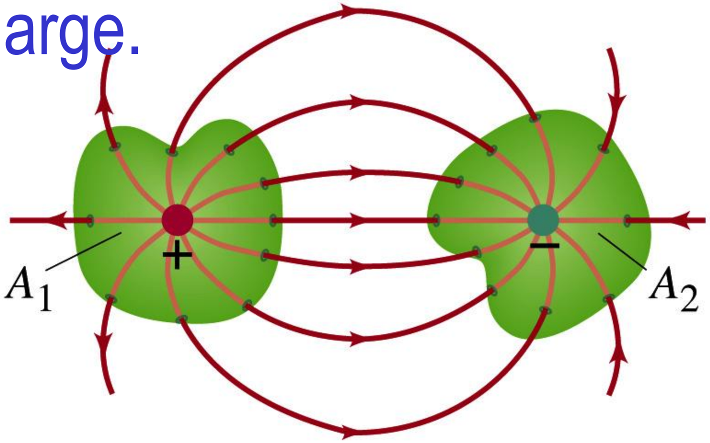
- We define the direction of the area vector as pointing outward from the enclosed volume.
  - For the line leaving the volume, $\theta<\pi/2$, so $\cos\theta>0$. The flux is positive.
  - For the line coming into the volume, $\theta>\pi/2$, so $\cos\theta<0$. The flux is negative.
  - If $\Phi_E>0$, there is a net flux out of the volume.
  - If $\Phi_E<0$, there is flux into the volume.

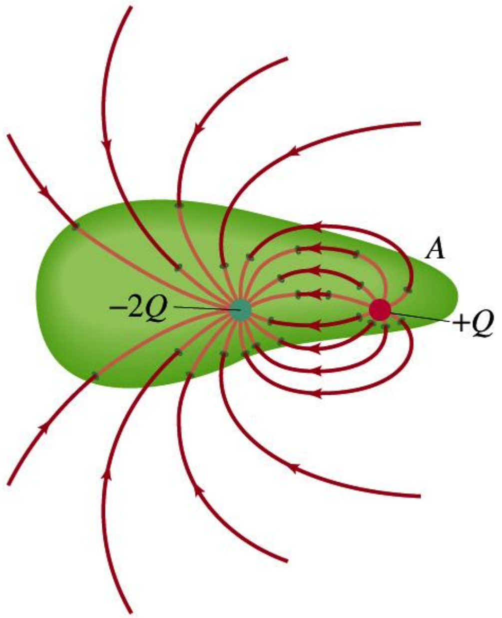


- In the above figures, each field that enters the volume also leaves the volume, so $\Phi_E = \oint \vec{E}\cdot d\vec{A} = 0.$
- The flux is non-zero only if one or more lines start or end inside the surface.

Generalization of the Electric Flux

- The field line starts or ends only on a charge.
- Sign of the net flux on the surface A_1?
  - Net outward flux (positive flux)
- How about A_2?
  - Net inward flux (negative flux)
- What is the flux in the bottom figure?
  - There should be a net inward flux (negative flux) since the total charge inside the volume is negative.
- The flux that crosses an enclosed surface is proportional to the total charge inside the surface. ➔ This is the crux of Gauss' law.

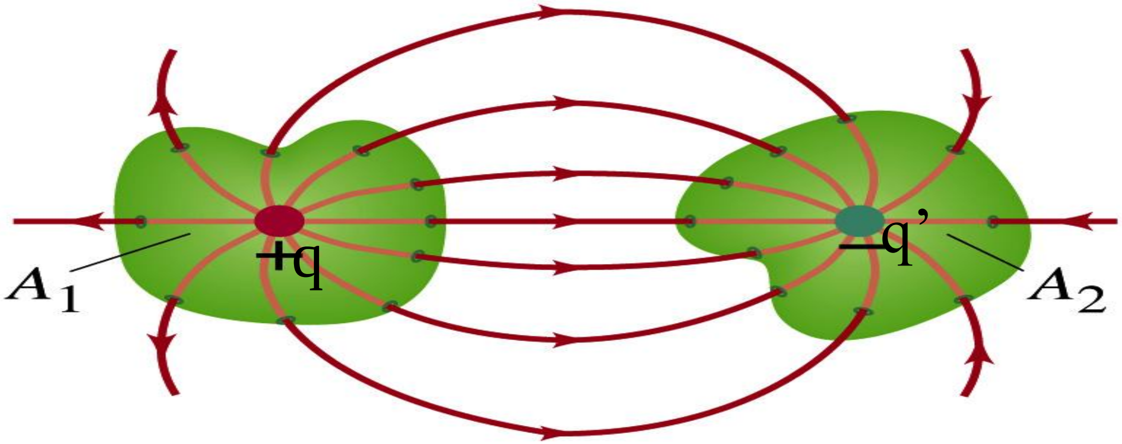




Gauss' Law

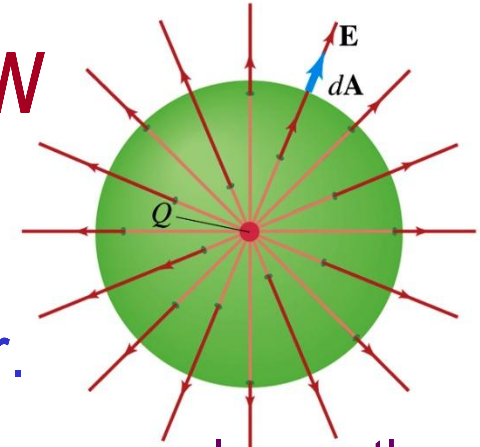
- The precise relation between flux and the enclosed charge is given by Gauss' Law $\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0}$
  - ε_0 is the permittivity of free space in the Coulomb's law
- A few important points on Gauss' Law
  - Freedom to choose!!
    - The integral is performed over the value of **E** on a closed surface of our choice in any given situation.
  - Test of existence of electrical charge!!
    - The charge Q_{encl} is the net charge enclosed by the arbitrary closed surface of our choice.
  - Universality of the law!
    - It does NOT matter where or how much charge is distributed inside the surface.
  - The charge outside the surface does not contribute to Q_{encl}. Why?
    - The charge outside the surface might impact field lines but not the total number of lines entering or leaving the surface

Gauss' Law



- Let's consider the case in the above figure.
- What are the results of the closed integral of the gaussian surfaces A_1 and A_2?
  - For A_1 $\oint \vec{E} \cdot d\vec{A} = \frac{+q}{\varepsilon_0}$
  - For A_2 $\oint \vec{E} \cdot d\vec{A} = \frac{-q'}{\varepsilon_0}$

Coulomb's Law from Gauss' Law



- Let's consider a charge Q enclosed inside our imaginary Gaussian surface of sphere of radius r.
  - Since we can choose any surface enclosing the charge, we choose the simplest possible one! ☺
- The surface is symmetric about the charge.
  - What does this tell us about the field E?
    - Must have the same magnitude at any point on the surface
    - Points radially outward (or inward) parallel to the surface vector d**A**.
- The Gaussian integral can be written as

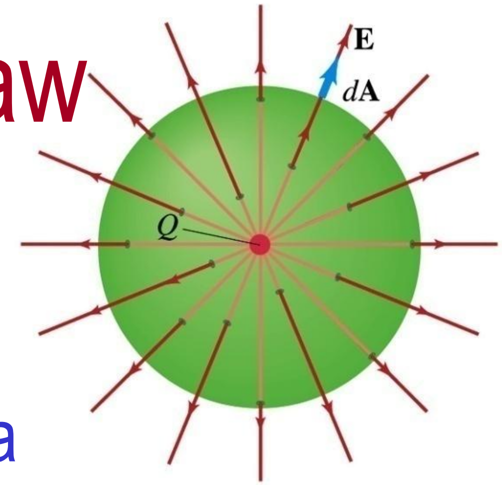
$$\oint \vec{E} \cdot d\vec{A} = \oint E dA = E \oint dA = E \; 4\pi r^2 \; = \frac{Q_{encl}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

Solve for E

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Electric Field of Coulomb's Law

Gauss' Law from Coulomb's Law



- Let's consider a single static point charge Q surrounded by an imaginary spherical surface.
- Coulomb's law tells us that the electric field at a spherical surface is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

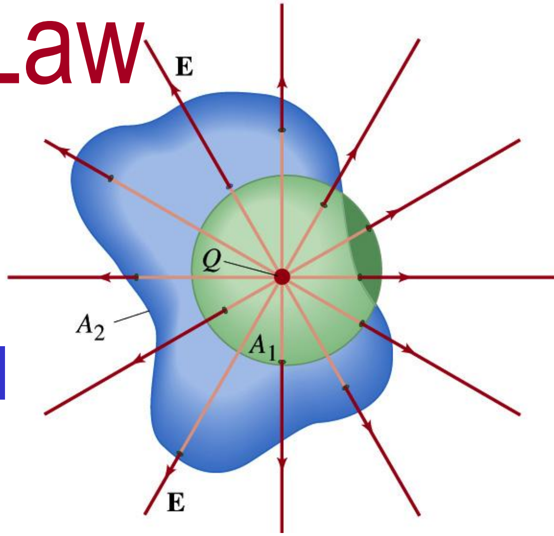
- Performing a closed integral over the surface, we obtain

$$\oint \vec{E}\cdot d\vec{A} = \oint \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}\cdot d\vec{A} = \oint \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}dA$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\oint dA = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\,4\pi r^2 = \frac{Q}{\varepsilon_0}$$

Gauss' Law

Gauss' Law from Coulomb's Law Irregular Surface



- Let's consider the same single static point charge Q surrounded by a symmetric spherical surface A_1 and a randomly shaped surface A_2.
- What is the difference in the number of field lines passing through the two surface due to the charge Q?
  - None. What does this mean?
    - The total number of field lines passing through the surface is the same no matter what the shape of the enclosed surface is.
  - So we can write: $\oint_{A_1} \vec{E} \cdot d\vec{A} = \oint_{A_2} \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$
  - What does this mean?
    - The flux due to the given enclosed charge is the same no matter what the shape of the surface enclosing it is. ➔ Gauss' law, $\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$, is valid for any surface surrounding a single point charge Q.

Gauss' Law w/ more than one charge

- Let's consider several charges inside a closed surface.
- For each charge, Q_i, inside the chosen closed surface,

$$\oint \vec{E}_i \cdot d\vec{A} = \frac{Q_i}{\varepsilon_0}$$

What is $\vec{E}_i$?

The electric field produced by Q_i alone!

- Since electric fields can be added vectorially, following the superposition principle, the total field **E** is equal to the sum of the fields due to each charge $\vec{E} = \sum \vec{E}_i$. So

$$\oint \vec{E} \cdot d\vec{A} = \oint \sum \vec{E}_i \cdot d\vec{A} = \frac{\sum Q_i}{\varepsilon_0} = \frac{Q_{encl}}{\varepsilon_0}$$

What is Q_{encl}?

The total enclosed charge!

- The value of the flux depends on the charge enclosed in the surface!! ➔ Gauss' law.

So what is Gauss' Law good for?

- Derivation of Gauss' law from Coulomb's law is only valid for **static electric charge**.
- Electric field can also be produced by changing magnetic fields.
  - Coulomb's law cannot describe this field, but Gauss' law is still valid
- Gauss' law is more general than Coulomb's law.
  - Can be used to obtain electric field, forces, or charges

Gauss' Law: Any **differences** between the input and output flux of the electric field over any enclosed surface is due to the charge within that surface!!!

21-7 Electric Field Calculations for Continuous Charge Distributions

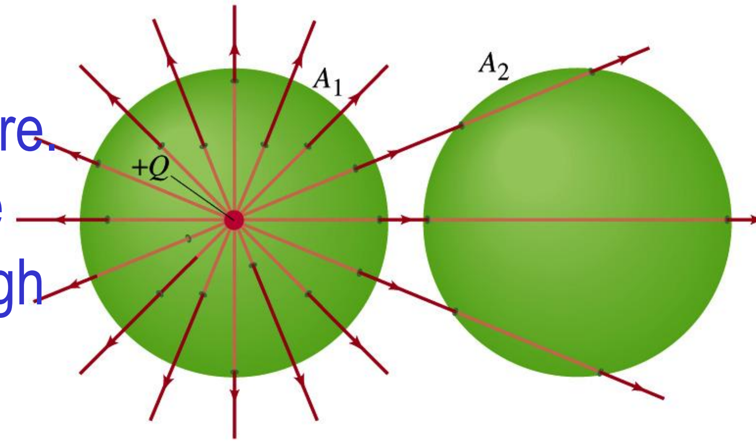
Conceptual Example 21-10: Charge at the center of a ring.

Imagine a small positive charge placed at the center of a nonconducting ring carrying a uniformly distributed negative charge. Is the positive charge in equilibrium if it is displaced slightly from the center along the axis of the ring, and if so is it stable? What if the small charge is negative? Neglect gravity, as it is much smaller than the electrostatic forces.

Solution: The positive charge is in stable equilibrium, as it is attracted uniformly by every part of the ring. The negative charge is also in equilibrium, but it is unstable; once it is displaced from its equilibrium position, it will accelerate away from the ring.

Example 22 – 2

Flux from Gauss' Law: Consider the two Gaussian surfaces, A_1 and A_2, shown in the figure. The only charge present is the charge +Q at the center of surface A_1. What is the net flux through each surface A_1 and A_2?



- The surface A_1 encloses the charge +Q, so from Gauss' law we obtain the total net flux

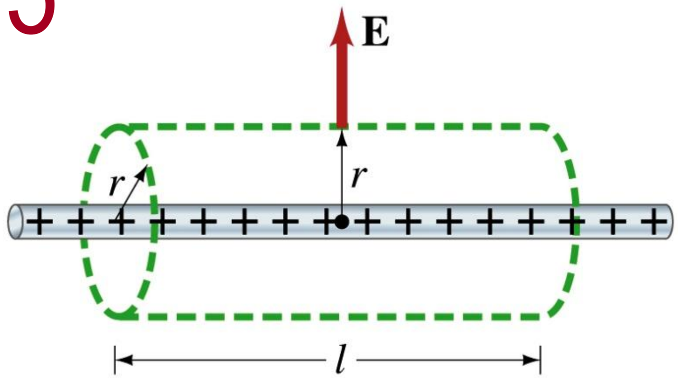
$$\oint \vec{E} \cdot d\vec{A} = \frac{+Q}{\varepsilon_0}$$

- For A_2 the charge, +Q, is outside the surface, so the total net flux is 0.

$$\oint \vec{E} \cdot d\vec{A} = \frac{0}{\varepsilon_0} = 0$$

Example 22 – 5

Long uniform line of charge: A very long straight wire possesses a uniform positive charge per unit length, λ. Calculate the electric field at points near but outside the wire, far from the ends.



- Which direction do you think the field due to the charge on the wire is?
  - Radially outward from the wire, the direction of radial vector r.
- Due to cylindrical symmetry, the field is constant anywhere on the Gaussian surface of a cylinder that surrounds the wire.
  - The end surfaces do not contribute to the flux at all. Why?
    - Because the field vector **E** is perpendicular to the surface vector d**A**.
- From Gauss' law

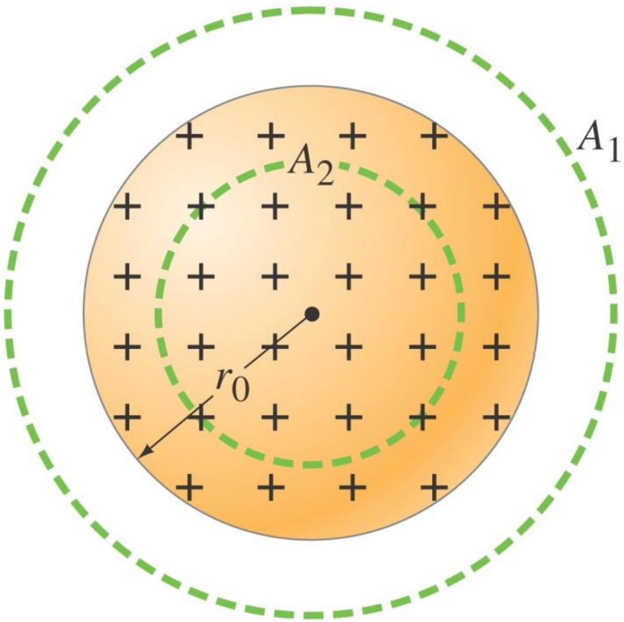
$$\oint \vec{E} \cdot d\vec{A} = E \oint dA = E \; 2\pi r l = \frac{Q_{encl}}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0}$$

Solving for E

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Example 22-4: Solid sphere of charge.

An electric charge Q is distributed uniformly throughout a nonconducting sphere of radius r_0. Determine the electric field (a) outside the sphere ($r > r_0$) and (b) inside the sphere ($r < r_0$).



Solution: a. Outside the sphere, a gaussian surface encloses the total charge Q. Therefore, $E = Q/(4\pi\varepsilon_0 r^2)$.

b. Within the sphere, a spherical gaussian surface encloses a fraction of the charge Qr^3/r_0^3 (the ratio of the volumes, as the charge density is constant). Integrating and solving for the field gives $E = Qr/(4\pi\varepsilon_0 r_0^3)$.