Physics 3313 - Lecture 10

Monday February 22, 2010
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1. HW4 on ch 5 is due Monday 3/1
2. HW5 on ch 6 will be assigned Friday 2/26 and due Monday 3/8
3. Monday 3/8 will be review
4. Weds 3/10 test on ch 1-6
4. Calculate the time, according to classical laws, it would take the electron of the hydrogen atom to radiate its energy and crash into the nucleus. [Hint: The radiated power \( P \) is given by \( \left( \frac{1}{4\pi \varepsilon_0} \right) \left( \frac{2Q^2}{3c^3} \right) \left( \frac{d^2 \vec{r}}{dt^2} \right)^2 \) where \( Q \) is the charge, \( c \) the speed of light, and \( \vec{r} \) the position vector of the electron from the center of the atom.]

The total energy of the atom is \(-\frac{e^2}{8\pi \varepsilon_0 r^2}\). Differentiating with respect to time:

\[
\frac{dE}{dt} = \frac{e^2}{8\pi \varepsilon_0 r^2} \frac{dr}{dt}
\]

Equating this result with the given equation from electromagnetic theory

\[
\frac{e^2}{8\pi \varepsilon_0 r^2} \frac{dr}{dt} = -\frac{1}{4\pi \varepsilon_0} \frac{2e^2}{3c^3} \left( \frac{d^2 \vec{r}}{dt^2} \right)^2
\]

\[
\frac{e^2}{2r^2} \frac{dr}{dt} = -\frac{2e^2}{3c^3} \left( \frac{d^2 \vec{r}}{dt^2} \right)^2
\]

In a circular orbit \( \frac{d^2 \vec{r}}{dt^2} \) is just the centripetal acceleration, which is also given by

\[
a = \frac{F}{m} = \frac{e^2}{4\pi \varepsilon_0 mr^2}
\]

Substituting:

\[
\frac{e^2}{2r^2} \frac{dr}{dt} = -\frac{2e^2}{3c^3} \left( \frac{e^2}{4\pi \varepsilon_0 mr^2} \right)^2
\]

\[
\frac{dr}{dt} = -\frac{4e^4}{(4\pi \varepsilon_0)^2 3m^2 e^3 r^2}
\]

By separation of variables:

\[
dt = -\left( 4\pi \varepsilon_0 \right)^2 \frac{3m^2 e^3}{4e^4} r^2 \, dr
\]

\[
t = -\left( 4\pi \varepsilon_0 \right)^2 \frac{3m^2 e^3}{4e^4} \int_{r_0}^{r} r^2 \, dr = \left( 4\pi \varepsilon_0 \right)^2 \frac{m^2 e^3}{4e^4} \alpha_0^3
\]

Using numerical values we find \( t = 1.55 \times 10^{-11} \) s.
I thus arrived at the overall concept which guided my studies: for both matter and radiations, light in particular, it is necessary to introduce the corpuscle concept and the wave concept at the same time.

- Louis de Broglie, 1929
Phase Velocity vs. Group Velocity

• The wave equation:  \( y = A \cos(\omega t - kx) \) represents an infinite series of waves with same amplitude and thus clearly does not represent a moving particle, which should be localized and represented by a wave packet or group

• A wave group is the superposition of waves with different wavelengths or frequencies
• If \( v_p(\lambda) \) different wavelengths do not proceed together \( v_g \neq v_p \) and there is dispersion
• An example of a wave group is beats: 2 waves with similar amplitude and slightly different frequency: example 440 and 442 Hz.
• We hear fluctuation sound of 441 Hz with two loudness peaks (beats) per second (tune violin)
Beats Example

• Consider two waves, \( y_1 \) and \( y_2 \) with slightly different frequency

\[
y_1 = A \cos(\omega t - kx) \quad y_2 = A \cos(\omega + \Delta \omega) t - [k + \Delta k] x
\]

• What happens if we add them?

• To evaluate this we need to use some trig identities:

• \( \cos \alpha + \cos \beta = 2 \cos \left( \frac{1}{2} (\alpha + \beta) \right) \cos \left( \frac{1}{2} (\alpha - \beta) \right) \) and \( \cos(-\theta) = \cos(\theta) \)

\[
y = y_1 + y_2 = 2A \cos \frac{1}{2} \left[ (2\omega + \Delta \omega)t - (2k + \Delta k)x \right] \cos \frac{1}{2} (\Delta \omega t - \Delta kx)
\]

\[
y = 2A \cos \left[ \omega t - kx \right] \cos(\Delta \omega t / 2 - \Delta kx / 2)
\]

• The first term is the sum of the two waves and basically has twice the amplitude and same frequency (more precisely it has the average frequency), while the second term is the modulation term of the original wave, which gives the wah wah sound
Principle of Superposition

- When two or more waves traverse the same region, they act independently of each other.

- As we saw, combining two waves yields:

  \[ \Psi(x,t) = \Psi_1(x,t) + \Psi_2(x,t) = 2A \left( \frac{\Delta k}{2} x - \frac{\Delta \omega}{2} t \right) \cos(k_{av} x - \omega_{av} t) \]

- The combined wave oscillates within an envelope that denotes the maximum displacement of the combined waves.

- When combining many waves with different amplitudes and frequencies, a pulse, or wave packet, is formed which moves at a group velocity:

  \[ u_{gr} = \frac{\Delta \omega}{\Delta k}. \]
Beats Applets

• http://www.lon-capa.org/~mmp/applist/beats/b.htm
• http://webphysics.davidson.edu/faculty/dmb/JavaSounddemos/beats.htm
More about Wave Velocities

- Definition of phase velocity: $v_p = f \lambda = 2\pi f \lambda / 2\pi = \omega / k$
- Definition of group velocity: $v_g = \frac{\Delta \omega}{\Delta k} = \frac{d\omega}{dk}$
- For light waves: $v_g = v_p = c$
- For de Broglie waves $v_g = \frac{d\omega}{dk} = \frac{d\omega}{dv} / \frac{dv}{dv}$ where $v$ is velocity of particle

$$\omega = 2\pi f = 2\pi \frac{\gamma mc^2}{h} \quad \frac{d\omega}{dv} = \frac{2\pi mv}{h(1 - v^2 / c^2)^\frac{3}{2}}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi \gamma mv}{h} \quad \frac{dk}{dv} = \frac{2\pi m}{h(1 - v^2 / c^2)^\frac{3}{2}}$$

Note that since both $\gamma$ and $v$ have velocity dependence, there are two terms that must be added together and simplified (as in Example on acceleration with a constant force)

$$v_g = \frac{d\omega}{dv} / \frac{dk}{dv} = v \quad \text{Wave group travels with velocity of the particle!}$$
Wave Properties

• The phase velocity is the velocity of a point on the wave that has a given phase (for example, the crest) and is given by

\[ v_{ph} = \frac{\lambda}{T} = \frac{\omega}{k} \]

• A phase constant \( \Phi \) shifts the wave:

\[ \Psi(x,t) = A \sin(kx - \omega t + \phi) \]

• This amplitude expression is a solution of the general wave equation

\[ \frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2} \]
Fourier Series

- The sum of many waves that form a wave packet is called a Fourier series:

\[ \Psi(x, t) = \sum_i A_i \cos(k_i x - \omega_i t) \]

- Summing an infinite number of waves yields the Fourier integral:

\[ \Psi(x, t) = \int \tilde{A}(k) \cos(kx - \omega t) \, dk \]
A Gaussian wave packet is often used to represent the position of particles.

\[ \Psi(x,0) = \Psi(x) = Ae^{-\frac{\Delta k^2 x^2}{2}} \cos(k_0 x) \]

- The envelope is the Gaussian.
Gaussian Distribution

Integrating over 2 standard deviations gives 95.4%, generally need 4-5σ for discovery (fluctuation probability less than 1/100000) Ex.: top quark discovery
Dispersion

• Considering the group velocity of a de Broglie wave packet:

\[ v_g = \frac{d\omega}{dk} = \frac{dE}{dp} \]

\[ E = hf = \hbar(2\pi f) = \hbar \omega \quad p = \frac{h}{\lambda} = \frac{\hbar(2\pi)}{\lambda} = \hbar k \]

with \( E^2 = p^2 c^2 + m^2 c^4 \) obtain \( v_g = \frac{dE}{dp} = \frac{pc^2}{E} \)

• The relationship between the phase velocity and the group velocity is

\[ u_{gr} = \frac{d\omega}{dk} = \frac{d}{dk}(\nu_{ph}k) = \nu_{ph} + k \frac{d\nu_{ph}}{dk} \]

• Hence the group velocity may be greater or less than the phase velocity. A medium is called non-dispersive when the phase velocity is the same for all frequencies and equal to the group velocity.
5.5: Waves or Particles?

- Young’s double-slit diffraction experiment demonstrates the wave property of light.
- However, dimming the light results in single flashes on the screen representative of particles.
Electron Double-Slit Experiment

- C. Jönsson of Tübingen, Germany, succeeded in 1961 in showing double-slit interference effects for electrons by constructing very narrow slits and using relatively large distances between the slits and the observation screen.

- This experiment demonstrated that precisely the same behavior occurs for both light (waves) and electrons (particles).
Which Slit?

- To determine which slit the electron went through: We set up a light shining on the double slit and use a powerful microscope to look at the region. After the electron passes through one of the slits, light bounces off the electron; we observe the reflected light, so we know which slit the electron came through. (Note the wavelength of light used must be smaller than the slit width d)
- Use a subscript “ph” to denote variables for light (photon). Therefore the momentum of the photon is

\[ p_{ph} = \frac{h}{\lambda_{ph}} > \frac{h}{d} \]

- The momentum of the electrons will be on the order of

\[ p_{el} = \frac{h}{\lambda_{el}} = \frac{h}{d} \]

- The difficulty is that the momentum of the photons used to determine which slit the electron went through is sufficiently great to strongly modify the momentum of the electron itself, thus changing the direction of the electron! The attempt to identify which slit the electron is passing through will in itself change the interference pattern.
Wave Particle Duality Solution

- The solution to the wave particle duality of an event is given by the following principle.

- **Bohr’s principle of complementarity**: It is not possible to describe physical observables simultaneously in terms of both particles and waves.

- **Physical observables** are those quantities such as position, velocity, momentum, and energy that can be experimentally measured. In any given instance we must use either the particle description or the wave description.
5.6 Uncertainty Principle

a) For a narrow wave group the position is accurately measured, but wavelength and thus momentum cannot be precisely determined.

b) Conversely for extended wave group it is easy to measure wavelength, but position uncertainty is large.

- Werner Heisenberg 1927, postulated that it is impossible to know exact momentum and position of an object at the same time:

\[
\Delta x \Delta p \gg \frac{\hbar}{2}, \quad \Delta E \Delta t \gg \frac{\hbar}{2}
\]

- Note this is not an apparatus error, but an unknowability of quantities.
- Can’t know perfectly where a particle is and where it’s going: future is not determined, just have probabilities! Sounds like philosophy!
- (Why is this not a problem for NASA?)