HW2 Solution

1) $$\frac{d\sigma}{d\Omega} = \left(\frac{zz'e^2}{4E}\right)^2 \frac{1}{\sin^4\theta/2}$$

$$= \left(\frac{79\times2\times e^2/\hbar c \times \hbar c}{4\times 10\,\text{keV}}\right)^2 \frac{1}{\sin^4\theta/2} \qquad \frac{e^2}{\hbar c} = \frac{1}{137} \quad \hbar c = 197\,\text{MeV fm}$$

$$= \left(\frac{79\times2\times\frac{1}{137}\times197\times10^{-13}\,\text{MeV cm}}{4\times 0.01\,\text{MeV}}\right)^2 \frac{1}{\sin^4\theta/2}$$

$$= \left(5.7\times10^{-10}\,\text{cm}\right)^2 = 32.5\times10^{-20}\,\text{cm}^2 \frac{1}{\sin^4\theta/2}$$

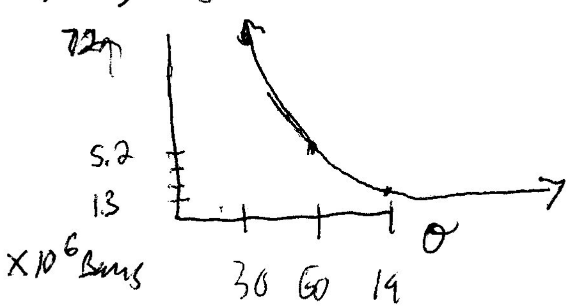
or $325{,}000$ Barns $/\sin^4\theta/2$

$\theta = 30 \Rightarrow \theta/2 = 15^\circ \quad \sin^4 15 = .0045 \qquad \frac{1}{.0045} = 220$

$= 7.2\times10^7$ Barns

$\theta = 60 \Rightarrow \theta/2 = 30^\circ \qquad = 5.2\times10^6$ Barns

$\theta = 90^\circ \Rightarrow \theta/2 = 45^\circ \qquad = 1.3\times10^6$ Barns



Note $$\left(\frac{zz'e^2}{4e}\right)^2 = \left(\frac{2\times79\times(4.8\times10^{-10}\,\text{esu})^2}{4\times1\times10^4\,\text{eV}\times\frac{1.6\times10^{-19}\,\text{J}}{\text{eV}}\times\frac{10^7\,\text{erg}}{\text{J}}}\right)^2$$

$\text{erg} = \frac{\text{g}\cdot\text{cm}^2}{\text{s}^2} \qquad \text{esu} = \frac{\text{g}^{1/2}\,\text{cm}^{3/2}}{\text{s}} \qquad = (5.7\times10^{-10}\,\text{cm})^2$ ✓

$\Rightarrow \frac{\text{esu}^2}{\text{erg}} = \text{cm}$ ✓

2.) $\sigma_{TOT} = \int_0^{2\pi} d\phi \int_0^{\pi} \frac{d\sigma}{d\Omega}(\theta) \sin\theta \, d\theta$

$= 2\pi \left(\frac{Z Z' e^2}{4E}\right)^2 \int_0^{\pi} \sin\theta \frac{1}{\sin^4 \theta/2} d\theta$

$= 2\pi \quad '' \quad \int_0^{\pi} 2\sin\theta/2 \cos\theta/2 \frac{1}{\sin^4 \theta/2} d\theta$

$u = \sin\theta/2$

$du = 1/2 \cos\theta/2$

$= \quad '' \quad '' \quad 4\int_0^{\pi} \frac{1}{2} \frac{\cos\theta/2}{\sin^3\theta/2} d\theta \qquad \int u^{-3} du = \frac{-1}{2} u^{-2}$

$= 8\pi \quad '' \quad -\frac{1}{2} (\sin\theta/2)^{-2} \Big|_{\theta_c}^{\pi}$

$= -4\pi \quad '' \quad (1 - 15) \quad \theta_c = 30$

$(1 - 4) \quad \theta_c = 60$

$(1 - 2) \quad \theta_c = 90$

$= -4\pi \times 3.25 \times 10^5$ Barns $\times$ $(-14) = 5.71 \times 10^7$ Barns $\theta = 30°$

$(-3) = 1.22 \times 10^7$ Barns $\theta = 60°$

$(-1) = 4.08 \times 10^6$ Barns $\theta = 90°$

Problem 1.10 *What is the minimum impact parameter needed to deflect 7.7 MeV α-particles from gold nuclei by at least 1°? What about by at least 30°? What is the ratio of probabilities for deflections of $\theta > 1°$ relative to $\theta > 30°$? (See the CRC Handbook for the density of gold.)*

For the scattering of a 7.7 MeV α-particle from gold, we have

$$Z = 2, \quad Z' = 79, \quad E = 7.7\,\text{MeV}, \tag{1.74}$$

so that we obtain

$$\begin{aligned} \frac{ZZ'e^2}{2E} &= ZZ' \times \frac{\hbar c}{2E} \times \frac{e^2}{\hbar c} \\ &= 2 \times 79 \times \frac{197\,\text{MeV} - \text{F}}{2 \times 7.7\,\text{MeV}} \times \frac{1}{137} \\ &\approx 14.5 \times 10^{-13}\,\text{cm} \approx 1.4 \times 10^{-12}\,\text{cm}. \end{aligned} \tag{1.75}$$

We know from Eq. (1.32) of the text that

$$b = \frac{ZZ'e^2}{2E} \cot \frac{\theta}{2}, \tag{1.76}$$

which leads to

$$b(\theta) \approx 1.4 \times 10^{-12} \cot \frac{\theta}{2}\,\text{cm}. \tag{1.77}$$

We note that for

$$\theta = 1° = \frac{\pi}{180} \approx \frac{1}{60} \ll 1, \tag{1.78}$$

we have

$$\tan\frac{\theta}{2} \approx \frac{\theta}{2} \approx \frac{1}{120}, \quad \cot\frac{\theta}{2} = \frac{1}{\tan\frac{\theta}{2}} \approx 120. \tag{1.79}$$

Similarly, for

$$\theta = 30^\circ = \frac{\pi}{6} \approx \frac{1}{2} \tag{1.80}$$

we have

$$\tan\frac{\theta}{2} \approx \frac{\theta}{2} \approx \frac{1}{4}, \quad \cot\frac{\theta}{2} = \frac{1}{\tan\frac{\theta}{2}} \approx 4. \tag{1.81}$$

Using these values in (1.77), we obtain

$$\begin{aligned} b(\theta = 1^\circ) &\approx 1.4 \times 10^{-12} \times 120\,\text{cm} = 1.7 \times 10^{-10}\,\text{cm}, \\ b(\theta = 30^\circ) &\approx 1.4 \times 10^{-12} \times 4\,\text{cm} = 5.6 \times 10^{-12}\,\text{cm}. \end{aligned} \tag{1.82}$$

As we have already seen in Problem 1.1, the probability of scattering for angles greater than θ_b goes as the area πb^2. Therefore, using (1.82) we have

$$\begin{aligned} \frac{\sigma(\theta > 1^\circ)}{\sigma(\theta > 30^\circ)} &= \frac{b^2(\theta = 1^\circ)}{b^2(\theta = 30^\circ)} \\ &\approx \left(\frac{1.7 \times 10^{-10}}{5.6 \times 10^{-12}}\right)^2 \approx 900. \end{aligned} \tag{1.83}$$

In other words, there will be approximately 900 more particle collisions for $\theta > 1^\circ$ than for $\theta > 30^\circ$.

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Problem 1.11 *Consider a collimated source of* 8 MeV *α-particles that provides* 10^4 *α/sec that impinge on a* 0.1 mm *gold foil. What counting rate would you expect in a detector that subtends an annular cone of* $\Delta\theta = 0.05$ *rad, at a scattering angle of* $\theta = 90^\circ$*? Compare this to the rate at* $\theta = 5^\circ$*. Is there a problem? Is it serious (see Problem* 1.12*). (Hint: You can use the small-angle approximation where appropriate. and find the density of gold in the CRC Handbook.)*

For scattering of 8 MeV α particles from gold, we have

$$\begin{aligned} Z &= 2, \quad Z' = 79, \quad E = 8\,\text{MeV}, \\ \rho &= \text{density of gold} \approx 19.3\,\text{g/cm}^3, \\ t &= \text{Thickness of gold foil} = 0.1\,\text{mm} = 10^{-2}\,\text{cm}, \\ N_0 &= \text{Incident flux} = 10^4/\text{sec}, \\ A &= \text{Atomic weight of gold} = 197, \\ A_0 &= \text{Avogadro's number} \approx 6 \times 10^{23}/\text{mole}, \\ \Delta\theta &= \text{Angle subtended by the detector} = 0.05\,\text{rad}. \end{aligned} \tag{1.84}$$

We can therefore calculate

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\theta) &= \left(\frac{ZZ'e^2}{4E}\right)^2 \text{cosec}^4\frac{\theta}{2}(\text{sr})^{-1} \\ &= \left(ZZ' \times \frac{\hbar c}{4E} \times \frac{e^2}{\hbar c}\right)^2 \text{cosec}^4\frac{\theta}{2}(\text{sr})^{-1} \\ &= \left(2 \times 79 \times \frac{197\ \text{MeV} - \text{F}}{4 \times 8\ \text{MeV}} \times \frac{1}{137}\right)^2 \text{cosec}^4\frac{\theta}{2}(\text{sr})^{-1} \\ &\approx 0.48 \times 10^{-24}\ \text{cosec}^4\frac{\theta}{2}\,\text{cm}^2/\text{sr}. \end{aligned} \tag{1.85}$$

Similarly, we have

$$\begin{aligned} \frac{A_0\rho t}{A} &\approx \frac{6 \times 10^{23} \times 19.3 \times 10^{-2}/\text{cm}^2}{197} \\ &\approx 6 \times 10^{20}/\text{cm}^2. \end{aligned} \tag{1.86}$$

From Eq. (1.40) of the text we therefore obtain the counting rate

$$\begin{aligned} dn(\theta) &= N_0\frac{A_0\rho t}{A}\frac{d\sigma}{d\Omega}(\theta)\,d\Omega \\ &\approx 10^4/\text{sec} \times 6\times10^{20}/\text{cm}^2 \times 0.48 \times 10^{-24}\text{cosec}^4\frac{\theta}{2}\,\text{cm}^2/\text{sr} \times d\Omega \\ &= 2.88\,\text{cosec}^4\frac{\theta}{2}\,d\Omega(\text{sec} - \text{sr})^{-1}. \end{aligned} \tag{1.87}$$

For scattering with azimuthal symmetry, we can write

$$d\Omega = 2\pi \sin\theta d\theta, \tag{1.88}$$

and if we identify $d\theta \approx \Delta\theta = 0.05\,\text{rad}$, we get

$$d\Omega \approx 2\pi \sin\theta \times 0.05\,\text{sr} \approx 0.3 \sin\theta\,\text{sr}. \tag{1.89}$$

Putting this back into (1.87), we obtain

$$dn(\theta) \approx 2.88 \text{ cosec}^4 \frac{\theta}{2} (\text{sec} - \text{sr})^{-1} \times 0.3 \sin\theta \text{ sr}$$
$$\approx 0.86 \sin\theta \text{ cosec}^4 \frac{\theta}{2} (\text{sec})^{-1}. \tag{1.90}$$

It follows that

$$dn\left(\theta = \frac{\pi}{2}\right) \approx 0.86 \times 1 \times (\sqrt{2})^4 \approx 3.4/\text{sec}. \tag{1.91}$$

On the other hand, for $\theta = 5^\circ = \frac{\pi}{36} \approx \frac{1}{12} \ll 1$, we have

$$\sin\theta \approx \theta \approx \frac{1}{12}, \quad \text{cosec}^4 \frac{\theta}{2} \approx \left(\frac{2}{\theta}\right)^4 \approx (24)^4, \tag{1.92}$$

and we obtain

$$dn(\theta = 5^\circ) \approx 0.86 \times \frac{1}{12} \times (24)^4/\text{sec} \approx 2.4 \times 10^4/\text{sec}. \tag{1.93}$$

This is, in fact, larger than the incident flux, and, if this were true, conservation of probability (particle number) would be violated, which is a serious problem! For one thing, we note that the approximation

$$d\theta \approx \Delta\theta, \tag{1.94}$$

is meaningful only when

$$\frac{\Delta\theta}{\theta} \ll 1, \tag{1.95}$$

which is clearly violated when $\Delta\theta = 0.05$ rad and $\theta = 5^\circ \approx \frac{1}{12} \approx 0.08$ rad. This is one of the sources of the difficulty. For other sources of this error, we turn to the solution of the next problem.

Problem 1.12 *Consider the expression Eq.* (1.41) *for Rutherford Scattering of α-particles from gold nuclei. Integrate this over all angles to obtain n. In principle, n cannot exceed* N_0, *the number of incident particles. Why? What cutoff value for θ would be required in the integral, that is, some* $\theta = \theta_0 > 0$, *to assure that n does not exceed* N_0 *in Problem* 1.4? (*Hint: After integrating, use the small-angle approximation to simplify the calculation.*) *Using the Heisenberg uncertainty principle* $\Delta p_x \Delta x \approx \hbar$, *where* Δx *is some*