PHYS 3446 – Lecture #5

Thurs., Feb. 5 2015

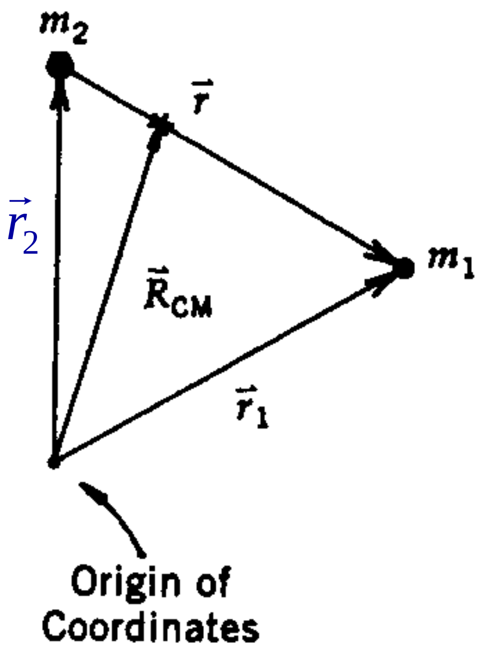
Brandt

1. Center of mass
2. Lab and CM frames
3. Scattering angles

Lab Frame and Center of Mass Frame

- So far, we have neglected the motion of target nuclei in Rutherford Scattering
- In reality, they recoil as a result of scattering
- This complication can best be handled using the Center of Mass frame under a central potential
- This description is also useful for scattering experiments with two beams of particles (moving target)

Lab Frame and CM Frame



$$\vec{r}_1 = r_1\,\hat{r}_1$$

$$\vec{r}_2 = r_2\,\hat{r}_2$$

$$\vec{r} = r\,\hat{r}$$

$$\vec{R}_{CM} = R_{CM}\,\hat{R}_{CM}$$

- The equations of motion can be written

$$m_1\ddot{\vec{r}}_1 = -\vec{\nabla}_1 V\left(\left|\vec{r}_1 - \vec{r}_2\right|\right)$$

$$m_2\ddot{\vec{r}}_2 = -\vec{\nabla}_2 V\left(\left|\vec{r}_1 - \vec{r}_2\right|\right)$$

where $\vec{\nabla}_i = \hat{r}_i \frac{\partial}{\partial r_i} + \frac{\hat{\theta}_i}{r_i}\frac{\partial}{\partial \theta_i} + \frac{\hat{\phi}_i}{r_i \sin\theta_i}\frac{\partial}{\partial \phi_i} \qquad i = 1, 2$

Since the potential depends only on relative separation of the particles, we redefine new variables, relative coordinates & coordinate of CM

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \quad \text{and} \quad \vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

Now some simple arithmetic

- From the equations of motion, we obtain

$$m_1\ddot{\vec{r}}_1 - m_2\ddot{\vec{r}}_2 = m_1\ddot{\vec{r}}_1 - m_2\left(\ddot{\vec{r}}_1 - \ddot{\vec{r}}\right) = \left(m_1 - m_2\right)\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}$$

$$= -\left(\hat{r}_1\frac{\partial}{\partial r_1} - \hat{r}_2\frac{\partial}{\partial r_2}\right)V\left(\left|\vec{r}\right|\right) = -2\hat{r}\frac{\partial}{\partial r}V\left(\left|\vec{r}\right|\right)$$

- Since the momentum of the system is conserved:

$$\frac{m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2}{m_1 + m_2} = \ddot{\vec{R}}_{CM} = 0 \Rightarrow m_1\ddot{\vec{r}}_1 = -m_2\ddot{\vec{r}}_2 = -m_2\left(\ddot{\vec{r}}_1 - \ddot{\vec{r}}\right)$$

- Rearranging the terms, we obtain

$$\left(m_1 + m_2\right)\ddot{\vec{r}}_1 = m_2\ddot{\vec{r}} \Rightarrow \ddot{\vec{r}}_1 = \frac{m_2}{\left(m_1 + m_2\right)}\ddot{\vec{r}}$$

$$\left(m_1 - m_2\right)\frac{m_2}{\left(m_1 + m_2\right)}\ddot{\vec{r}} + m_2\ddot{\vec{r}} = \frac{2m_1m_2}{\left(m_1 + m_2\right)}\ddot{\vec{r}} \Rightarrow \boxed{\frac{m_1m_2}{\left(m_1 + m_2\right)}\ddot{\vec{r}} = -\hat{r}\frac{\partial}{\partial r}V\left(\left|\vec{r}\right|\right)}$$

Lab Frame and CM Frame

- From the equations in previous slides

$$\frac{m_1 m_2}{m_1 + m_2}\ddot{\vec{r}} \equiv \mu\ddot{\vec{r}} = -\vec{\nabla}V\left(|\vec{r}|\right) = -\frac{\partial V\left(|\vec{r}|\right)}{\partial r}\hat{r}$$

Reduced Mass

and

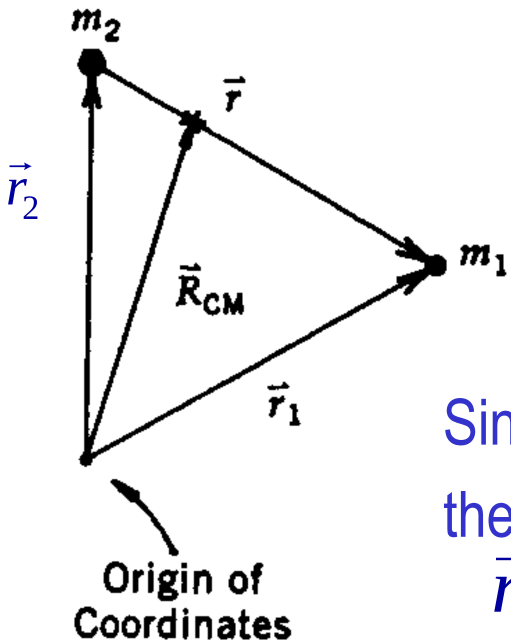
$$\left(m_1 + m_2\right)\ddot{\vec{R}}_{CM} = M\ddot{\vec{R}}_{CM} = 0$$

Thus

$$\dot{\vec{R}}_{CM} = \text{constant } \hat{R}$$

- What do we learn from this exercise?
- For a central potential, the motion of the two particles can be decoupled when re-written in terms of
  - a relative coordinate
  - The coordinate of center of mass

Lab Frame and CM Frame



$\vec{r}_1 = r_1 \hat{r}_1$

$\vec{r}_2 = r_2 \hat{r}_2$

$\vec{r} = r \hat{r}$

$\vec{R}_{CM} = R_{CM} \hat{R}_{CM}$

$$\vec{\nabla}_i = \hat{r}_i \frac{\partial}{\partial r_i} + \frac{\hat{\theta}_i}{r_i} \frac{\partial}{\partial \theta_i} + \frac{\hat{\phi}_i}{r_i \sin\theta_i} \frac{\partial}{\partial \phi_i}$$

$$i = 1, 2$$

Since the potential depends only on relative separation of the particles, we define new variables:

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

relative coordinates

$$\vec{R}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

coordinate of CM

F=ma= m(dp/dt)=-dU/dx

For a central potential, the motion of the two particles can be decoupled when re-written in terms of a relative coordinate and the CM coordinate

$$\frac{m_1 m_2}{(m_1 + m_2)} \ddot{\vec{r}} = -\hat{r} \frac{\partial}{\partial r} V(|\vec{r}|)$$

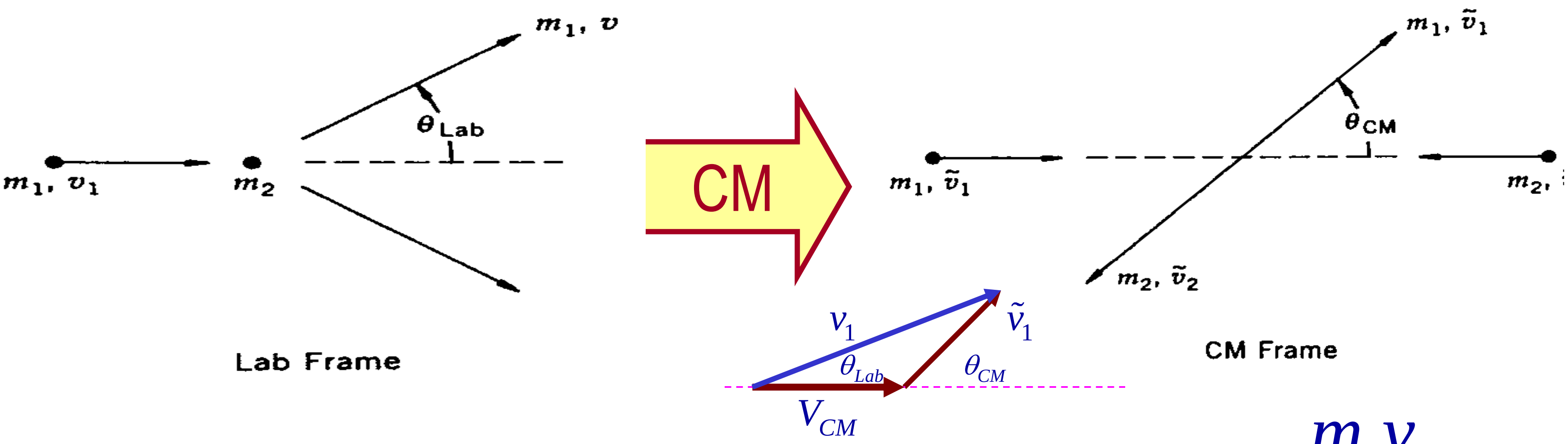
$$\mu \ddot{\vec{r}} = -\vec{\nabla} V(|\vec{r}|)$$

Lab Frame and CM Frame

- The CM is moving at a constant velocity in the lab frame independent of the form of the central potential
- How do we know this? Momentum is conserved so

  $dp/dt=0$ so $p=mv$ is constant and ->v is constant
- The motion is that of a fictitious particle with mass μ (the reduced mass) and coordinate **r**.
- Frequently we define the Center of Mass frame as the frame where the sum of the momenta of all the interacting particles is 0.

Relationship of variables in Lab and CM



- The speed of CM in lab frame $$v_{CM} = \dot{R}_{CM} = \frac{m_1 v_1}{m_1 + m_2}$$ (Since v_2=0 m_2v_2=0)
- Speeds of the particles in CM frame are

$$\tilde{v}_1 = v_1 - v_{CM} = \frac{m_2 v_1}{m_1 + m_2} \quad \text{and} \quad \tilde{v}_2 = v_{CM} = \frac{m_1 v_1}{m_1 + m_2}$$

$m_1 \times$ $\qquad$ $m_2 \times$

- The momenta of the two particles are equal and opposite!!

Differential cross sections in Lab and CM

- The particles that scatter in lab at an angle θ_{Lab} into solid angle $d\Omega_{Lab}$ scatter at θ_{CM} into solid angle $d\Omega_{CM}$ in CM.
- Since ϕ is invariant, $d\phi_{Lab} = d\phi_{CM}$.
  - Why?
  - ϕ is perpendicular to the direction of boost, thus is invariant.
- Thus, the differential cross section becomes:

$$\frac{d\sigma}{d\Omega_{Lab}}(\theta_{Lab})\sin\theta_{Lab}\,d\theta_{Lab} = \frac{d\sigma}{d\Omega_{CM}}(\theta_{CM})\sin\theta_{CM}\,d\theta_{CM}$$

reorganize →

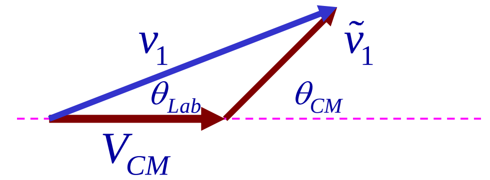
$$\frac{d\sigma}{d\Omega_{Lab}}(\theta_{Lab}) = \frac{d\sigma}{d\Omega_{CM}}(\theta_{CM})\frac{d(\cos\theta_{CM})}{d(\cos\theta_{Lab})}$$

Using Eq. 1.53 →

$$\frac{d\sigma}{d\Omega_{Lab}}(\theta_{Lab}) = \frac{d\sigma}{d\Omega_{CM}}(\theta_{CM})\frac{\left(1+2\zeta\cos\theta_{CM}+\zeta^2\right)^{3/2}}{\left|1+\zeta\cos\theta_{CM}\right|}$$

Scattering angles in Lab and CM

- θ_{CM} represents the change in the direction of the relative position vector **r** as a result of the collision in CM frame



- Thus, it must be identical to the scattering angle for a particle with reduced mass, μ.
- Z components of the velocities of scattered particle with m_1 in lab and CM are:

$$v\cos\theta_{Lab} - v_{CM} = \tilde{v}_1 \cos\theta_{CM}$$

- The perpendicular components of the velocities are:

$$v\sin\theta_{Lab} = \tilde{v}_1 \sin\theta_{CM}$$ (boost is only in the z direction)

- Thus, the angles are related (for elastic scattering only) as:

$$\tan\theta_{Lab} = \frac{\sin\theta_{CM}}{\cos\theta_{CM} + v_{CM}/\tilde{v}_1} = \frac{\sin\theta_{CM}}{\cos\theta_{CM} + m_1/m_2} = \frac{\sin\theta_{CM}}{\cos\theta_{CM} + \zeta}$$

with $v_{CM} = \frac{m_1 v_1}{m_1 + m_2}$ $\quad \tilde{v}_1 = \frac{m_2 v_1}{m_1 + m_2}$ (Later we use a version of this equation solving for cos)