PHYS 3446 – Lecture #6

Tuesday, Feb. 10 2015 Dr. ***Brandt***

1. Relativistic Variables
2. Invariant Scalars
3. Feynman Diagram

Special Relativity Variables

- Fractional velocity $\vec{\beta} = \vec{v}/c$
- Lorentz γ factor $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$
- Relative momentum and the total energy of the particle moving at a velocity $\vec{v} = \vec{\beta} c$ is

$$\vec{P} = \gamma M \vec{v} = \gamma M \vec{\beta} c$$

$$E = \sqrt{T^2 + E_{Rest}^2} = \sqrt{P^2 c^2 + M^2 c^4} = \gamma M c^2$$

- Square of four momentum P=(E/c,**p**), rest mass E

Appendix 1 review

Relativistic Variables

- Velocity of CM in the scattering of two particles with rest mass m_1 and m_2 is:

$$\vec{\beta}_{CM} = \frac{\vec{v}_{CM}}{c} = \frac{\left(\vec{P}_1 + \vec{P}_2\right)c}{E_1 + E_2}$$

- If m_1 is the mass of the projectile and m_2 is that of the target, for a fixed target we obtain

$$\vec{\beta}_{CM} = \frac{\vec{P}_1 c}{E_1 + m_2 c^2} = \frac{\vec{P}_1 c}{\sqrt{P_1^2 c^2 + m_1^2 c^4} + m_2 c^2}$$

Relativistic Variables-Special Cases

- At very low energies where $m_1c^2 >> P_1c$, the velocity reduces to:

$$\vec{\beta}_{CM} = \frac{m_1 \vec{v}_1 c}{m_1 c^2 + m_2 c^2} = \frac{m_1 \vec{v}_1}{(m_1 + m_2)c}$$

- At very high energies where $m_1c^2 << P_1c$ and $m_2c^2 << P_1c$, the velocity can be written as:

$$\beta_{CM} = \left|\vec{\beta}_{CM}\right| = \frac{1}{\sqrt{1 + \left(\frac{m_1 c^2}{P_1 c}\right)^2 + \frac{m_2 c^2}{P_1 c}}} \approx 1 - \frac{m_2 c}{P_1} - \frac{1}{2}\left(\frac{m_1 c}{P_1}\right)^2$$

Expansion

Relativistic Variables

- For high energies, if $m_1 \sim m_2$,

$$\beta_{CM} \approx 1 - \frac{m_2 c}{P_1}$$

- γ_{CM} becomes:

$$\gamma_{CM} = \left(1 - \beta_{CM}^2\right)^{-1/2} = \left[\left(1 - \beta_{CM}\right)\left(1 + \beta_{CM}\right)\right]^{-1/2} \approx \left[2\left(\frac{m_2 c}{P_1}\right)\right]^{-1/2} = \sqrt{\frac{P_1}{2 m_2 c}}$$

- In general, for fixed target we showed

$$\vec{\beta}_{CM} = \frac{\vec{P}_1 c}{E_1 + m_2 c^2}$$

$$1 - \beta_{CM}^2 = \frac{m_1^2 c^4 + 2 E_1 m_2 c^2 + m_2^2 c^4}{\left(E_1 + m_2 c^2\right)^2}$$

- Thus γ_{CM} becomes

$$\gamma_{CM} = \left(1 - \beta_{CM}^2\right)^{-1/2} = \frac{E_1 + m_2 c^2}{\sqrt{m_1^2 c^4 + 2 E_1 m_2 c^2 + m_2^2 c^4}}$$

Invariant Scalars

Relativistic Variables

- The invariant scalar, s, is defined as:

$$s = (P_1 + P_2)^2 = (E_1 + E_2)^2 - (\vec{P}_1 + \vec{P}_2)^2 c^2$$

$$= m_1^2 c^4 + m_2^2 c^4 + 2E_1 E_2 - 2 p_1 p_2 c^2$$

- Evaluate in lab frame where $p_2=0$ $E_2=m_2c^2$

Using $E^2-P^2=M^2$

$$s = m_1^2 c^4 + m_2^2 c^4 + 2E_1 m_2 c^2$$

- So what is this in the CM frame?

$$= (E_{1CM} + E_{2CM})^2 - (\vec{P}_{1CM} + \vec{P}_{2CM})^2 c^2$$

0

$$= (E_{1CM} + E_{2CM})^2 = (E_{ToT}^{CM})^2$$

$$= m_1^2 c^4 + m_2^2 c^4 + 2E_1 m_2 c^2$$

- Thus, $\sqrt{s}$ represents the total available energy in the CM; At the LHC eventually $\sqrt{s} = 14\text{TeV}$ (root-s is now 8 TeV)

Useful Invariant Scalar Variables

- Another invariant scalar, t, the momentum transfer (difference in four momenta), is useful for scattering:

$$t = \left(P_f - P_i\right)^2 = \left(E_1^f - E_1^i\right)^2 - \left(\vec{P}_1^{\,f} - \vec{P}_1^{\,i}\right)^2 c^2$$

- Since momentum and total energy are conserved in all collisions, t can be expressed in terms of target variables

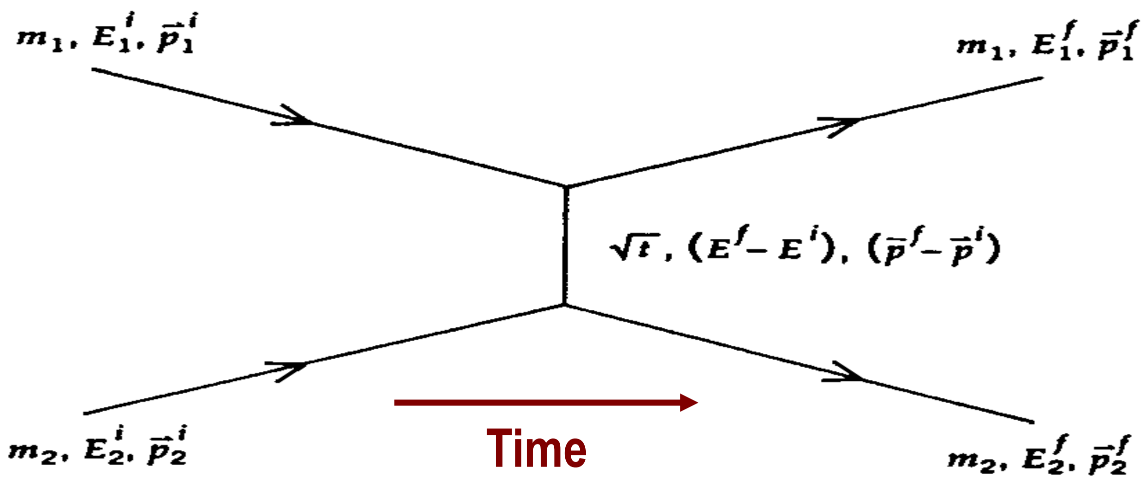
$$t = \left(E_2^f - E_2^i\right)^2 - \left(\vec{P}_2^{\,f} - \vec{P}_2^{\,i}\right)^2 c^2$$

- In CM frame for an elastic scattering, where $P^i_{CM} = P^f_{CM} = P_{CM}$ and $E^i_{CM} = E^f_{CM}$:

$$t = -\left(P_{CM}^{f\,2} + P_{CM}^{i\,2} - 2\vec{P}_{CM}^{\,f} \cdot \vec{P}_{CM}^{\,i}\right) c^2 = -2P_{CM}^2 c^2\left(1 - \cos\theta_{CM}\right).$$

Feynman Diagram

- The variable t is always negative for elastic scattering
- The variable t could be viewed as the square of the invariant mass of a particle with $E_2^f - E_2^i$ and $\vec{P}_2^f - \vec{P}_2^i$ exchanged in the scattering



t-channel diagram

Momentum of the carrier is the difference between the two particles.

- While the virtual particle cannot be detected in the scattering, the consequence of its exchange can be calculated and observed!!!
  - A virtual particle is a particle whose mass is less than the rest mass of an equivalent free particle

Useful Invariant Scalar Variables

- For convenience we define a variable q^2, $q^2c^2 = -t$
- In the lab frame, $\vec{P}^i_{2Lab} = 0$, thus we obtain:

$$q^2c^2 = -\left[\left(E^f_{2Lab} - m_2c^2\right)^2 - \left(P^f_{2Lab}c\right)^2\right]$$
$$= 2m_2c^2\left(E^f_{2Lab} - m_2c^2\right) = 2m_2c^2T^f_{2Lab}$$

- In the non-relativistic limit: $T^f_{2Lab} \approx \frac{1}{2}m_2v_2^2$
- q^2 represents "hardness of the collision". High q^2 is more violent collision Small θ_{CM} corresponds to small q^2.
- $$\frac{d\sigma}{dq^2} = \frac{4\pi\left(kZZ'e^2\right)^2}{v^2}\frac{1}{q^4}$$ Divergence at q^2~0 characteristic of a Coulomb field

Problem 1.10 *What is the minimum impact parameter needed to deflect 7.7 MeV α-particles from gold nuclei by at least 1°? What about by at least 30°? What is the ratio of probabilities for deflections of $\theta > 1°$ relative to $\theta > 30°$? (See the CRC Handbook for the density of gold.)*

For the scattering of a 7.7 MeV α-particle from gold, we have

$$Z = 2, \quad Z' = 79, \quad E = 7.7\,\text{MeV}, \tag{1.74}$$

so that we obtain

$$\begin{aligned}\frac{ZZ'e^2}{2E} &= ZZ' \times \frac{\hbar c}{2E} \times \frac{e^2}{\hbar c} \\ &= 2 \times 79 \times \frac{197\,\text{MeV} - \text{F}}{2 \times 7.7\,\text{MeV}} \times \frac{1}{137} \\ &\approx 14.5 \times 10^{-13}\,\text{cm} \approx 1.4 \times 10^{-12}\,\text{cm}.\end{aligned} \tag{1.75}$$

We know from Eq. (1.32) of the text that

$$b = \frac{ZZ'e^2}{2E}\cot\frac{\theta}{2}, \tag{1.76}$$

which leads to

$$b(\theta) \approx 1.4 \times 10^{-12}\cot\frac{\theta}{2}\,\text{cm}. \tag{1.77}$$

We note that for

$$\theta = 1° = \frac{\pi}{180} \approx \frac{1}{60} \ll 1, \tag{1.78}$$

we have

$$\tan\frac{\theta}{2} \approx \frac{\theta}{2} \approx \frac{1}{120}, \quad \cot\frac{\theta}{2} = \frac{1}{\tan\frac{\theta}{2}} \approx 120. \tag{1.79}$$

Similarly, for

$$\theta = 30° = \frac{\pi}{6} \approx \frac{1}{2} \tag{1.80}$$

we have

$$\tan\frac{\theta}{2} \approx \frac{\theta}{2} \approx \frac{1}{4}, \quad \cot\frac{\theta}{2} = \frac{1}{\tan\frac{\theta}{2}} \approx 4. \tag{1.81}$$

Using these values in (1.77), we obtain

$$\begin{aligned}b(\theta = 1°) &\approx 1.4 \times 10^{-12} \times 120\,\text{cm} = 1.7 \times 10^{-10}\,\text{cm}, \\ b(\theta = 30°) &\approx 1.4 \times 10^{-12} \times 4\,\text{cm} = 5.6 \times 10^{-12}\,\text{cm}.\end{aligned} \tag{1.82}$$

As we have already seen in Problem 1.1, the probability of scattering for angles greater than θ_b goes as the area πb^2. Therefore, using (1.82) we have

$$\begin{aligned}\frac{\sigma(\theta > 1°)}{\sigma(\theta > 30°)} &= \frac{b^2(\theta = 1°)}{b^2(\theta = 30°)} \\ &\approx \left(\frac{1.7 \times 10^{-10}}{5.6 \times 10^{-12}}\right)^2 \approx 900.\end{aligned} \tag{1.83}$$

In other words, there will be approximately 900 more particle collisions for $\theta > 1°$ than for $\theta > 30°$.

CGS Units

- Review cgs units. See for example:

  http://en.wikipedia.org/wiki/Statcoulomb

  10^7 erg =J erg=g cm/s^2

  e=4.8×10^{-10} stat-coulombs (or esu) if esu^2/erg^2 =cm ->
  esu= $g^{1/2}cm^{3/2}/s^2$