PHYS 1443 – Section 003
Lecture #13

Monday, Oct. 28, 2002
Dr. Jaehoon Yu

1. Rotational Kinetic Energy
2. Calculation of Moment of Inertia
3. Relationship Between Angular and Linear Quantities
4. Review

There is no homework today!! Prepare well for the exam!!

Announcements

- 2^{nd} Term exam
  - **This Wednesday, Oct. 30**, in the class
  - Covers **chapters 6 – 10**
  - No need to bring blue book
  - Some fundamental formulae will be given
  - Bring your calculators but delete all the formulae

Rotational Kinematics

The first type of motion we have learned in linear kinematics was under a constant acceleration. We will learn about the rotational motion under constant acceleration, because these are the simplest motions in both cases.

Just like the case in linear motion, one can obtain

Angular Speed under constant angular acceleration:

$$w_f = w_i + at$$

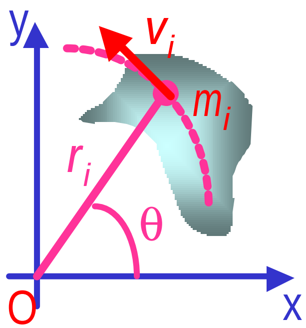
Angular displacement under constant angular acceleration:

$$q_f = q_i + w_i t + \frac{1}{2} a t^2$$

One can also obtain

$$w_f^2 = w_i^2 + 2a\left(q_f - q_i\right)$$

Rotational Energy



What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, m_i, moving at a tangential speed, v_i, is

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 w^2$$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

$$K_R = \sum_i K_i = \frac{1}{2} \sum_i m_i r_i^2 w^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right) w^2$$

By defining a new quantity called, Moment of Inertia, *I*, as

$$\boxed{I = \sum_i m_i r_i^2}$$

The above expression is simplified as

$$\boxed{K_R = \frac{1}{2} I w^2}$$

What are the dimension and unit of Moment of Inertia? $kg \cdot m^2$ $\left[ML^2\right]$

What do you think the moment of inertia is?

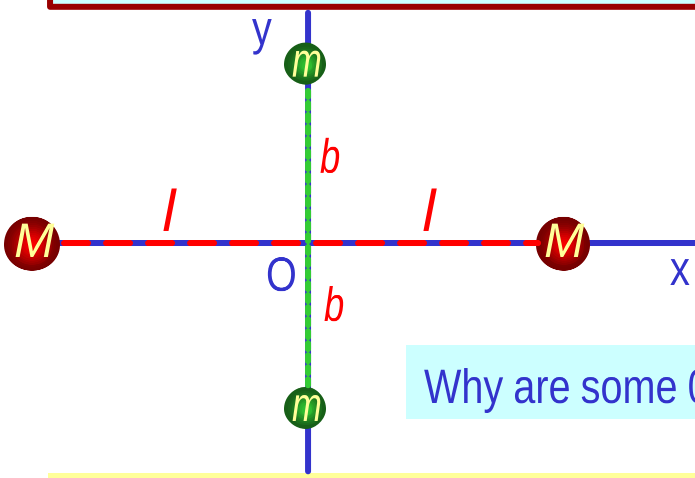
Measure of resistance of an object to changes in its rotational motion.

What similarity do you see between rotational and linear kinetic energies?

Mass and speed in linear kinetic energy are replaced by moment of inertia and angular speed.

Example 10.4

In a system consists of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at ω.



Since the rotation is about y axis, the moment of inertia about y axis, I_y, is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m \cdot 0^2 + m \cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2\right)\omega^2 = Ml^2\omega^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2\left(Ml^2 + mb^2\right)$$

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(2Ml^2 + 2mb^2\right)\omega^2 = \left(Ml^2 + mb^2\right)\omega^2$$

Calculation of Moments of Inertia

Moments of inertia for large objects can be computed, if we assume the object consists of small volume elements with mass, Δm_i.

The moment of inertia for the large rigid object is $I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$

It is sometimes easier to compute moments of inertia in terms of volume of the elements rather than their mass

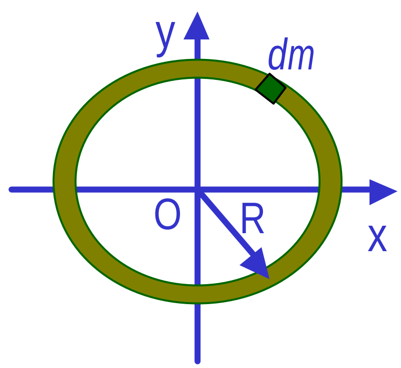
How can we do this?

Using the volume density, ρ, replace d*m* in the above equation with dV.

$$\boxed{r = \frac{dm}{dV}} \quad \boxed{dm = r dV}$$

The moments of inertia becomes $\boxed{I = \int r r^2 dV}$

Example 10.5: Find the moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.



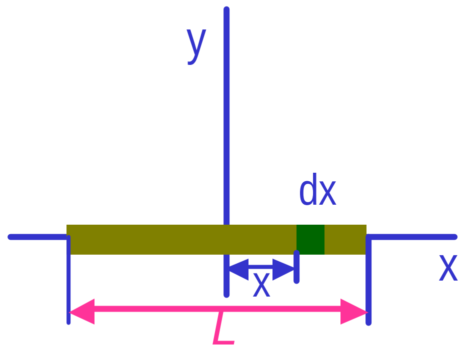
The moment of inertia is $I = \int r^2 dm = R^2 \int dm = MR^2$

What do you notice from this result?

The moment of inertia for this object is the same as that of a point of mass M at the distance R.

Example 10.6

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing through its center of mass.



The line density of the rod is $l = \frac{M}{L}$

so the masslet is $dm = l\,dx = \frac{M}{L}dx$

The moment of inertia is

$$I = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{M}{3L}\left(\frac{L^3}{4}\right) = \frac{ML^2}{12}$$

What is the moment of inertia when the rotational axis is at one end of the rod.

$$I = \int r^2 dm = \int_0^L \frac{x^2 M}{L} dx = \frac{M}{L}\left[\frac{1}{3}x^3\right]_0^L$$

$$= \frac{M}{3L}\left[(L)^3 - 0\right] = \frac{M}{3L}\left(L^3\right) = \frac{ML^2}{3}$$

Will this be the same as the above. Why or why not?

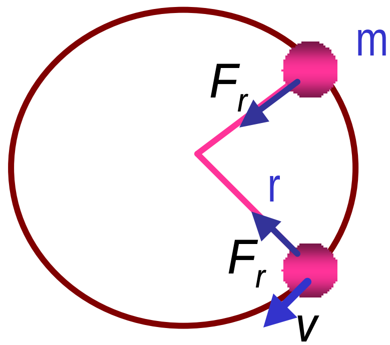
Since the moment of inertia is resistance to motion, it makes perfect sense for it to be harder to move when it is rotating about the axis at one end.

Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Similar Quantity	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = \int r^2 dm$
Length of motion	Distance L	Angle q (Radian)
Speed	$v = \frac{dr}{dt}$	$w = \frac{dq}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$a = \frac{dw}{dt}$
Force	Force $F = ma$	Torque $t = Ia$
Work	Work $W = \int_{x_i}^{x_f} F dx$	Work $W = \int_{q_i}^{q_f} t dq$
Power	$P = \vec{F} \cdot \vec{v}$	$P = tw$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{w}$
Kinetic Energy	Kinetic $K = \frac{1}{2}mv^2$	Rotational $K_R = \frac{1}{2}Iw^2$

Newton's Second Law & Uniform Circular Motion



The centripetal acceleration is always perpendicular to velocity vector, v, for uniform circular motion.

$$a_r = \frac{v^2}{r}$$

Are there forces in this motion? If so, what do they do?

*The force that causes the centripetal acceleration acts toward the center of the circular path and causes a change in the direction of the velocity vector. This force is called **centripetal force**.*

$$\sum F_r = ma_r = m\frac{v^2}{r}$$

What do you think will happen to the ball if the string that holds the ball breaks? Why?

Based on Newton's 1st law, since the external force no longer exist, the ball will continue its motion without change and will fly away following the tangential direction to the circle.

Motion in Accelerated Frames

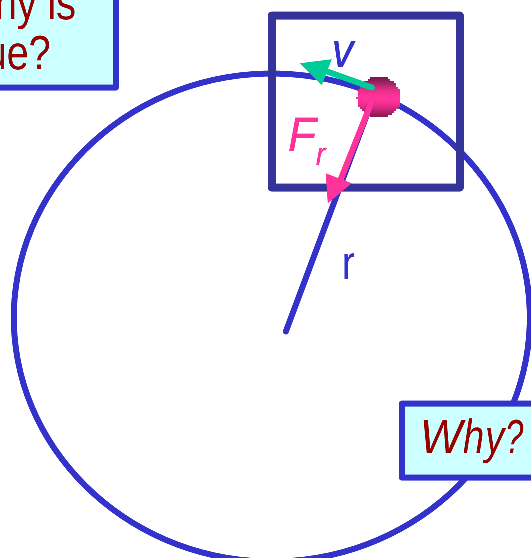
Newton's laws are valid only when observations are made in an inertial frame of reference. What happens in a non-inertial frame?

Fictitious forces are needed to apply Newton's second law in an accelerated frame.

This force does not exist when the observations are made in an inertial reference frame.

What does this mean and why is this true?

Let's consider a free ball inside a box under uniform circular motion.



How does this motion look like in an inertial frame (or frame outside a box)?

We see that the box has a radial force exerted on it but none on the ball directly, until…

How does this motion look like in the box?

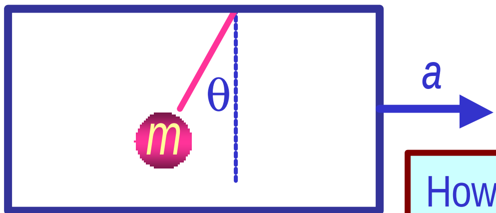
The ball is tumbled over to the wall of the box and feels that it is getting force that pushes it toward the wall.

Why?

According to Newton's first law, the ball wants to continue on its original movement tangentially but since the box is turning, the ball feels like it is being pushed toward the wall relative to everything else in the box.

Example 6.9

A ball of mass m is hung by a cord to the ceiling of a boxcar that is moving with an acceleration *a*. What do the inertial observer at rest and the non-inertial observer traveling inside the car conclude? How do they differ?

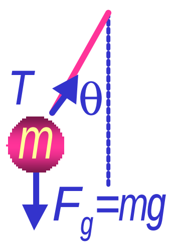


This is how the ball looks like no matter which frame you are in.

How do the free-body diagrams look for two frames?

How do the motions interpreted in these two frames? Any differences?

Inertial Frame



$$\sum \vec{F} = \vec{F}_g + \vec{T}$$

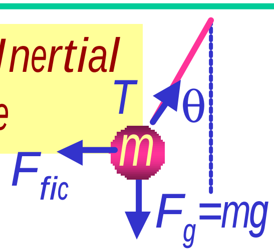
$$\sum F_x = ma_x = ma_c = T\sin\theta$$

$$\sum F_y = T\cos\theta - mg = 0$$

$$T = \frac{mg}{\cos\theta} \qquad a_c = g\tan\theta$$

For an inertial frame observer, the forces being exerted on the ball are only **T** and **F**$_g$. The acceleration of the ball is the same as that of the box car and is provided by the x component of the tension force.

Non-Inertial Frame



$$\sum \vec{F} = \vec{F}_g + \vec{T} + \vec{F}_{fic}$$

$$\sum F_x = T\sin\theta - F_{fic} = 0 \qquad F_{fic} = ma_{fic} = T\sin\theta$$

$$\sum F_y = T\cos\theta - mg = 0$$

$$T = \frac{mg}{\cos\theta} \qquad a_{fic} = g\tan\theta$$

In the non-inertial frame observer, the forces being exerted on the ball are **T**, **F**$_g$, and **F**$_{fic}$.

For some reason the ball is under a force, **F**$_{fic}$, that provides acceleration to the ball.

While the mathematical expression of the acceleration of the ball is identical to that of inertial frame observer's, the cause of the force, or physical law is dramatically different.

Work Done by a Constant Force

Work in physics is done only when a sum of forces exerted on an object made a motion to the object.

Which force did the work? Force $\vec{F}$

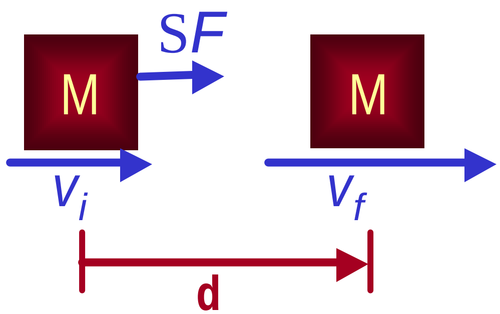
How much work did it do? $\boxed{W = \sum \vec{F} \cdot \vec{d} = Fd\cos\theta}$ Unit? $\boxed{N \bullet m = J \text{ (for Joule)}}$

What does this mean? **Physical work is done only by the component of of the force along the movement of the object.**

Work is energy transfer!!

Kinetic Energy and Work-Kinetic Energy Theorem

- Some problems are hard to solve using Newton's second law
  - If forces exerting on the object during the motion are so complicated
  - Relate the work done on the object by the net force to the change of the speed of the object



Suppose net force ΣF was exerted on an object for displacement d to increase its speed from v_i to v_f.

The work on the object by the net force ΣF is

$$W = \left(\sum \vec{F}\right)\cdot \vec{d} = (ma)d\cos 0 = (ma)d$$

Displacement $$d = \frac{1}{2}\left(v_f + v_i\right)t$$

Acceleration $$a = \frac{v_f - v_i}{t}$$

Work $$W = (ma)d = \left[m\left(\frac{v_f - v_i}{t}\right)\right]\frac{1}{2}\left(v_f + v_i\right)t = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

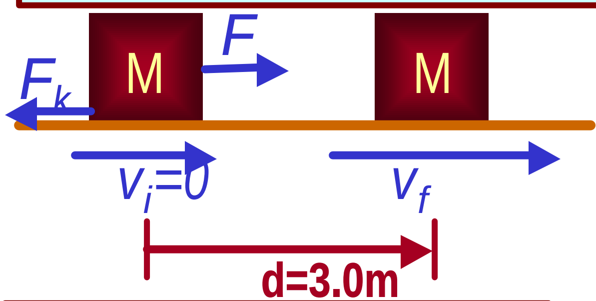
Kinetic Energy $$KE = \frac{1}{2}mv^2$$

Work $$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = KE_f - KE_i = \Delta KE$$

The work done by the net force caused change of the object's kinetic energy.

Example 7.8

A 6.0kg block initially at rest is pulled to East along a horizontal surface with coefficient of kinetic friction μ_k=0.15 by a constant horizontal force of 12N. Find the speed of the block after it has moved 3.0m.



Work done by the force F is

$$W_F = \vec{F}\cdot\vec{d} = \left|\vec{F}\right|\left|\vec{d}\right|\cos\theta = 12\times 3.0\cos 0 = 36(J)$$

Work done by friction F_k is

$$W_k = \vec{F}_k\cdot\vec{d} = \left|\mu_k mg\right|\left|\vec{d}\right|\cos\theta$$
$$= 0.15\times 6.0\times 9.8\times 3.0\cos 180 = -26(J)$$

Thus the total work is

$$W = W_F + W_k = 36 - 26 = 10(J)$$

Using work-kinetic energy theorem and the fact that initial speed is 0, we obtain

$$W = W_F + W_k = \frac{1}{2}mv_f^2$$

Solving the equation for v_f, we obtain

$$v_f = \sqrt{\frac{2W}{m}} = \sqrt{\frac{2\times 10}{6.0}} = 1.8m/s$$

Work and Kinetic Energy

Work in physics is done only when a sum of forces exerted on an object made a motion to the object.

What does this mean?

However much tired your arms feel, if you were just holding an object without moving it you have not done any physical work.

Mathematically, work is written in scalar product of force vector and the displacement vector

$$W = \sum \vec{F}_i \cdot \vec{d} = Fd\cos\theta$$

Kinetic Energy is the energy associated with motion and capacity to perform work. Work requires change of energy after the completion ← **Work-Kinetic energy theorem**

$$K = \frac{1}{2}mv^2$$

$$\sum W = K_f - K_i = \Delta K$$

Nm=Joule

Power

- Rate at which work is done
  - What is the difference for the same car with two different engines (4 cylinder and 8 cylinder) climbing the same hill? ➔ 8 cylinder car climbs up faster

Is the amount of work done by the engines different? NO

Then what is different? The rate at which the same amount of work performed is higher for 8 cylinder than 4.

Average power $\boxed{\overline{P} = \frac{W}{\Delta t}}$

Instantaneous power $\boxed{P \equiv \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} = \vec{F} \cdot \frac{d}{dt}\left(\vec{s}\right) = \vec{F} \cdot \vec{v} = Fv \cos q}$

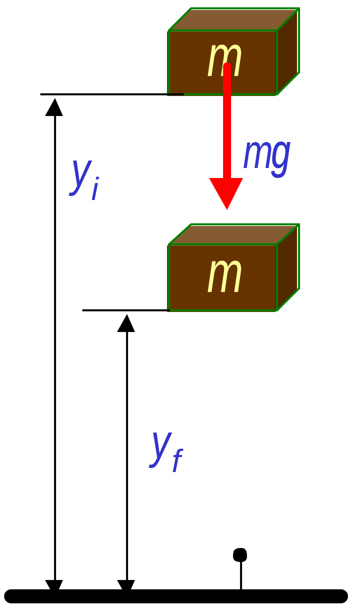
Unit? $\boxed{J/s = Watts}$ $\boxed{1HP = 746\,Watts}$

What do power companies sell? $\boxed{1kWH = 1000\,Watts \times 3600\,s = 3.6 \times 10^{6}\,J}$

Energy

Gravitational Potential Energy

Potential energy given to an object by gravitational field in the system of Earth due to its height from the surface



When an object is falling, gravitational force, Mg, performs work on the object, increasing its kinetic energy. The potential energy of an object at a height y which is the potential to work is expressed as

$$U_g = \vec{F}_g \cdot \vec{y} = mg\left(-\vec{j}\right) \cdot y\left(-\vec{j}\right) = mgy$$

$$U_g \equiv mgy$$

Work performed on the object by the gravitational force as the brick goes from y_i to y_f is:

$$W_g = U_i - U_f$$
$$= mgy_i - mgy_f = -\Delta U_g$$

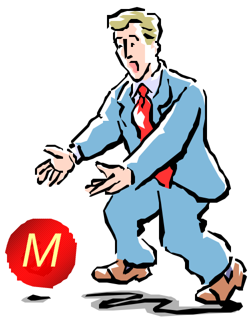
What does this mean?

Work by the gravitational force as the brick goes from y_i to y_f is negative of the change in the system's potential energy

➔ Potential energy was lost in order for gravitational force to increase the brick's kinetic energy.

Example 8.1

A bowler drops bowling ball of mass 7kg on his toe. Choosing floor level as y=0, estimate the total work done on the ball by the gravitational force as the ball falls.



Let's assume the top of the toe is 0.03m from the floor and the hand was 0.5m above the floor.

$$U_i = mgy_i = 7\times9.8\times0.5 = 34.3J \qquad U_f = mgy_f = 7\times9.8\times0.03 = 2.06J$$

$$\Delta U = -\left(U_f - U_i\right) = 32.24\,J \cong 30\,J$$

b) Perform the same calculation using the top of the bowler's head as the origin.

What has to change?

First we must re-compute the positions of ball at the hand and of the toe.

Assuming the bowler's height is 1.8m, the ball's original position is –1.3m, and the toe is at –1.77m.

$$U_i = mgy_i = 7\times9.8\times\left(-1.3\right) = -89.2J \qquad U_f = mgy_f = 7\times9.8\times\left(-1.77\right) = -121.4J$$

$$\Delta U = -\left(U_f - U_i\right) = 32.2\,J \cong 30\,J$$

Elastic Potential Energy

Potential energy given to an object by a spring or an object with elasticity in the system consists of the object and the spring without friction.

The force spring exerts on an object when it is distorted from its equilibrium by a distance x is

$$F_s = -kx$$

The work performed on the object by the spring is

$$W_s = \int_{x_i}^{x_f} (-kx)dx = \left[-\frac{1}{2}kx^2\right]_{x_i}^{x_f} = -\frac{1}{2}kx_f^2 + \frac{1}{2}kx_i^2 = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

The potential energy of this system is

$$U_s \equiv \frac{1}{2}kx^2$$

What do you see from the above equations?

The work done on the object by the spring depends only on the initial and final position of the distorted spring.

Where else did you see this trend?

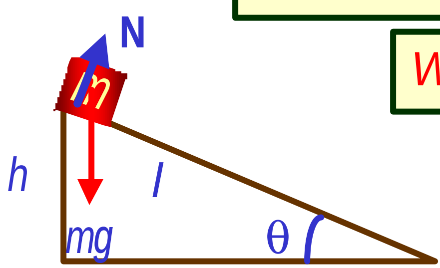
The gravitational potential energy, U_g

So what does this tell you about the elastic force?

A conservative force!!!

Conservative and Non-conservative Forces

The work done on an object by the gravitational force does not depend on the object's path.



When directly falls, the work done on the object is

$$W_g = mgh$$

When sliding down the hill of length l, the work is

$$W_g = F_{g-incline} \times l = mg \sin\theta \times l$$

$$W_g = mg\left(l \sin\theta\right) = mgh$$

How about if we lengthen the incline by a factor of 2, keeping the height the same??

Still the same amount of work☺

$$W_g = mgh$$

So the work done by the gravitational force on an object is independent on the path of the object's movements. It only depends on the difference of the object's initial and final position in the direction of the force.

The forces like gravitational or elastic forces are called conservative forces

1. *If the work performed by the force does not depend on the path*
2. *If the work performed on a closed path is 0.*

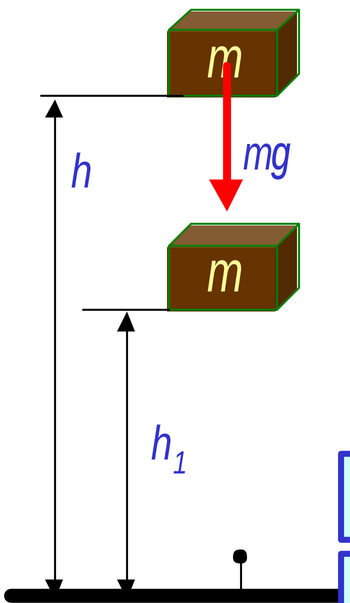
Total mechanical energy is conserved!!

$$E_M \equiv KE_i + PE_i = KE_f + PE_f$$

Conservation of Mechanical Energy

Total mechanical energy is the sum of kinetic and potential energies

$$E \equiv K + U$$



Let's consider a brick of mass *m* at a height *h* from the ground

What is its potential energy?

$$U_g = mgh$$

What happens to the energy as the brick falls to the ground?

$$\Delta U = U_f - U_i = -\int_{x_i}^{x_f} F_x dx$$

The brick gains speed

By how much?

$$v = gt$$

So what?

The brick's kinetic energy increased

$$K = \frac{1}{2}mv^2 = \frac{1}{2}mg^2t^2$$

And?

The lost potential energy converted to kinetic energy

What does this mean?

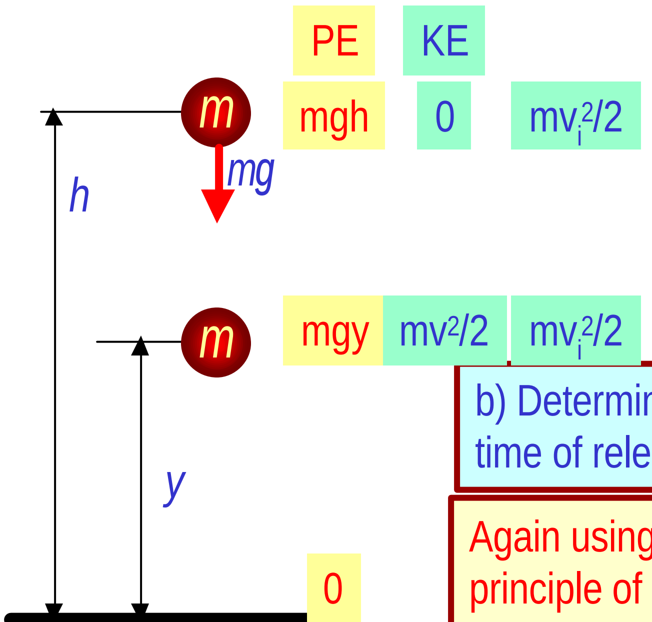
The total mechanical energy of a system remains constant in any isolated system of objects that interacts only through conservative forces: ***Principle of mechanical energy conservation***

$$E_i = E_f$$

$$K_i + \sum U_i = K_f + \sum U_f$$

Example 8.2

A ball of mass *m* is dropped from a height *h* above the ground. Neglecting air resistance determine the speed of the ball when it is at a height *y* above the ground.



Using the principle of mechanical energy conservation

$$K_i + U_i = K_f + U_f$$

$$0 + mgh = \frac{1}{2}mv^2 + mgy$$

$$\frac{1}{2}mv^2 = mg(h-y)$$

$$\therefore v = \sqrt{2g(h-y)}$$

b) Determine the speed of the ball at *y* if it had initial speed v_i at the time of release at the original height *h*.

Again using the principle of mechanical energy conservation but with non-zero initial kinetic energy!!!

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mv_i^2 + mgh = \frac{1}{2}mv_f^2 + mgy$$

$$\frac{1}{2}m\left(v_f^2 - v_i^2\right) = mg(h-y)$$

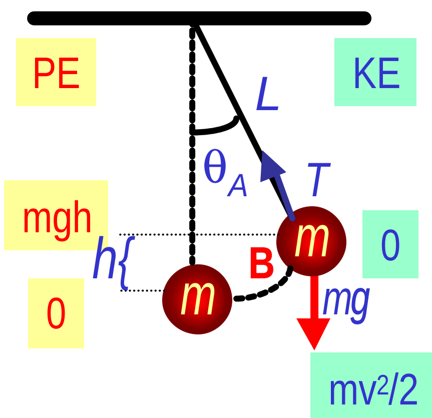
$$\therefore v_f = \sqrt{v_i^2 + 2g(h-y)}$$

This result look very similar to a kinematic expression, doesn't it? Which one is it?

Ex08-03.ip

Example 8.3

A ball of mass *m* is attached to a light cord of length L, making up a pendulum. The ball is released from rest when the cord makes an angle θ_A with the vertical, and the pivoting point P is frictionless. Find the speed of the ball when it is at the lowest point, B.



Compute the potential energy at the maximum height, *h*. Remember where the 0 is.

$$h = L - L\cos\theta_A = L(1-\cos\theta_A)$$

$$U_i = mgh = mgL(1-\cos\theta_A)$$

Using the principle of mechanical energy conservation

$$K_i + U_i = K_f + U_f$$

$$0 + mgh = mgL(1-\cos\theta_A) = \frac{1}{2}mv^2$$

$$v^2 = 2gL(1-\cos\theta_A) \quad \therefore v = \sqrt{2gL(1-\cos\theta_A)}$$

b) Determine tension T at the point B.

Using Newton's 2nd law of motion and recalling the centripetal acceleration of a circular motion

$$\sum F_r = T - mg = ma_r = m\frac{v^2}{r} = m\frac{v^2}{L}$$

$$T = mg + m\frac{v^2}{L} = m\left(g + \frac{v^2}{L}\right) = m\left(g + \frac{2gL(1-\cos\theta_A)}{L}\right)$$

$$= m\frac{gL + 2gL(1-\cos\theta_A)}{L}$$

$$\therefore T = mg(3 - 2\cos\theta_A)$$

Cross check the result in a simple situation. What happens when the initial angle θ_A is 0? $T = mg$

Work Done by Non-conserve Forces

Mechanical energy of a system is not conserved when any one of the forces in the system is a non-conservative force.

Two kinds of non-conservative forces:

Applied forces: Forces that are external to the system. These forces can take away or add energy to the system. So the mechanical energy of the system is no longer conserved.

If you were to carry around a ball, the force you apply to the ball is external to the system of ball and the Earth. Therefore, you add kinetic energy to the ball-Earth system.

$$W_{you} + W_g = \Delta K; \quad W_g = -\Delta U$$

$$W_{you} = W_{app} = \Delta K + \Delta U$$

Kinetic Friction: Internal non-conservative force that causes irreversible transformation of energy. The friction force causes the kinetic and potential energy to transfer to internal energy

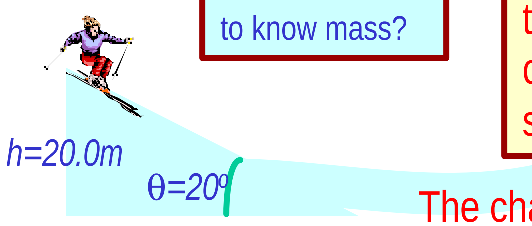
$$W_{friction} = \Delta K_{friction} = -f_k d$$

$$\Delta E = E_f - E_i = \Delta K + \Delta U = -f_k d$$

Example 8.6

Ex08-06.ip

A skier starts from rest at the top of frictionless hill whose vertical height is 20.0*m* and the inclination angle is 20°. Determine how far the skier can get on the snow at the bottom of the hill with a coefficient of kinetic friction between the ski and the snow is 0.210.



Don't we need to know mass?

Compute the speed at the bottom of the hill, using the mechanical energy conservation on the hill before friction starts working at the bottom

$$ME = mgh = \frac{1}{2}mv^2$$

$$v = \sqrt{2gh}$$

$$v = \sqrt{2\times 9.8\times 20.0} = 19.8m/s$$

The change of kinetic energy is the same as the work done by kinetic friction.

What does this mean in this problem?

$$\Delta K = K_f - K_i = -f_k d$$

Since $K_f = 0$

$$-K_i = -f_k d; \quad f_k d = K_i$$

$$f_k = \mu_k n = \mu_k mg$$

$$d = \frac{K_i}{\mu_k mg} = \frac{\frac{1}{2}mv^2}{\mu_k mg} = \frac{v^2}{2\mu_k g} = \frac{(19.8)^2}{2\times 0.210\times 9.80} = 95.2m$$

Since we are interested in the distance the skier can get to before stopping, the friction must do as much work as the available kinetic energy.

Well, it turns out we don't need to know mass.

What does this mean?

No matter how heavy the skier is he will get as far as anyone else has gotten.

General Energy Conservation and Mass-Energy Equivalence

General Principle of Energy Conservation

The total energy of an isolated system is conserved as long as all forms of energy are taken into account.

What about friction?

Friction is a non-conservative force and causes mechanical energy to change to other forms of energy.

However, if you add the new form of energy altogether the system as a whole did not lose any energy, as long as it is self-contained or isolated.

In the grand scale of the universe, no energy can be destroyed or created but just transformed or transferred from one place to another. Total energy of universe is constant.

Principle of Conservation of Mass

In any physical or chemical process, mass is neither created nor destroyed. Mass before a process is identical to the mass after the process.

Einstein's Mass-Energy equality.

$$\boxed{E_R = mc^2}$$

How many joules does your body correspond to?

Linear Momentum and Forces

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(m\vec{v}\right)$$

What can we learn from this Force-momentum relationship?

- The rate of the change of particle's momentum is the same as the net force exerted on it.
- When net force is 0, the particle's linear momentum is constant.
- If a particle is isolated, the particle experiences no net force, therefore its momentum does not change and is conserved.

Something else we can do with this relationship. What do you think it is?

The relationship can be used to study the case where the mass changes as a function of time.

Can you think of a few cases like this?

Motion of a meteorite

Trajectory a satellite

Conservation of Linear Momentum in a Two Particle System

Consider a system with two particles that does not have any external forces exerting on it. What is the impact of Newton's 3rd Law?

If particle#1 exerts force on particle #2, there must be another force that the particle #2 exerts on #1 as the reaction force. Both the forces are internal forces and the net force in the SYSTEM is still 0.

Now how would the momenta of these particles look like?

Let say that the particle #1 has momentum p_1 and #2 has p_2 at some point of time.

Using momentum-force relationship

$$\vec{F}_{21} = \frac{d\vec{p}_1}{dt}$$ and $$\vec{F}_{12} = \frac{d\vec{p}_2}{dt}$$

And since net force of this system is 0

$$\sum \vec{F} = \vec{F}_{12} + \vec{F}_{21} = \frac{d\vec{p}_2}{dt} + \frac{d\vec{p}_1}{dt} = \frac{d}{dt}\left(\vec{p}_2 + \vec{p}_1\right) = 0$$

Therefore $\vec{p}_2 + \vec{p}_1 = const$

The total linear momentum of the system is conserved!!!

Impulse and Linear Momentum

Net force causes change of momentum → Newton's second law

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$d\vec{p} = \vec{F}dt$$

By integrating the above equation in a time interval t_i to t_f, one can obtain impulse ***I***.

$$\int_{t_i}^{t_f} d\vec{p} = \vec{p}_f - \vec{p}_i = \Delta\vec{p} = \int_{t_i}^{t_f} \vec{F}dt$$

$$\vec{I} \equiv \int_{t_i}^{t_f} \vec{F}dt = \Delta\vec{p}$$

So what do you think an impulse is?

*Impulse of the force **F** acting on a particle over the time interval $\Delta t = t_f - t_i$ is equal to the change of the momentum of the particle caused by that force. Impulse is the degree of which an external force changes momentum.*

The above statement is called the impulse-momentum theorem and is equivalent to Newton's second law.

What are the dimension and unit of Impulse? What is the direction of an impulse vector?

Defining a time-averaged force

$$\overline{\vec{F}} \equiv \frac{1}{\Delta t}\int_{t_i}^{t_f} \vec{F}dt$$

Impulse can be rewritten

$$\vec{I} \equiv \overline{\vec{F}}\Delta t$$

If force is constant

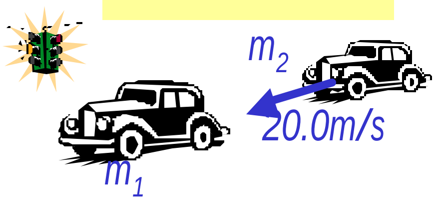
$$\vec{I} \equiv \vec{F}\Delta t$$

It is generally approximated that the impulse force exerted acts on a short time but much greater than any other forces present.

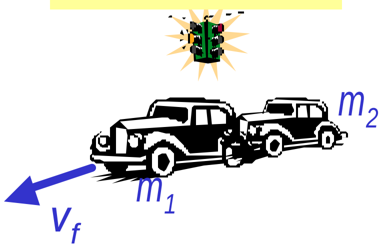
Example 9.5

A car of mass 1800kg stopped at a traffic light is rear-ended by a 900kg car, and the two become entangled. If the lighter car was moving at 20.0m/s before the collision what is the velocity of the entangled cars after the collision?

Before collision



After collision



The momenta before and after the collision are

$$\vec{p}_i = m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = 0 + m_2\vec{v}_{2i}$$

$$\vec{p}_f = m_1\vec{v}_{1f} + m_2\vec{v}_{2f} = (m_1+m_2)\vec{v}_f$$

Since momentum of the system must be conserved

$$\vec{p}_i = \vec{p}_f \Rightarrow (m_1+m_2)\vec{v}_f = m_2\vec{v}_{2i}$$

$$\vec{v}_f = \frac{m_2\vec{v}_{2i}}{(m_1+m_2)} = \frac{900\times 20.0\vec{i}}{900+1800} = 6.67\vec{i}\ m/s$$

What can we learn from these equations on the direction and magnitude of the velocity before and after the collision?

The cars are moving in the same direction as the lighter car's original direction to conserve momentum.

The magnitude is inversely proportional to its own mass.

Elastic and Perfectly Inelastic Collisions

In perfectly Inelastic collisions, the objects stick together after the collision, moving together. Momentum is conserved in this collision, so the final velocity of the stuck system is

$$m_1\vec{v}_{1i}+m_2\vec{v}_{2i} = (m_1+m_2)\vec{v}_f$$

$$\vec{v}_f = \frac{m_1\vec{v}_{1i}+m_2\vec{v}_{2i}}{(m_1+m_2)}$$

How about elastic collisions?

In elastic collisions, both the momentum and the kinetic energy are conserved. Therefore, the final speeds in an elastic collision can be obtained in terms of initial speeds as

$$m_1\vec{v}_{1i}+m_2\vec{v}_{2i} = m_1\vec{v}_{1f}+m_2\vec{v}_{2f}$$

$$\frac{1}{2}m_1v_{1i}^2+\frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2+\frac{1}{2}m_2v_{2f}^2$$

$$m_1\left(v_{1i}^2-v_{1f}^2\right) = m_2\left(v_{2i}^2-v_{2f}^2\right)$$

$$m_1\left(v_{1i}-v_{1f}\right)\left(v_{1i}+v_{1f}\right) = m_2\left(v_{2i}-v_{2f}\right)\left(v_{2i}+v_{2f}\right)$$

From momentum conservation above

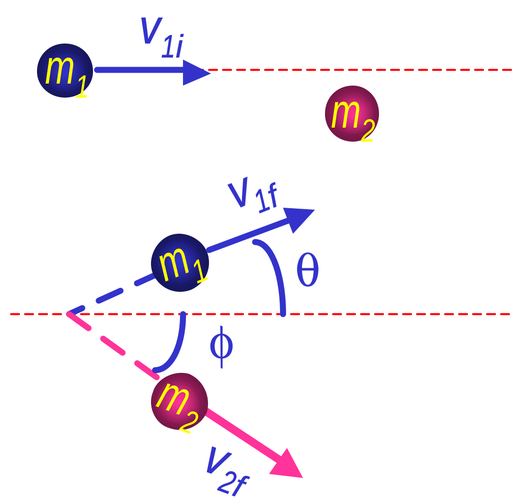
$$m_1\left(v_{1i}-v_{1f}\right) = m_2\left(v_{2i}-v_{2f}\right)$$

$$v_{1f} = \left(\frac{m_1-m_2}{m_1+m_2}\right)v_{1i}+\left(\frac{2m_2}{m_1+m_2}\right)v_{2i}$$

$$v_{2f} = \left(\frac{2m_1}{m_1+m_2}\right)v_{1i}+\left(\frac{m_1-m_2}{m_1+m_2}\right)v_{2i}$$

Example 9.9

Proton #1 with a speed 3.50×10^5 m/s collides elastically with proton #2 initially at rest. After the collision, proton #1 moves at an angle of 37^o to the horizontal axis and proton #2 deflects at an angle ϕ to the same axis. Find the final speeds of the two protons and the scattering angle of proton #2, ϕ.



Since both the particles are protons $m_1=m_2=m_p$.

Using momentum conservation, one obtains

x-comp. $m_p v_{1i} = m_p v_{1f} \cos q + m_p v_{2f} \cos f$

y-comp. $m_p v_{1f} \sin q - m_p v_{2f} \sin f = 0$

Canceling m_p and put in all known quantities, one obtains

$$\boxed{v_{1f} \cos 37^\circ + v_{2f} \cos f = 3.50 \times 10^5 \quad (1)}$$

$$\boxed{v_{1f} \sin 37^\circ = v_{2f} \sin f \quad (2)}$$

From kinetic energy conservation:

$$\boxed{\left(3.50 \times 10^5\right)^2 = v_{1f}^2 + v_{2f}^2 \quad (3)}$$

Solving Eqs. 1-3 equations, one gets

$$\boxed{v_{1f} = 2.80 \times 10^5 \ m/s}$$

$$\boxed{v_{2f} = 2.11 \times 10^5 \ m/s}$$

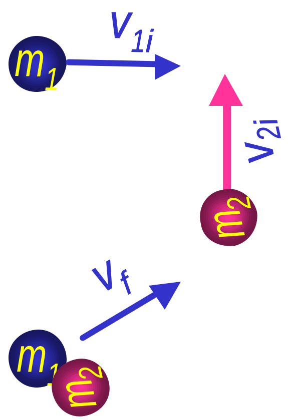
$$\boxed{f = 53.0^\circ}$$

Do this at home☺

Ex09-09.ip

Example 9.9

A 1500kg car traveling east with a speed of 25.0 m/s collides at an interaction with a 2500kg van traveling north at a speed of 20.0 m/s. After the collision the two cars stuck to each other, and the wreckage is moving together. Determine the velocity of the wreckage after the collision, assuming the vehicles underwent a perfectly inelastic collision.



The initial momentum of the two car system before the collision is

$$\vec{p}_i = m_1 v_{1i}\vec{i} + m_2 v_{2i}\vec{j} = 1500\times 25.0\vec{i} + 2500\times 20.0\vec{j}$$

$$= 3.75\times 10^4\vec{i} + 5.0\times 10^4\vec{j}$$

The final momentum of the two car system after the perfectly inelastic collision is

$$\vec{p}_f = (m_1+m_2)\left(v_{fx}\vec{i} + v_{fy}\vec{j}\right) = 4.0\times 10^3 v_{fx}\vec{i} + 4.0\times 10^3 v_{fy}\vec{j}$$

Using momentum conservation

$$\boxed{\vec{p}_f = \vec{p}_i}$$

X-comp. $\boxed{p_{fx} = p_{ix}}$ $\boxed{(m_1+m_2)v_{fx} = m_1 v_{1x} + 0}$ $\boxed{v_{fx} = \frac{(m_1 v_{1x}+0)}{m_1+m_2} = \frac{3.75\times 10^4}{1500+2500} = 9.38 m/s}$

Y-comp. $\boxed{p_{fy} = p_{iy}}$ $\boxed{(m_1+m_2)v_{fy} = 0 + m_2 v_{2y}}$ $\boxed{v_{fy} = \frac{(0+m_2 v_{2y})}{m_1+m_2} = \frac{5.0\times 10^4}{1500+2500} = 12.5 m/s}$

$$\vec{v}_f = v_{fx}\vec{i} + v_{fy}\vec{j} = \left(9.38\vec{i} + 12.5\vec{j}\right) m/s$$

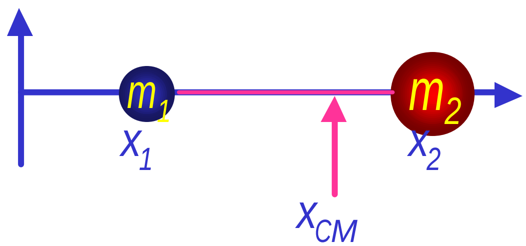
Center of Mass

We've been solving physical problems treating objects as sizeless points with masses, but in realistic situation objects have shapes with masses distributed throughout the body.

Center of mass of a system is the average position of the system's mass and represents the motion of the system as if all the mass is on the point.

What does above statement tell you concerning forces being exerted on the system?

The total external force exerted on the system of total mass M causes the center of mass to move at an acceleration given by $\vec{a} = \sum \vec{F} / M$ *as if all the mass of the system is concentrated on the center of mass.*



Consider a massless rod with two balls attached at either end.

The position of the center of mass of this system is the mass averaged position of the system

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

CM is closer to the heavier object

Center of Mass of a Rigid Object

The formula for CM can be expanded to Rigid Object or a system of many particles

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

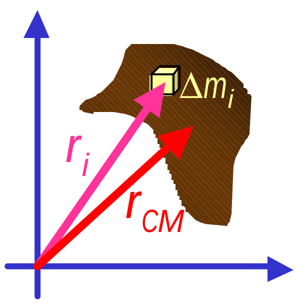
$$y_{CM} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

$$z_{CM} = \frac{\sum_i m_i z_i}{\sum_i m_i}$$

The position vector of the center of mass of a many particle system is

$$\vec{r}_{CM} = x_{CM}\vec{i} + y_{CM}\vec{j} + z_{CM}\vec{k} = \frac{\sum_i m_i x_i \vec{i} + \sum_i m_i y_i \vec{j} + \sum_i m_i z_i \vec{k}}{\sum_i m_i}$$

$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{M}$$



A rigid body – an object with shape and size with mass spread throughout the body, ordinary objects – can be considered as a group of particles with mass m_i densely spread throughout the given shape of the object

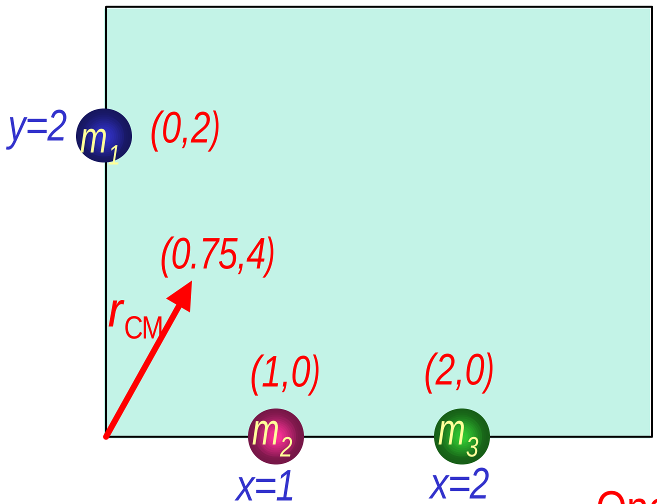
$$x_{CM} \approx \frac{\sum_i \Delta m_i x_i}{M}$$

$$x_{CM} = \lim_{\Delta m_i \to 0} \frac{\sum_i \Delta m_i x_i}{M} = \frac{1}{M}\int x\,dm$$

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\,dm$$

Example 9.12

A system consists of three particles as shown in the figure. Find the position of the center of mass of this system.



Using the formula for CM for each position vector component

$$x_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i} \qquad y_{CM} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

One obtains

$$\vec{r}_{CM} = x_{CM}\vec{i} + y_{CM}\vec{j} = \frac{(m_2 + 2m_3)\vec{i} + 2m_1\vec{j}}{m_1 + m_2 + m_3}$$

$$x_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{m_2 + 2m_3}{m_1 + m_2 + m_3}$$

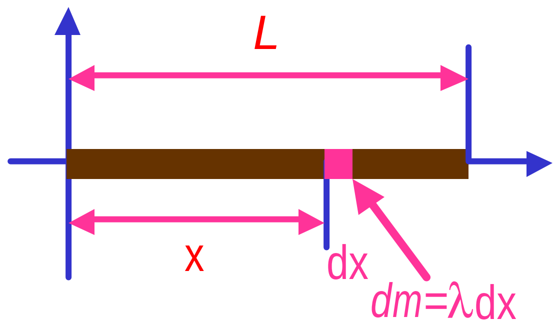
$$y_{CM} = \frac{\sum_i m_i y_i}{\sum_i m_i} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{2m_1}{m_1 + m_2 + m_3}$$

If $m_1 = 2kg; m_2 = m_3 = 1kg$

$$\vec{r}_{CM} = \frac{3\vec{i} + 4\vec{j}}{4} = 0.75\vec{i} + \vec{j}$$

Example 9.13

Show that the center of mass of a rod of mass *M* and length *L* lies in midway between its ends, assuming the rod has a uniform mass per unit length.



The formula for CM of a continuous object is

$$x_{CM} = \frac{1}{M}\int_{x=0}^{x=L} x\,dm$$

Since the density of the rod (λ) is constant; $l = M/L$

The mass of a small segment $dm = l\,dx$

Therefore $$x_{CM} = \frac{1}{M}\int_{x=0}^{x=L} l x dx = \frac{1}{M}\left[\frac{1}{2} l x^2\right]_{x=0}^{x=L} = \frac{1}{M}\left(\frac{1}{2} l L^2\right) = \frac{1}{M}\left(\frac{1}{2} ML\right) = \frac{L}{2}$$

Find the CM when the density of the rod non-uniform but varies linearly as a function of x, λ=α x

$$M = \int_{x=0}^{x=L} l\,dx = \int_{x=0}^{x=L} a x dx$$

$$= \left[\frac{1}{2} a x^2\right]_{x=0}^{x=L} = \frac{1}{2} a L^2$$

$$x_{CM} = \frac{1}{M}\int_{x=0}^{x=L} l\,x dx = \frac{1}{M}\int_{x=0}^{x=L} a x^2 dx = \frac{1}{M}\left[\frac{1}{3} a x^3\right]_{x=0}^{x=L}$$

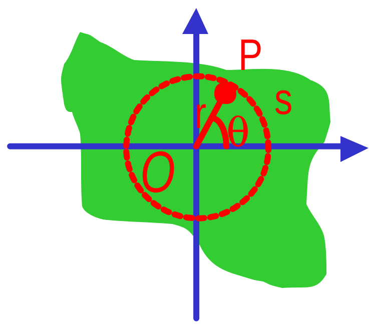
$$x_{CM} = \frac{1}{M}\left(\frac{1}{3} a L^3\right) = \frac{1}{M}\left(\frac{2}{3} ML\right) = \frac{2L}{3}$$

Fundamentals on Rotation

Linear motions can be described as the motion of the center of mass with all the mass of the object concentrated on it.

Is this still true for rotational motions?

No, because different parts of the object have different linear velocities and accelerations.



Consider a motion of a rigid body – an object that does not change its shape – rotating about the axis protruding out of the slide.

The arc length, or sergita, is $s = r\theta$

Therefore the angle, θ, is $\theta = \frac{s}{r}$*. And the unit of the angle is in radian.*

One radian is the angle swept by an arc length equal to the radius of the arc.

Since the circumference of a circle is 2πr, $360^\circ = 2\pi r / r = 2\pi$

The relationship between radian and degrees is $1\,\mathrm{rad} = 360^\circ / 2\pi = 180^\circ / \pi$