PHYS 1443 – Section 003
Lecture #15

Wednesday, Nov. 6, 2002
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1. Rolling Motion of a Rigid Body
2. Total Kinetic Energy of a Rolling Rigid Body
3. Kinetic Energy of a Rolling Sphere
4. Torque and Vector Product
5. Properties of Vector Product
6. Angular Momentum

Today's homework is homework #15 due 12:00pm, Wednesday, Nov. 13!!

Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Similar Quantity	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = \int r^2 dm$
Length of motion	Displacement r	Angle q (Radian)
Speed	$v = \frac{dr}{dt}$	$w = \frac{dq}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$a = \frac{dw}{dt}$
Force	Force $F = ma$	Torque $t = Ia$
Work	Work $W = \int_{x_i}^{x_f} F dx$	Work $W = \int_{q_i}^{q_f} t dq$
Power	$P = \vec{F} \cdot \vec{v}$	$P = tw$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{w}$
Kinetic Energy	Kinetic $K = \frac{1}{2} mv^2$	Rotational $K_R = \frac{1}{2} Iw^2$

Rolling Motion of a Rigid Body

What is a rolling motion?

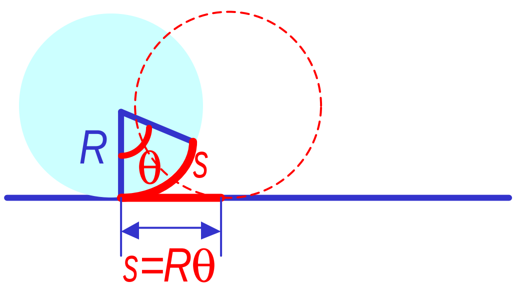
A more generalized case of a motion where the rotational axis moves together with the object

A rotational motion about the moving axis

To simplify the discussion, let's make a few assumptions

1. Limit our discussion on very symmetric objects, such as cylinders, spheres, etc
2. The object rolls on a flat surface

Let's consider a cylinder rolling without slipping on a flat surface



Under what condition does this "Pure Rolling" happen?

The total linear distance the CM of the cylinder moved is

$$s = R\theta$$

Thus the linear speed of the CM is

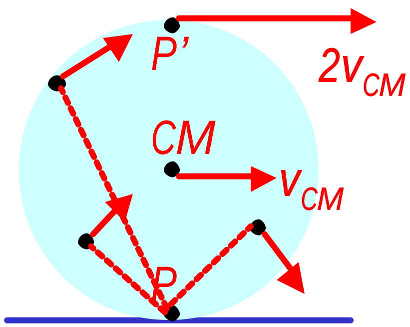
$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

Condition for "Pure Rolling"

More Rolling Motion of a Rigid Body

The magnitude of the linear acceleration of the CM is

$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{dw}{dt} = Ra$$



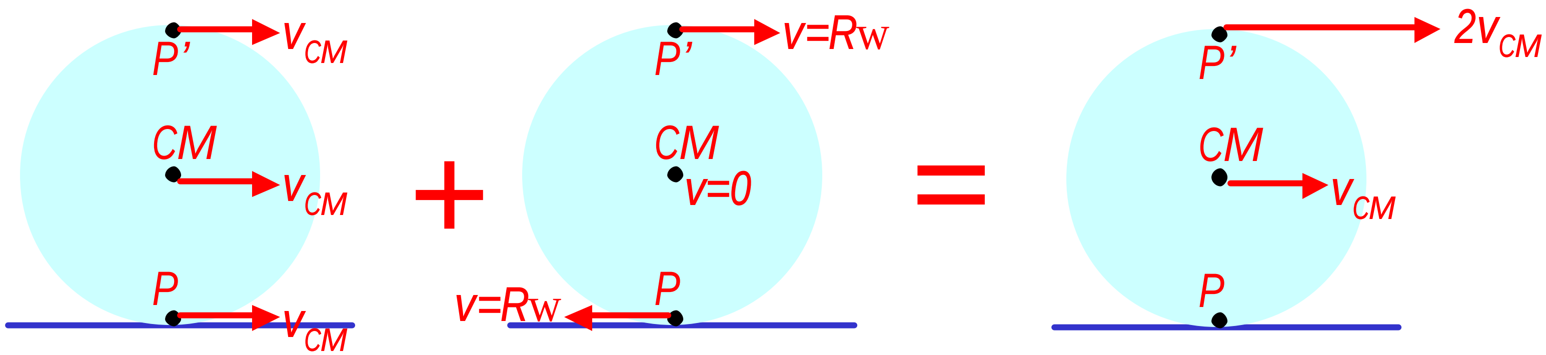
As we learned in the rotational motion, all points in a rigid body moves at the same angular speed but at a different linear speed.

At any given time the point that comes to P has 0 linear speed while the point at P' has twice the speed of CM

Why??

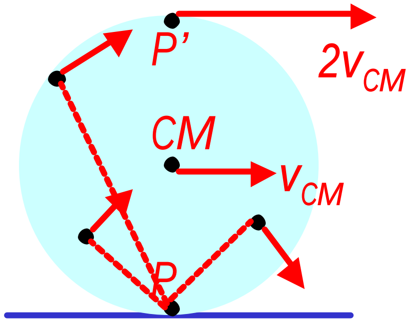
CM is moving at the same speed at all times.

A rolling motion can be interpreted as the sum of Translation and Rotation



Total Kinetic Energy of a Rolling Body

What do you think the total kinetic energy of the rolling cylinder is?



Since it is a rotational motion about the point P, we can writ the total kinetic energy

$$K = \frac{1}{2} I_P \omega^2$$

Where, I_P, is the moment of inertia about the point P.

Using the parallel axis theorem, we can rewrite

$$K = \frac{1}{2} I_P \omega^2 = \frac{1}{2}\left(I_{CM} + MR^2\right)\omega^2 = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} MR^2\omega^2$$

Since $v_{CM}=R\omega$, the above relationship can be rewritten as

$$K = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}^2$$

Rotational kinetic energy about the CM

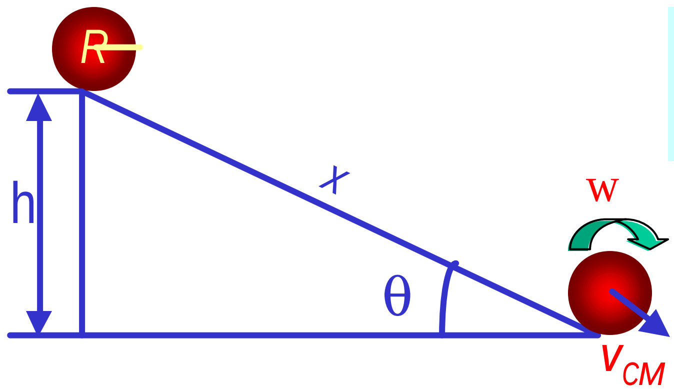
Translational Kinetic energy of the CM

What does this equation mean?

Total kinetic energy of a rolling motion is the sum of the rotational kinetic energy about the CM

And the translational kinetic of the CM

Kinetic Energy of a Rolling Sphere



Let's consider a sphere with radius R rolling down a hill without slipping.

$$K = \frac{1}{2} I_{CM} w^2 + \frac{1}{2} M R^2 w^2$$

$$= \frac{1}{2} I_{CM} \left(\frac{v_{CM}}{R} \right)^2 + \frac{1}{2} M v_{CM}^2$$

$$= \frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}^2$$

Since $v_{CM} = R\omega$

What is the speed of the CM in terms of known quantities and how do you find this out?

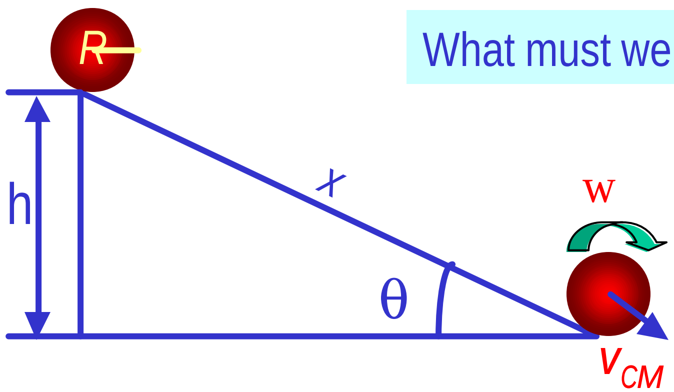
Since the kinetic energy at the bottom of the hill must be equal to the potential energy at the top of the hill

$$K = \frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}^2 = Mgh$$

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM} / MR^2}}$$

Example 11.1

For solid sphere as shown in the figure, calculate the linear speed of the CM at the bottom of the hill and the magnitude of linear acceleration of the CM.



What must we know first?

The moment of inertia the sphere with respect to the CM!!

$$I_{CM} = \int r^2 dm = \frac{2}{5} MR^2$$

Thus using the formula in the previous slide

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM}/MR^2}} = \sqrt{\frac{2gh}{1 + 2/5}} = \sqrt{\frac{10}{7} gh}$$

Since h=xsinθ, one obtains

$$v_{CM}^2 = \frac{10}{7} gx\sin\theta$$

Using kinematic relationship

$$v_{CM}^2 = 2a_{CM}x$$

The linear acceleration of the CM is

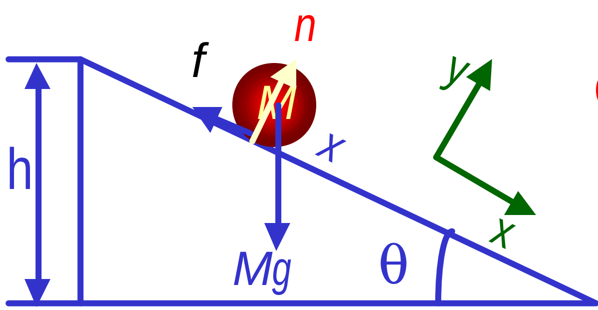
$$a_{CM} = \frac{v_{CM}^2}{2x} = \frac{5}{7} g\sin\theta$$

What do you see?

Linear acceleration of a sphere does not depend on anything but g and θ.

Example 11.2

For solid sphere as shown in the figure, calculate the linear speed of the CM at the bottom of the hill and the magnitude of linear acceleration of the CM. Solve this problem using Newton's second law, the dynamic method.



What are the forces involved in this motion?

Gravitational Force, Frictional Force, Normal Force

Newton's second law applied to the CM gives

$$\sum F_x = Mg\sin\theta - f = Ma_{CM}$$

$$\sum F_y = n - Mg\cos\theta = 0$$

Since the forces *Mg* and **n** go through the CM, their moment arm is 0 and do not contribute to torque, while the static friction ***f*** causes torque

$$\tau_{CM} = fR = I_{CM}\alpha$$

We know that

$$I_{CM} = \frac{2}{5}MR^2$$

$$a_{CM} = R\alpha$$

We obtain

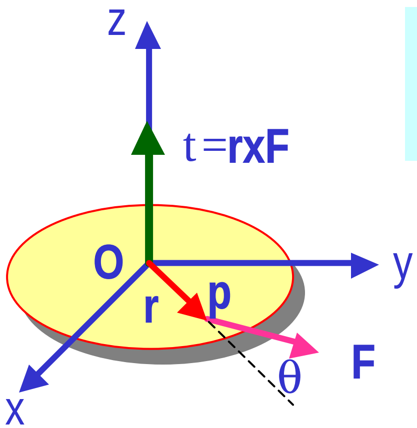
$$f = \frac{I_{CM}\alpha}{R} = \frac{\frac{2}{5}MR^2}{R}\left(\frac{a_{CM}}{R}\right) = \frac{2}{5}Ma_{CM}$$

Substituting ***f*** in dynamic equations

$$Mg\sin\theta = \frac{7}{5}Ma_{CM}$$

$$a_{CM} = \frac{5}{7}g\sin\theta$$

Torque and Vector Product



Let's consider a disk fixed onto the origin O and the force *F* exerts on the point p. What happens?

The disk will start rotating counter clockwise about the Z axis

The magnitude of torque given to the disk by the force *F* is

$$t = Fr\sin f$$

But torque is a vector quantity, what is the direction?
How is torque expressed mathematically?

$$\boxed{\vec{t} \equiv \vec{r} \times \vec{F}}$$

What is the direction? The direction of the torque follows the right-hand rule!!

The above quantity is called
Vector product or Cross product

$$\vec{C} \equiv \vec{A} \times \vec{B}$$

$$\boxed{\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right| \sin q}$$

What is the result of a vector product?

Another vector

What is another vector operation we've learned?

Scalar product

$$C \equiv \vec{A} \cdot \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \cos q$$

Result? A scalar

Properties of Vector Product

Vector Product is Non-commutative

What does this mean?

If the order of operation changes the result changes $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$

Following the right-hand rule, the direction changes $\boxed{\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}}$

Vector Product of two parallel vectors is 0.

$$\left|\vec{C}\right| = \left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin = \left|\vec{A}\right|\left|\vec{B}\right|\sin 0 = 0$$

Thus, $\boxed{\vec{A}\times\vec{A} = 0}$

If two vectors are perpendicular to each other

$$\left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta = \left|\vec{A}\right|\left|\vec{B}\right|\sin 90^\circ = \left|\vec{A}\right|\left|\vec{B}\right| = AB$$

Vector product follows distribution law

$$\vec{A}\times\left(\vec{B}+\vec{C}\right) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

The derivative of a Vector product with respect to a scalar variable is

$$\frac{d\left(\vec{A}\times\vec{B}\right)}{dt} = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$$

More Properties of Vector Product

The relationship between unit vectors, $\vec{i}, \vec{j}$ and $\vec{k}$

$$\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0$$

$$\vec{i}\times\vec{j}=-\vec{j}\times\vec{i}=\vec{k}$$

$$\vec{j}\times\vec{k}=-\vec{k}\times\vec{j}=\vec{i}$$

$$\vec{k}\times\vec{i}=-\vec{i}\times\vec{k}=\vec{j}$$

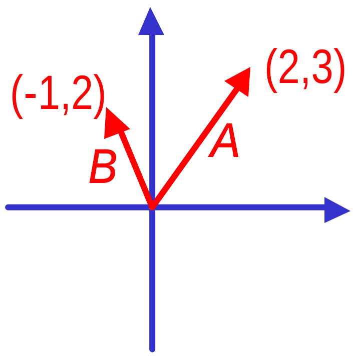
Vector product of two vectors can be expressed in the following determinant form

$$\vec{A}\times\vec{B}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\vec{i}\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}-\vec{j}\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}+\vec{k}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}$$

$$=\left(A_yB_z-A_zB_y\right)\vec{i}-\left(A_xB_z-A_zB_x\right)\vec{j}+\left(A_xB_y-A_yB_x\right)\vec{k}$$

Example 11.3

Two vectors lying in the xy plane are given by the equations *A=2i+3j* and *B=-i+2j*, verify that *AxB= -BxA*



$$\vec{A}\times\vec{B} = \left(2\vec{i}+3\vec{j}\right)\times\left(-\vec{i}+2\vec{j}\right) = 4\vec{i}\times\vec{j} - 3\vec{j}\times\vec{i}$$

Since $\vec{i}\times\vec{j}=\vec{k}$

$$\vec{A}\times\vec{B} = 4\vec{k} + 3\vec{j}\times\left(-\vec{i}\right) = 4\vec{k} + 3\vec{k} = 7\vec{k}$$

$$\vec{B}\times\vec{A} = \left(-\vec{i}+2\vec{j}\right)\times\left(2\vec{i}+3\vec{j}\right) = -3\vec{i}\times\vec{j} + 4\vec{j}\times\vec{i}$$

Using the same unit vector relationship as above

$$\vec{B}\times\vec{A} = -3\vec{k} - 4\vec{k} = -7\vec{k}$$

Therefore, *AxB= -BxA*

Now prove this using determinant method