PHYS 1443 – Section 003
Lecture #18

Monday, Nov. 18, 2002
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1. Elastic Properties of Solids
2. Simple Harmonic Motion
3. Equation of Simple Harmonic Motion
4. Oscillatory Motion of a Block Spring System

Today's homework is homework #18 due 12:00pm, Monday, Nov. 25!!

How did we solve equilibrium problems?

1. Identify all the forces and their directions and locations
2. Draw a free-body diagram with forces indicated on it
3. Write down vector force equation for each x and y component with proper signs
4. Select a rotational axis for torque calculations → Selecting the axis such that the torque of one of the unknown forces become 0.
5. Write down torque equation with proper signs
6. Solve the equations for unknown quantities

Elastic Properties of Solids

We have been assuming that the objects do not change their shapes when external forces are exerting on it. It this realistic?

No. In reality, the objects get deformed as external forces act on it, though the internal forces resist the deformation as it takes place.

Deformation of solids can be understood in terms of Stress and Strain

Stress: A quantity proportional to the force causing deformation.
Strain: Measure of degree of deformation

It is empirically known that for small stresses, strain is proportional to stress

The constants of proportionality are called Elastic Modulus

$$\text{Elastic Modulus} \equiv \frac{\text{stress}}{\text{strain}}$$

Three types of Elastic Modulus

1. **Young's modulus**: Measure of the elasticity in length
2. **Shear modulus**: Measure of the elasticity in plane
3. **Bulk modulus**: Measure of the elasticity in volume

Young's Modulus

Let's consider a long bar with cross sectional area A and initial length L_i.

Tensile stress $\text{Tensile Stress} \equiv \frac{F_{ex}}{A}$ Tensile strain $\text{Tensile Strain} \equiv \frac{\Delta L}{L_i}$

Young's Modulus is defined as $$Y \equiv \frac{\text{Tensile Stress}}{\text{Tensile Strain}} = \frac{F_{ex}/A}{\Delta L/L_i}$$

Used to characterize a rod or wire stressed under tension or compression

What is the unit of Young's Modulus?

Force per unit area

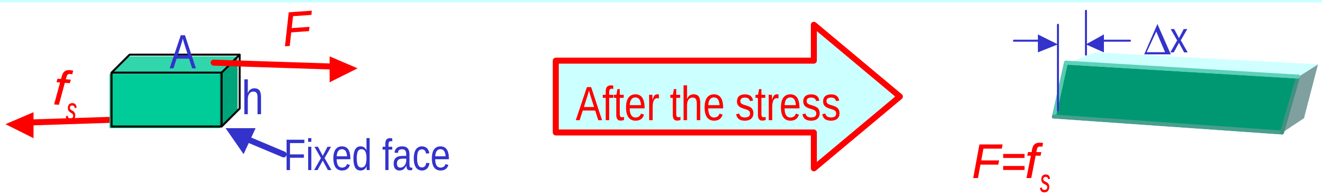
Experimental Observations

1. For fixed external force, the change in length is proportional to the original length
2. The necessary force to produce a given strain is proportional to the cross sectional area

Elastic limit: Maximum stress that can be applied to the substance before it becomes permanently deformed

Shear Modulus

Another type of deformation occurs when an object is under a force tangential to one of its surfaces while the opposite face is held fixed by another force.



Shear stress

$$\text{Shear Stress} \equiv \frac{\text{Tangential Force}}{\text{Surface Area the force applies}} = \frac{F}{A}$$

Shear strain

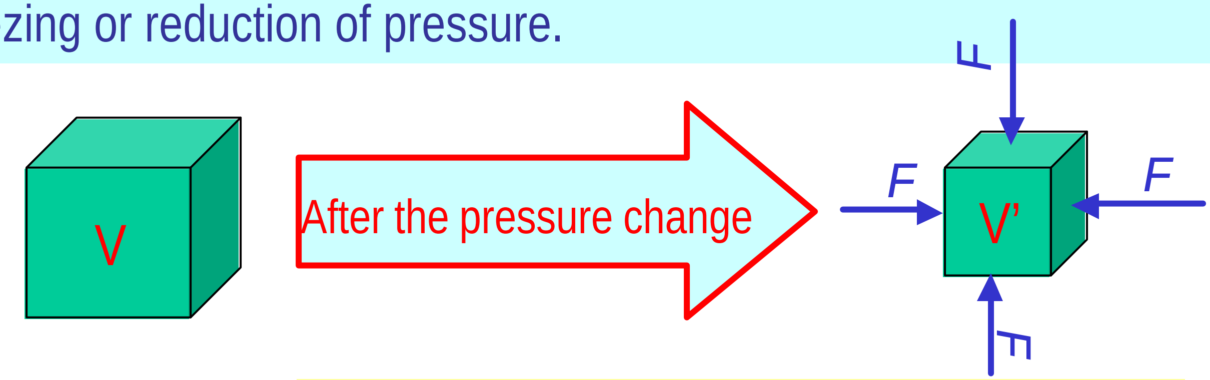
$$\text{Shear Strain} \equiv \frac{\Delta x}{h}$$

Shear Modulus is defined as

$$S \equiv \frac{\text{Shear Stress}}{\text{Shear Strain}} = \frac{F/A}{\Delta x/h}$$

Bulk Modulus

Bulk Modulus characterizes the response of a substance to uniform squeezing or reduction of pressure.



Volume stress =pressure

$$\text{Pressure} \equiv \frac{\text{Normal Force}}{\text{Surface Area the force applies}} = \frac{F}{A}$$

If the pressure on an object changes by $\Delta P = \Delta F/A$, the object will undergo a volume change ΔV.

Bulk Modulus is defined as

$$B \equiv \frac{\text{Volume Stress}}{\text{Volume Strain}} = -\frac{\Delta F / A}{\Delta V / V_i} = -\frac{\Delta P}{\Delta V / V_i}$$

Because the change of volume is reverse to change of pressure.

Compressibility is the reciprocal of Bulk Modulus

Example 12.7

A solid brass sphere is initially under normal atmospheric pressure of $1.0 \times 10^5 N/m^2$. The sphere is lowered into the ocean to a depth at which the pressures is $2.0 \times 10^7 N/m^2$. The volume of the sphere in air is $0.5 m^3$. By how much its volume change once the sphere is submerged?

Since bulk modulus is

$$B = -\frac{\Delta P}{\Delta V / V_i}$$

The amount of volume change is

$$\Delta V = -\frac{\Delta P V_i}{B}$$

From table 12.1, bulk modulus of brass is 6.1×10^{10} N/m^2

The pressure change ΔP is

$$\Delta P = P_f - P_i = 2.0\times10^7 - 1.0\times10^5 \approx 2.0\times10^7$$

Therefore the resulting volume change ΔV is

$$\Delta V = V_f - V_i = -\frac{2.0\times10^7 \times 0.5}{6.1\times10^{10}} = -1.6\times10^{-4} m^3$$

The volume has decreased.

Simple Harmonic Motion

What do you think a harmonic motion is?

Motion that occurs by the force that depends on displacement, and the force is always directed toward the system's equilibrium position.

What is a system that has this kind of character? A system consists of a mass and a spring

When a spring is stretched from its equilibrium position by a length x, the force acting on the mass is

$$F = -kx$$

It's negative, because the force resists against the change of length, directed toward the equilibrium position.

From Newton's second law $F = ma = -kx$ we obtain $a = -\frac{k}{m}x$

This is a second order differential equation that can be solved but it is beyond the scope of this class.

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Condition for simple harmonic motion

What do you observe from this equation?

Acceleration is proportional to displacement from the equilibrium

Acceleration is opposite direction to displacement

Equation of Simple Harmonic Motion

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

The solution for the 2nd order differential equation

$$x = A\cos(\omega t + \phi)$$

Amplitude

Phase

Angular Frequency

Phase constant

Generalized expression of a simple harmonic motion

Let's think about the meaning of this equation of motion

What happens when t=0 and ϕ=0?

$$x = A\cos(0+0) = A$$

What is ϕ if x is not A at t=0?

$$x = A\cos(\phi) = x'$$

$$\phi = \cos^{-1}(x')$$

What are the maximum/minimum possible values of x?

A/-A

An oscillation is fully characterized by its:

- Amplitude
- Period or frequency
- Phase constant

More on Equation of Simple Harmonic Motion

What is the time for full cycle of oscillation?

Since after a full cycle the position must be the same

$$x = A\cos\left(w\left(t+T\right)+f\right) = A\cos\left(wt+2p+f\right)$$

The period $T = \dfrac{2p}{w}$

One of the properties of an oscillatory motion

How many full cycles of oscillation does this undergo per unit time?

$$f = \frac{1}{T} = \frac{w}{2p}$$

Frequency

What is the unit?

1/s=Hz

Let's now think about the object's speed and acceleration.

$$x = A\cos\left(wt+f\right)$$

Speed at any given time $v = \dfrac{dx}{dt} = -wA\sin\left(wt+f\right)$

Max speed $v_{\max} = wA$

Acceleration at any given time $a = \dfrac{dv}{dt} = -w^2A\cos\left(wt+f\right) = -w^2x$

Max acceleration $a_{\max} = w^2A$

What do we learn about acceleration?

Acceleration is reverse direction to displacement

Acceleration and speed are $\pi/2$ off phase:

When *v* is maximum, *a* is at its minimum

Simple Harmonic Motion continued

Phase constant determines the starting position of a simple harmonic motion.

$$x = A\cos(wt + f)$$ At t=0 $$x\big|_{t=0} = A\cos f$$

This constant is important when there are more than one harmonic oscillation involved in the motion and to determine the overall effect of the composite motion

Let's determine phase constant and amplitude

At t=0 $$x_i = A\cos f \qquad v_i = -wA\sin f$$

By taking the ratio, one can obtain the phase constant

$$\boxed{f = \tan^{-1}\left(-\frac{v_i}{wx_i}\right)}$$

By squaring the two equation and adding them together, one can obtain the amplitude

$$x_i^2 = A^2\cos^2 f$$

$$v_i^2 = w^2A^2\sin^2 f$$

$$A^2\left(\cos^2 f + \sin^2 f\right) = A^2 = x_i^2 + \left(\frac{v_i}{w}\right)^2$$

$$\boxed{A = \sqrt{x_i^2 + \left(\frac{v_i}{w}\right)^2}}$$

Example 13.1

An object oscillates with simple harmonic motion along the x-axis. Its displacement from the origin varies with time according to the equation; $x=(4.00m)\cos\left(pt+\frac{p}{4}\right)$ where t is in seconds and the angles is in the parentheses are in radians. a) Determine the amplitude, frequency, and period of the motion.

From the equation of motion: $X=A\cos(wt+f)=(4.00m)\cos\left(pt+\frac{p}{4}\right)$

The amplitude, A, is $A=4.00m$ The angular frequency, ω, is $w=p$

Therefore, frequency and period are $T=\frac{2p}{w}=\frac{2p}{p}=2s$ $f=\frac{1}{T}=\frac{w}{2p}=\frac{p}{2p}=\frac{1}{2}s^{-1}$

b)Calculate the velocity and acceleration of the object at any time t.

Taking the first derivative on the equation of motion, the velocity is

$$v=\frac{dx}{dt}=-(4.00\times p)\sin\left(pt+\frac{p}{4}\right)m/s$$

By the same token, taking the second derivative of equation of motion, the acceleration, a, is

$$a=\frac{d^2x}{dt^2}=-(4.00\times p^2)\cos\left(pt+\frac{p}{4}\right)m/s^2$$

Fig13-10.ip

Simple Block-Spring System

A block attached at the end of a spring on a frictionless surface experiences acceleration when the spring is displaced from an equilibrium position.

$$a = -\frac{k}{m}x$$

This becomes a second order differential equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

If we denote

$$w^2 = \frac{k}{m}$$

The resulting differential equation becomes

$$\frac{d^2x}{dt^2} = -w^2 x$$

Since this satisfies condition for simple harmonic motion, we can take the solution

$$\boxed{x = A\cos\left(wt + f\right)}$$

Does this solution satisfy the differential equation?

Let's take derivatives with respect to time

$$\frac{dx}{dt} = A\frac{d}{dt}\left(\cos\left(wt + f\right)\right) = -w\mathrm{A}\sin\left(wt + f\right)$$

Now the second order derivative becomes

$$\frac{d^2x}{dt^2} = -w\mathrm{A}\frac{d}{dt}\left(\sin\left(wt + f\right)\right) = -w^2\mathrm{A}\cos\left(wt + f\right) = -w^2 x$$

Whenever the force acting on a particle is linearly proportionalto the displacement from some equilibrium position and is in the opposite direction, the particle moves in simple harmonic motion.

More Simple Block-Spring System

How do the period and frequency of this harmonic motion look?

Since the angular frequency ω is $w = \sqrt{\frac{k}{m}}$

The period, T, becomes $T = \frac{2p}{w} = 2p\sqrt{\frac{m}{k}}$

So the frequency is $f = \frac{1}{T} = \frac{w}{2p} = \frac{1}{2p}\sqrt{\frac{k}{m}}$

What can we learn from these?

- Frequency and period do not depend on amplitude
- Period is inversely proportional to spring constant and proportional to mass

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Special case #1

Let's consider that the spring is stretched to distance A and the block is let go from rest, giving 0 initial speed; x_i=A, v_i=0,

$$x = \mathrm{A}\cos wt \quad v = \frac{dx}{dt} = -w\mathrm{A}\sin wt \quad a = \frac{d^2x}{dt^2} = -w^2\mathrm{A}\cos wt \quad a_i = -w^2\mathrm{A} = -kA/m$$

This equation of motion satisfies all the conditions. So it isthe solution for this motion.

Special case #2

Suppose block is given non-zero initial velocity v_i to positive x at the instant it is at the equilibrium, x_i=0

$$f = \tan^{-1}\left(-\frac{v_i}{wx_i}\right) = \tan^{-1}(-\infty) = -\frac{p}{2} \quad x = \mathrm{A}\cos\left(wt - \frac{p}{2}\right) = A\sin(wt)$$

Is this a good solution?

Example 13.2

A car with a mass of 1300kg is constructed so that its frame is supported by four springs. Each spring has a force constant of 20,000N/m. If two peoploe riding in the car have a combined mass of 160kg, find the frequency of vibration of the car after it is driven over a pothole in the road.

Let's assume that mass is evenly distributed to all four springs.

The total mass of the system is 1460kg.

Therefore each spring supports 365kg each.

From the frequency relationship based on Hook's law

$$f = \frac{1}{T} = \frac{w}{2p} = \frac{1}{2p}\sqrt{\frac{k}{m}}$$

Thus the frequency for vibration of each spring is

$$f = \frac{1}{2p}\sqrt{\frac{k}{m}} = \frac{1}{2p}\sqrt{\frac{20000}{365}} = 1.18\ s^{-1} = 1.18\ Hz$$

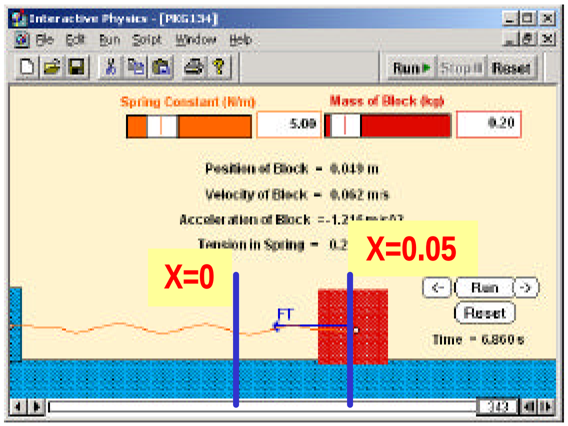
How long does it take for the car to complete two full vibrations?

The period is $T = \frac{1}{f} = 2p\sqrt{\frac{m}{k}} = 0.849\ s$ For two cycles $2T = 1.70\ s$

Ex13-03.ip

Example 13.3

A block with a mass of 200g is connected to a light spring for which the force constant is 5.00 N/m and is free to oscillate on a horizontal, frictionless surface. The block is displaced 5.00 cm from equilibrium and released from reset. Find the period of its motion.



From the Hook's law, we obtain

$$W = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.00}{0.20}} = 5.00\ s^{-1}$$

As we know, period does not depend on the amplitude or phase constant of the oscillation, therefore the period, T, is simply

$$T = \frac{2p}{w} = \frac{2p}{5.00} = 1.26\ s$$

Determine the maximum speed of the block.

From the general expression of the simple harmonic motion, the speed is

$$v_{\max} = \frac{dx}{dt} = -wA\sin\left(wt+f\right)$$

$$= wA = 5.00 \times 0.05 = 0.25\, m/s$$