

PHYS 1443 – Section 003

Lecture #20

Monday, Nov. 25, 2002

Dr. Jaehoon Yu

1. Simple Harmonic and Uniform Circular Motions
2. Damped Oscillation
3. Newton's Law of Universal Gravitation
4. Free Fall Acceleration and Gravitational Force
5. Kepler's Laws
6. Gravitation Field and Potential Energy

Today's homework is homework #20 due 12:00pm, Monday, Dec. 2!!



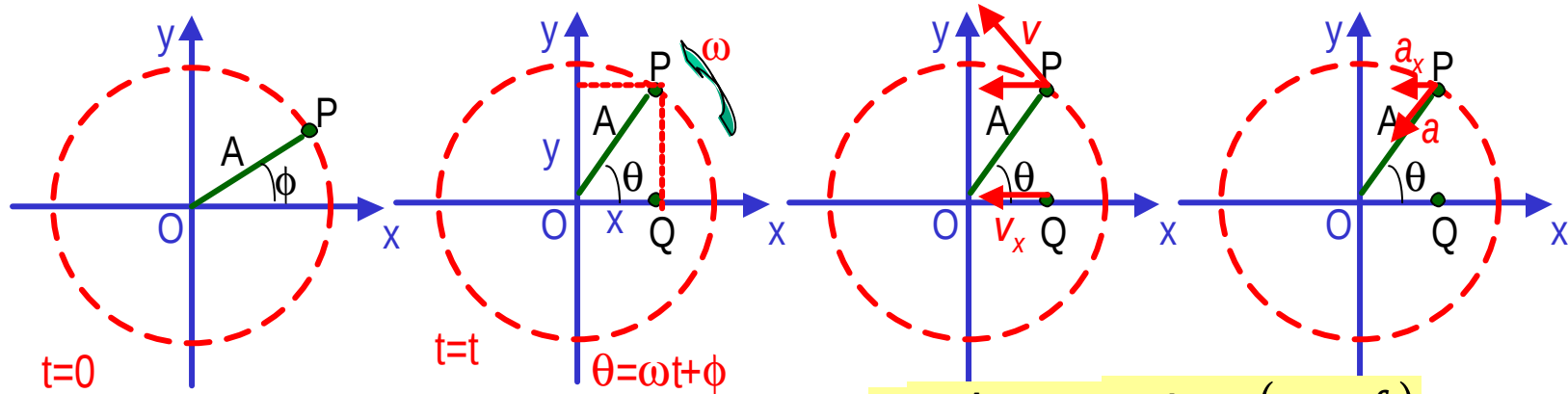
Announcements

- Class on Wednesday
- Remember the Term Exam on Monday, Dec. 9 in the class
 - Covers chapters 11 – 15
 - Review on Wednesday, Dec. 4



Simple Harmonic and Uniform Circular Motions

Uniform circular motion can be understood as a superposition of two simple harmonic motions in x and y axis.



When the particle rotates at a uniform angular speed ω , x and y coordinate position become

Since the linear velocity in a uniform circular motion is $A\omega$, the velocity components are

Since the radial acceleration in a uniform circular motion is $v^2/A = \omega^2 A$, the components are

$$x = A \cos q = A \cos(\omega t + f)$$

$$y = A \sin q = A \sin(\omega t + f)$$

$$v_x = -v \sin q = -A\omega \sin(\omega t + f)$$

$$v_y = +v \cos q = A\omega \cos(\omega t + f)$$

$$a_x = -a \cos q = -A\omega^2 \cos(\omega t + f)$$

$$a_y = -a \sin q = -A\omega^2 \sin(\omega t + f)$$



Example 13.7

A particle rotates counterclockwise in a circle of radius 3.00m with a constant angular speed of 8.00 rad/s. At $t=0$, the particle has an x coordinate of 2.00m and is moving to the right. A) Determine the x coordinate as a function of time.

Since the radius is 3.00m, the amplitude of oscillation in x direction is 3.00m. And the angular frequency is 8.00rad/s. Therefore the equation of motion in x direction is

$$X = A \cos q = (3.00m) \cos(8.00t + f)$$

Since $x=2.00$, when $t=0$

$$2.00 = (3.00m) \cos f \quad f = \cos^{-1}\left(\frac{2.00}{3.00}\right) = 48.2^\circ$$

However, since the particle was moving to the right $\phi = -48.2^\circ$,

$$x = (3.00m) \cos(8.00t - 48.2^\circ)$$

Find the x components of the particle's velocity and acceleration at any time t.

Using the displacement

$$v_x = \frac{dx}{dt} = -(3.00 \cdot 8.00) \sin(8.00t - 48.2) = (-24.0m/s) \sin(8.00t - 48.2^\circ)$$

Likewise, from velocity

$$a_x = \frac{dv}{dt} = (-24.0 \cdot 8.00) \cos(8.00t - 48.2) = (-192m/s^2) \cos(8.00t - 48.2^\circ)$$



Damped Oscillation

More realistic oscillation where an oscillating object loses its mechanical energy in time by a retarding force such as friction or air resistance.

Let's consider a system whose retarding force is air resistance $R = -bv$ (b is called damping coefficient) and restoration force is $-kx$

$$\sum F_x = -kx - bv = ma_x$$

$$-kx - b \frac{dx}{dt} = m \frac{d^2 x}{dt^2}$$

The solution for the above 2nd order differential equation is

$$x = e^{-\frac{b}{2m}t} A \cos(\omega t + f)$$

The angular frequency ω for this motion is

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

Damping Term

This equation of motion tells us that when the retarding force is much smaller than restoration force, the system oscillates but the amplitude decreases, and ultimately, the oscillation stops.

We express the angular frequency as

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

Where the natural frequency ω_0

$$\omega_0 = \sqrt{\frac{k}{m}}$$



More on Damped Oscillation

The motion is called **Underdamped** when the magnitude of the maximum retarding force $R_{\max} = bv_{\max} < kA$

How do you think the damping motion would change as retarding force changes?

$$-bv_{\max} \rightarrow -kA$$

As the retarding force becomes larger, the amplitude reduces more rapidly, eventually stopping at its equilibrium position

Under what condition this system does not oscillate?

$$w = 0 \quad w_0 = \frac{b}{2m}$$

The system is **Critically damped**

$$b = 2mw_0 = 2\sqrt{mk}$$

What do you think happen?

Once released from non-equilibrium position, the object would return to its equilibrium position and stops.

If the retarding force is larger than restoration force

$$R_{\max} = bv_{\max} > kA \quad \text{The system is } \mathbf{Overdamped}$$

Once released from non-equilibrium position, the object would return to its equilibrium position and stops, but a lot slower than before

Newton's Law of Universal Gravitation

People have been very curious about the stars in the sky, making observations for a long time. But the data people collected have not been explained until Newton has discovered the law of gravitation.

Every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

How would you write this principle mathematically?

$$F_g \propto \frac{m_1 m_2}{r_{12}^2}$$

With G

$$F_g = G \frac{m_1 m_2}{r_{12}^2}$$

G is the universal gravitational constant, and its value is

$$G = 6.673 \times 10^{-11}$$

Unit?

$$N \cdot m^2 / kg^2$$

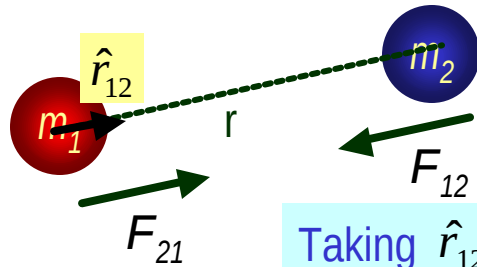
This constant is not given by the theory but must be measured by experiment.

This form of forces is known as an inverse-square law, because the magnitude of the force is inversely proportional to the square of the distances between the objects.



More on Law of Universal Gravitation

Consider two particles exerting gravitational forces to each other.



Two objects exert gravitational force on each other following Newton's 3rd law.

Taking $\hat{r}_{12}$ as the unit vector, we can write the force m_2 experiences as

$$\vec{F}_{12} = -G \frac{m_1 m_2}{r^2} \hat{r}_{12}$$

What do you think the negative sign mean?

It means that the force exerted on the particle 2 by particle 1 is attractive force, pulling #2 toward #1.

Gravitational force is a field force: Forces act on object without physical contact between the objects at all times, independent of medium between them.

The gravitational force exerted by a finite size, spherically symmetric mass distribution on a particle outside the distribution is the same as if the entire mass of the distributions was concentrated at the center.

How do you think the gravitational force on the surface of the earth look?

$$F_g = G \frac{M_E m}{R_E^2}$$



Free Fall Acceleration & Gravitational Force

Weight of an object with mass m is mg . Using the force exerting on a particle of mass m on the surface of the Earth, one can get

$$mg = G \frac{M_E m}{R_E^2}$$

$$g = G \frac{M_E}{R_E^2}$$

What would the gravitational acceleration be if the object is at an altitude h above the surface of the Earth?

$$F_g = mg' = G \frac{M_E m}{r^2} = G \frac{M_E m}{(R_E + h)^2}$$

$$g' = G \frac{M_E}{(R_E + h)^2}$$

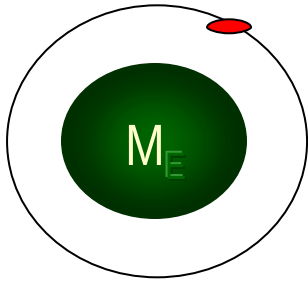
What do these tell us about the gravitational acceleration?

- The gravitational acceleration is independent of the mass of the object
- The gravitational acceleration decreases as the altitude increases
- If the distance from the surface of the Earth gets infinitely large, the weight of the object approaches 0.



Example 14.2

The international space station is designed to operate at an altitude of 350km. When completed, it will have a weight (measured on the surface of the Earth) of $4.22 \times 10^6 \text{ N}$. What is its weight when in its orbit?



The total weight of the station on the surface of the Earth is

$$F_{GE} = mg = G \frac{M_E m}{R_E^2} = 4.22 \times 10^6 \text{ N}$$

Since the orbit is at 350km above the surface of the Earth, the gravitational force at that height is

$$F_O = mg = G \frac{M_E m}{(R_E + h)^2} = \frac{R_E^2}{(R_E + h)^2} F_{GE}$$

Therefore the weight in the orbit is

$$F_O = \frac{R_E^2}{(R_E + h)^2} F_{GE} = \frac{(6.37 \times 10^6)^2}{(6.37 \times 10^6 + 3.50 \times 10^5)^2} \times 4.22 \times 10^6 = 3.80 \times 10^6 \text{ N}$$



Example 14.3

Using the fact that $g=9.80\text{m/s}^2$ at the Earth's surface, find the average density of the Earth.

Since the gravitational acceleration is

$$g = G \frac{M_E}{R_E^2} = 6.67 \times 10^{-11} \frac{M_E}{R_E^2}$$

So the mass of the Earth is

$$M_E = \frac{R_E^2 g}{G}$$

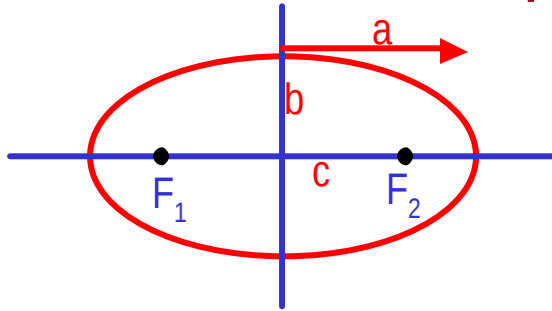
Therefore the density of the Earth is

$$\rho = \frac{M_E}{V_E} = \frac{\frac{R_E^2 g}{G}}{\frac{4\pi}{3} R_E^3} = \frac{3g}{4\pi G R_E}$$

$$= \frac{3 \times 9.80}{4\pi \times 6.67 \times 10^{-11} \times 6.37 \times 10^6} = 5.50 \times 10^3 \text{ kg/m}^3$$



Kepler's Laws & Ellipse



Ellipses have two different axis, major (long) and minor (short) axis, and two focal points, F_1 & F_2

a is the length of a semi-major axis

b is the length of a semi-minor axis

Kepler lived in Germany and discovered the law's governing planets' movement some 70 years before Newton, by analyzing data.

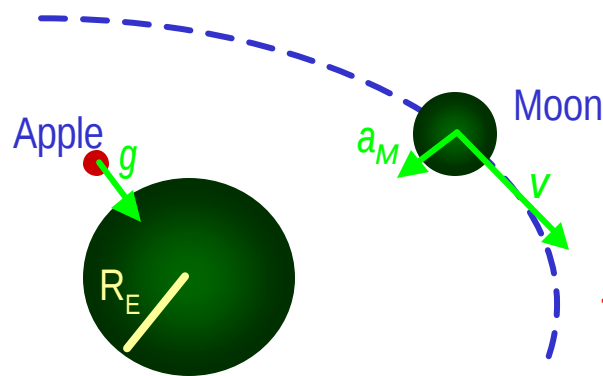
- All planets move in elliptical orbits with the Sun at one focal point.
- The radius vector drawn from the Sun to a planet sweeps out equal area in equal time intervals. (*Angular momentum conservation*)
- The square of the orbital period of any planet is proportional to the cube of the semi-major axis of the elliptical orbit.

Newton's laws explain the cause of the above laws. Kepler's third law is the direct consequence of law of gravitation being inverse square law.



The Law of Gravity and the Motion of Planets

- Newton assumed that the law of gravitation applies the same whether it is on the Moon or the apple on the surface of the Earth.
- The interacting bodies are assumed to be point like particles.



Newton predicted that the ratio of the Moon's acceleration a_M to the apple's acceleration g would be

$$\frac{a_M}{g} = \frac{(1/r_M)^2}{(1/R_E)^2} = \left(\frac{R_E}{r_M}\right)^2 = \left(\frac{6.37 \times 10^6}{3.84 \times 10^8}\right)^2 = 2.75 \times 10^{-4}$$

Therefore the centripetal acceleration of the Moon, a_M , is

$$a_M = 2.75 \times 10^{-4} \times 9.80 = 2.70 \times 10^{-3} \text{ m/s}^2$$

Newton also calculated the Moon's orbital acceleration a_M from the knowledge of its distance from the Earth and its orbital period, $T=27.32 \text{ days}=2.36 \times 10^6 \text{ s}$

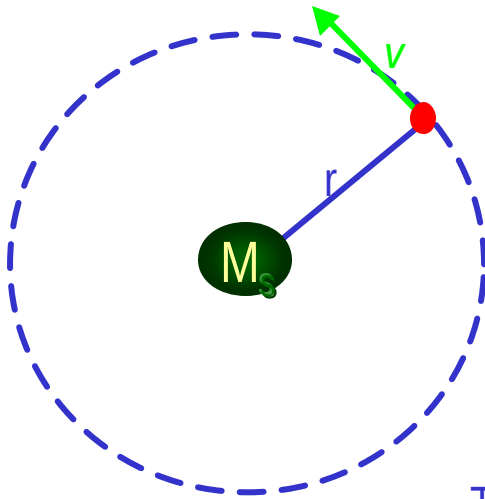
$$a_M = \frac{v^2}{r_M} = \frac{(2\pi r_M / T)^2}{r_M} = \frac{4\pi^2 r_M}{T^2} = \frac{4\pi^2 \times 3.84 \times 10^8}{(2.36 \times 10^6)^2} = 2.72 \times 10^{-3} \text{ m/s}^2 \approx \frac{9.80}{(60)^2}$$

This means that the Moon's distance is about 60 times that of the Earth's radius, its acceleration is reduced by the square of the ratio. This proves that the inverse square law is valid.



Kepler's Third Law

It is crucial to show that Kepler's third law can be predicted from the inverse square law for circular orbits.



Since the gravitational force exerted by the Sun is radially directed toward the Sun to keep the planet circle, we can apply Newton's second law

$$\frac{GM_s M_p}{r^2} = \frac{M_p v^2}{r}$$

Since the orbital speed, v , of the planet with period T is $v = \frac{2\pi r}{T}$

The above can be written $\frac{GM_s M_p}{r^2} = \frac{M_p (2\pi r / T)^2}{r}$

Solving for T
one can obtain

$$T^2 = \left(\frac{4\pi^2}{GM_s} \right) r^3 = K_s r^3 \quad \text{and} \quad K_s = \left(\frac{4\pi^2}{GM_s} \right) = 2.97 \times 10^{-19} \text{ s}^2 / \text{m}^3$$

This is Kepler's third law. It's also valid for ellipse for r being the length of the semi-major axis. The constant K_s is independent of mass of the planet.



Example 14.4

Calculate the mass of the Sun using the fact that the period of the Earth's orbit around the Sun is 3.16×10^7 s, and its distance from the Sun is 1.496×10^{11} m.

Using Kepler's third law.

$$T^2 = \left(\frac{4\pi^2}{GM_s} \right) r^3 = K_s r^3$$

The mass of the Sun, M_s , is

$$M_s = \left(\frac{4\pi^2}{GT} \right) r^3$$

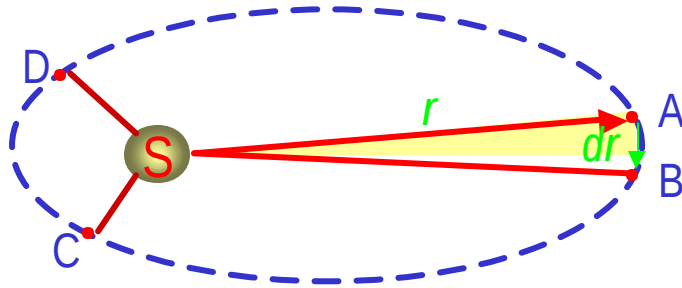
$$= \left(\frac{4\pi^2}{6.67 \times 10^{-11} \times 3.16 \times 10^7} \right) \times (1.496 \times 10^{11})^3$$

$$= 1.99 \times 10^{30} \text{ kg}$$



Kepler's Second Law and Angular Momentum Conservation

Consider a planet of mass M_p moving around the Sun in an elliptical orbit.



Since the gravitational force acting on the planet is always toward radial direction, it is a *central force*

Therefore the torque acting on the planet by this force is always 0.

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times F \hat{r} = 0$$

Since torque is the time rate change of angular momentum L , the angular momentum is constant.

$$\frac{d\vec{L}}{dt} = 0 \quad \vec{L} = \text{const}$$

Because the gravitational force exerted on a planet by the Sun results in no torque, the angular momentum L of the planet is constant.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times M_p \vec{v} = M_p \vec{r} \times \vec{v} = \text{const}$$

Since the area swept by the motion of the planet is

$$dA = \frac{1}{2} |\vec{r} \times d\vec{r}| = \frac{1}{2} |\vec{r} \times \vec{v} dt| = \frac{L}{2M_p} dt \quad \Rightarrow \quad \frac{dA}{dt} = \frac{L}{2M_p} = \text{const}$$

This is Kepler's second law which states that the radius vector from the Sun to a planet sweeps out equal areas in equal time intervals.