PHYS 1443 – Section 003
Lecture #23

Wednesday, Dec. 4, 2002
Dr. Jaehoon Yu

Review of Chapters 11 - 15

Announcements

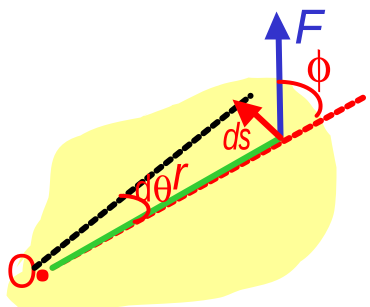
- Final Term Exam
  - Monday, Dec. 9, between 12:00pm – 1:30pm for 1.5 hours in the class room
  - Covers chapters 11 – 15

Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Similar Quantity	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = \int r^2 dm$
Length of motion	Distance L	Angle q (Radian)
Speed	$v = \frac{dr}{dt}$	$w = \frac{dq}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$a = \frac{dw}{dt}$
Force	Force $F = ma$	Torque $t = Ia$
Work	Work $W = \int_{x_i}^{x_f} F dx$	Work $W = \int_{q_i}^{q_f} t dq$
Power	$P = \vec{F} \cdot \vec{v}$	$P = tw$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{w}$
Kinetic Energy	Kinetic $K = \frac{1}{2}mv^2$	Rotational $K_R = \frac{1}{2}Iw^2$

Work, Power, and Energy in Rotation



Let's consider a motion of a rigid body with a single external force *F* exerting on the point P, moving the object by *ds*.

The work done by the force *F* as the object rotates through the infinitesimal distance ds=rdθ is

$$dW = \vec{F}\cdot d\vec{s} = (F \sin f)\, r\, dq$$

What is *F*sinφ? — The tangential component of force *F*.

What is the work done by radial component *F*cosφ? — Zero, because it is perpendicular to the displacement.

Since the magnitude of torque is r*F*sinφ,

$$dW = t\, dq$$

The rate of work, or power becomes

$$P = \frac{dW}{dt} = \frac{t\, dq}{dt} = t w$$

How was the power defined in linear motion?

$$\sum t = Ia = I\left(\frac{dw}{dt}\right) = I\left(\frac{dw}{dq}\right)\left(\frac{dq}{dt}\right)$$

The rotational work done by an external force equals the change in rotational energy.

$$dW = \sum t\, dq = Iw\,dw$$

The work put in by the external force then

$$W = \int_{q_i}^{q_f} \sum t\, dq = \int_{w_i}^{w_f} Iw\,dw = \frac{1}{2} I w_f^2 - \frac{1}{2} I w_i^2$$

Rolling Motion of a Rigid Body

What is a rolling motion?

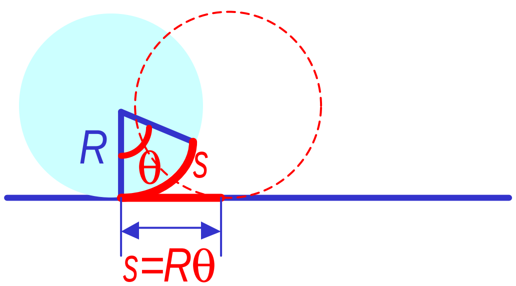
A more generalized case of a motion where the rotational axis moves together with the object

A rotational motion about the moving axis

To simplify the discussion, let's make a few assumptions

1. Limit our discussion on very symmetric objects, such as cylinders, spheres, etc
2. The object rolls on a flat surface

Let's consider a cylinder rolling without slipping on a flat surface



Under what condition does this "Pure Rolling" happen?

The total linear distance the CM of the cylinder moved is

$$s = R\theta$$

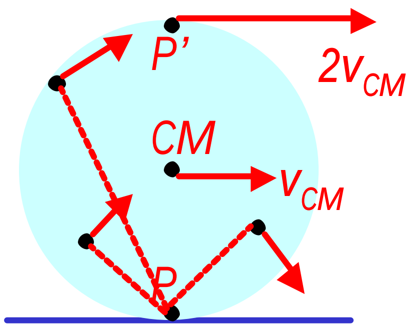
Thus the linear speed of the CM is

$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

Condition for "Pure Rolling"

Total Kinetic Energy of a Rolling Body

What do you think the total kinetic energy of the rolling cylinder is?



Since it is a rotational motion about the point P, we can writ the total kinetic energy

$$K = \frac{1}{2} I_P w^2$$

Where, I_P, is the moment of inertia about the point P.

Using the parallel axis theorem, we can rewrite

$$K = \frac{1}{2} I_P w^2 = \frac{1}{2}\left(I_{CM} + MR^2\right) w^2 = \frac{1}{2} I_{CM} w^2 + \frac{1}{2} MR^2 w^2$$

Since $v_{CM}=R\omega$, the above relationship can be rewritten as

$$K = \frac{1}{2} I_{CM} w^2 + \frac{1}{2} M v_{CM}^2$$

Rotational kinetic energy about the CM

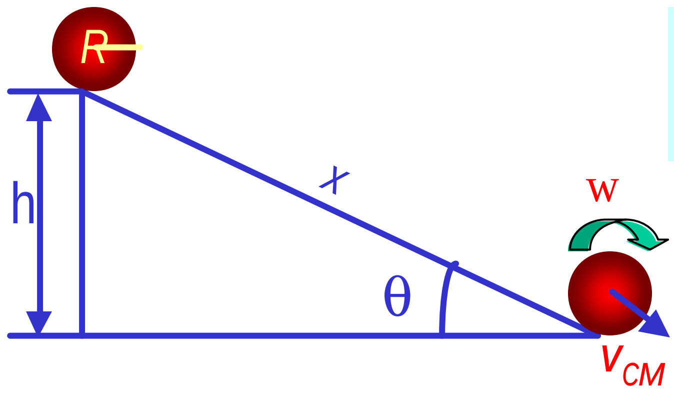
Translational Kinetic energy of the CM

What does this equation mean?

Total kinetic energy of a rolling motion is the sum of the rotational kinetic energy about the CM

And the translational kinetic of the CM

Kinetic Energy of a Rolling Sphere



Let's consider a sphere with radius R rolling down a hill without slipping.

$$K = \frac{1}{2} I_{CM} w^2 + \frac{1}{2} M R^2 w^2$$

$$= \frac{1}{2} I_{CM} \left(\frac{v_{CM}}{R} \right)^2 + \frac{1}{2} M v_{CM}^2$$

$$= \frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}^2$$

Since $v_{CM} = R\omega$

What is the speed of the CM in terms of known quantities and how do you find this out?

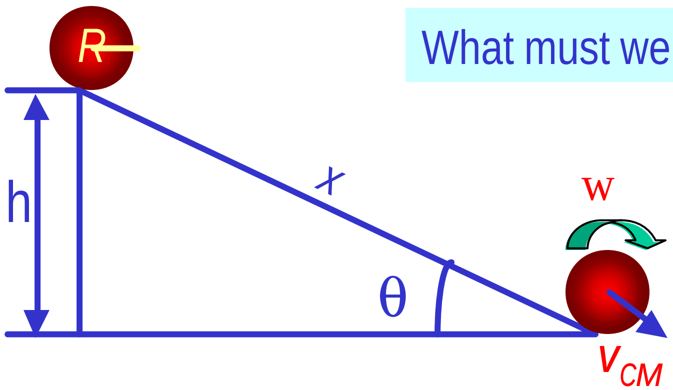
Since the kinetic energy at the bottom of the hill must be equal to the potential energy at the top of the hill

$$K = \frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}^2 = Mgh$$

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM} / MR^2}}$$

Example 11.1

For solid sphere as shown in the figure, calculate the linear speed of the CM at the bottom of the hill and the magnitude of linear acceleration of the CM.



What must we know first?

The moment of inertia the sphere with respect to the CM!!

$$I_{CM} = \int r^2 dm = \frac{2}{5} MR^2$$

Thus using the formula in the previous slide

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM}/MR^2}} = \sqrt{\frac{2gh}{1+2/5}} = \sqrt{\frac{10}{7}gh}$$

Since h=xsinθ, one obtains

$$v_{CM}^2 = \frac{10}{7} gx\sin\theta$$

Using kinematic relationship

$$v_{CM}^2 = 2a_{CM}x$$

The linear acceleration of the CM is

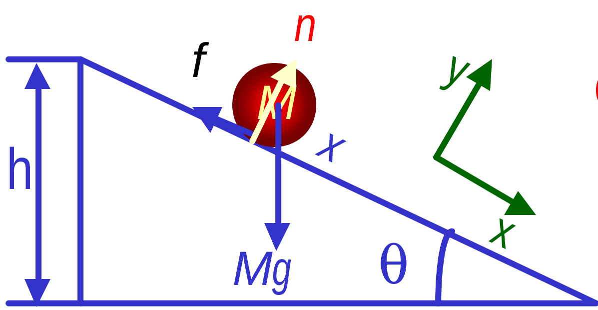
$$a_{CM} = \frac{v_{CM}^2}{2x} = \frac{5}{7} g\sin\theta$$

What do you see?

Linear acceleration of a sphere does not depend on anything but g and θ.

Example 11.2

For solid sphere as shown in the figure, calculate the linear speed of the CM at the bottom of the hill and the magnitude of linear acceleration of the CM. Solve this problem using Newton's second law, the dynamic method.



What are the forces involved in this motion?

Gravitational Force, Frictional Force, Normal Force

Newton's second law applied to the CM gives

$$\sum F_x = Mg\sin\theta - f = Ma_{CM}$$

$$\sum F_y = n - Mg\cos\theta = 0$$

Since the forces *Mg* and **n** go through the CM, their moment arm is 0 and do not contribute to torque, while the static friction *f* causes torque

$$\tau_{CM} = fR = I_{CM}\alpha$$

We know that

$$I_{CM} = \frac{2}{5}MR^2$$

$$a_{CM} = R\alpha$$

We obtain

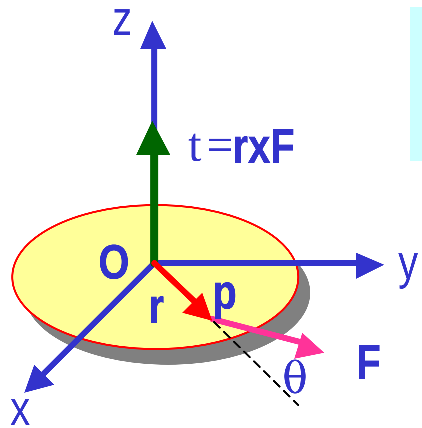
$$f = \frac{I_{CM}\alpha}{R} = \frac{\frac{2}{5}MR^2}{R}\left(\frac{a_{CM}}{R}\right) = \frac{2}{5}Ma_{CM}$$

Substituting *f* in dynamic equations

$$Mg\sin\theta = \frac{7}{5}Ma_{CM}$$

$$a_{CM} = \frac{5}{7}g\sin\theta$$

Torque and Vector Product



Let's consider a disk fixed onto the origin O and the force *F* exerts on the point p. What happens?

The disk will start rotating counter clockwise about the Z axis

The magnitude of torque given to the disk by the force *F* is

$$t = Fr\sin f$$

But torque is a vector quantity, what is the direction?
How is torque expressed mathematically?

$$\boxed{\vec{t} \equiv \vec{r} \times \vec{F}}$$

What is the direction? The direction of the torque follows the right-hand rule!!

The above quantity is called
Vector product or Cross product

$$\vec{C} \equiv \vec{A} \times \vec{B}$$

$$\boxed{\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right| \sin q}$$

What is the result of a vector product?

Another vector

What is another vector operation we've learned?

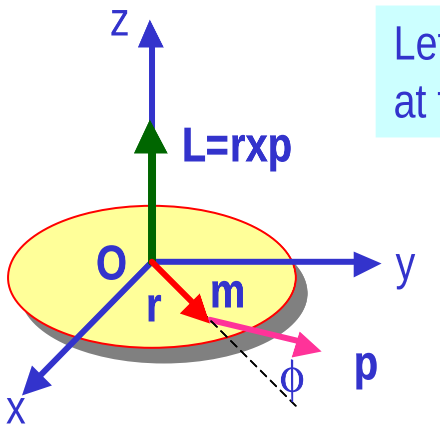
Scalar product

$$C \equiv \vec{A} \cdot \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \cos q$$

Result? A scalar

Angular Momentum of a Particle

If you grab onto a pole while running, your body will rotate about the pole, gaining angular momentum. We've used linear momentum to solve physical problems with linear motions, angular momentum will do the same for rotational motions.



Let's consider a point-like object (particle) with mass *m* located at the vector location ***r*** and moving with linear velocity ***v***

The instantaneous angular momentum ***L*** of this particle relative to origin O is

$$\boxed{\vec{L} \equiv \vec{r} \times \vec{p}}$$

What is the unit and dimension of angular momentum? $kg \cdot m^2/s^2$

Note that ***L*** depends on origin O. Why? Because ***r*** changes

What else do you learn? The direction of ***L*** is +z

Since ***p*** is *m**v***, the magnitude of ***L*** becomes $L = mvr \sin f$

What do you learn from this? If the direction of linear velocity points to the origin of rotation, the particle does not have any angular momentum.

If the linear velocity is perpendicular to position vector, the particle moves exactly the same way as a point on a rim.

The point O has to be inertial.

Angular Momentum and Torque

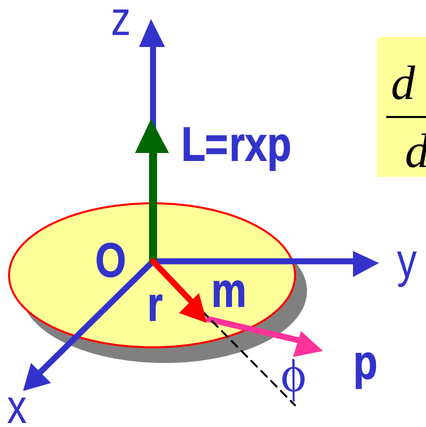
Can you remember how net force exerting on a particle and the change of its linear momentum are related?

$$\boxed{\sum \vec{F} = \frac{d\vec{p}}{dt}}$$

Total external forces exerting on a particle is the same as the change of its linear momentum.

The same analogy works in rotational motion between torque and angular momentum.

Net torque acting on a particle is $\sum \vec{t} = \vec{r} \times \sum \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$



$$\frac{d\vec{L}}{dt} = \frac{d\left(\vec{r}\times\vec{p}\right)}{dt} = \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt} = 0 + \vec{r}\times\frac{d\vec{p}}{dt}$$

Why does this work?

Because *v* is parallel to the linear momentum

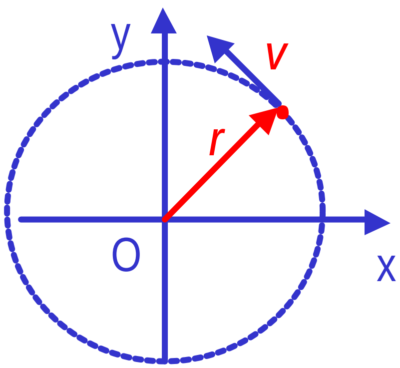
Thus the torque-angular momentum relationship

$$\boxed{\sum \vec{t} = \frac{d\vec{L}}{dt}}$$

The net torque acting on a particle is the same as the time rate change of its angular momentum

Example 11.4

A particle of mass *m* is moving in the xy plane in a circular path of radius r and linear velocity v about the origin O. Find the magnitude and direction of angular momentum with respect to O.



Using the definition of angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m\vec{r} \times \vec{v}$$

Since both the vectors, r and v, are on x-y plane and using right-hand rule, the direction of the angular momentum vector is +z (coming out of the screen)

The magnitude of the angular momentum is $\left\|\vec{L}\right\| = \left|m\vec{r} \times \vec{v}\right| = mrv\sin f = mrv\sin 90^{\circ} = mrv$

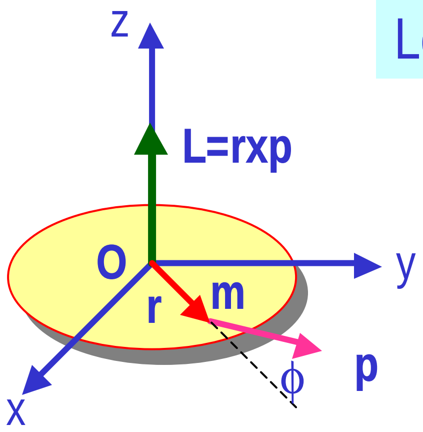
So the angular momentum vector can be expressed as $\vec{L} = mrv\vec{k}$

Find the angular momentum in terms of angular velocity w.

Using the relationship between linear and angular speed

$$\vec{L} = mrv\vec{k} = mr^2 w\vec{k} = mr^2\vec{w} = I\vec{w}$$

Angular Momentum of a Rotating Rigid Body



Let's consider a rigid body rotating about a fixed axis

Each particle of the object rotates in the xy plane about the z-axis at the same angular speed, ω

Magnitude of the angular momentum of a particle of mass m_i about origin O is $m_i v_i r_i$

$$L_i = m_i r_i v_i = m_i r_i^2 w$$

Summing over all particle's angular momentum about z axis

$$L_z = \sum_i L_i = \sum_i \left(m_i r_i^2 w\right)$$

What do you see?

$$L_z = \sum_i \left(m_i r_i^2\right) w = Iw$$

Since I is constant for a rigid body

$$\frac{dL_z}{dt} = I\frac{dw}{dt} = Ia$$

α is angular acceleration

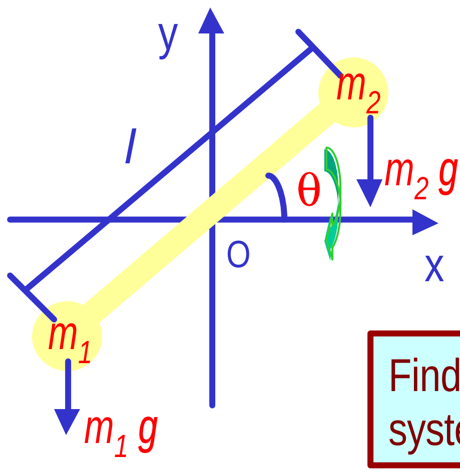
Thus the torque-angular momentum relationship becomes

$$\boxed{\sum t_{ext} = \frac{dL_z}{dt} = Ia}$$

Thus the net external torque acting on a rigid body rotating about a fixed axis is equal to the moment of inertia about that axis multiplied by the object's angular acceleration with respect to that axis.

Example 11.6

A rigid rod of mass *M* and length *l* pivoted without friction at its center. Two particles of mass m_1 and m_2 are connected to its ends. The combination rotates in a vertical plane with an angular speed of ω. Find an expression for the magnitude of the angular momentum.



The moment of inertia of this system is

$$I = I_{rod} + I_{m_1} + I_{m_2} = \frac{1}{12}Ml^2 + \frac{1}{4}m_1 l^2 + \frac{1}{4}m_2 l^2$$

$$= \frac{l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)$$

$$\boxed{L = Iw = \frac{wl^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)}$$

Find an expression for the magnitude of the angular acceleration of the system when the rod makes an angle θ with the horizon.

If $m_1 = m_2$, no angular momentum because net torque is 0.

If θ=+/−π/2, at equilibrium so no angular momentum.

First compute net external torque

$$t_1 = m_1 g \frac{l}{2}\cos q \qquad t_2 = -m_2 g \frac{l}{2}\cos q$$

$$t_{ext} = t_1 + t_2 = \frac{gl\cos q\,(m_1 - m_2)}{2}$$

Thus α becomes

$$a = \frac{\sum t_{ext}}{I} = \frac{\frac{1}{2}(m_1 - m_1)gl\cos q}{\frac{l^2}{4}\left(\frac{1}{3}M + m_1 + m_2\right)} = \frac{2(m_1 - m_1)\cos q}{\left(\frac{1}{3}M + m_1 + m_2\right)} g/l$$

Conservation of Angular Momentum

Remember under what condition the linear momentum is conserved?

Linear momentum is conserved when the net external force is 0.

$$\sum \vec{F} = 0 = \frac{d\vec{p}}{dt}$$
$$\vec{p} = const$$

By the same token, the angular momentum of a system is constant in both magnitude and direction, if the resultant external torque acting on the system is 0.

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt} = 0$$

$$\vec{L} = const$$

What does this mean?

Angular momentum of the system before and after a certain change is the same.

$$\vec{L}_i = \vec{L}_f = \text{constant}$$

Three important conservation laws for isolated system that does not get affected by external forces

$$\begin{cases} K_i + U_i = K_f + U_f & \textbf{Mechanical Energy} \\ \vec{p}_i = \vec{p}_f & \textbf{Linear Momentum} \\ \vec{L}_i = \vec{L}_f & \textbf{Angular Momentum} \end{cases}$$

Example 11.8

A star rotates with a period of 30days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0×10^{4}km, collapses into a neutron start of radius 3.0km. Determine the period of rotation of the neutron star.

What is your guess about the answer?

The period will be significantly shorter, because its radius got smaller.

Let's make some assumptions:

1. There is no torque acting on it
2. The shape remains spherical
3. Its mass remains constant

Using angular momentum conservation

$$L_i = L_f$$

$$I_i\omega_i = I_f\omega_f$$

The angular speed of the star with the period T is

$$\omega = \frac{2\pi}{T}$$

Thus

$$\omega_f = \frac{I_i\omega_i}{I_f} = \frac{mr_i^2}{mr_f^2}\frac{2\pi}{T_i}$$

$$T_f = \frac{2\pi}{\omega_f} = \left(\frac{r_f^2}{r_i^2}\right)T_i = \left(\frac{3.0}{1.0\times10^4}\right)^2\times30\ days = 2.7\times10^{-6}\ days = 0.23\ s$$

Conditions for Equilibrium

What do you think does the term "An object is at its equilibrium" mean?

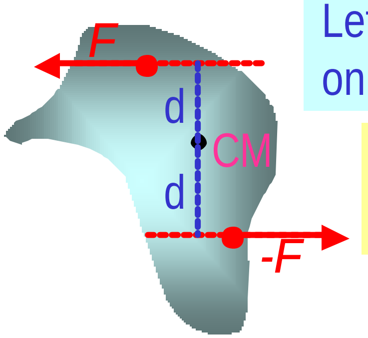
The object is either at rest (<u>Static Equilibrium</u>) or its center of mass is moving with a constant velocity (<u>Dynamic Equilibrium</u>).

When do you think an object is at its equilibrium?

Translational Equilibrium: Equilibrium in linear motion $\sum \vec{F} = 0$

Is this it?

The above condition is sufficient for a point-like particle to be at its static equilibrium. However for object with size this is not sufficient. One more condition is needed. What is it?



Let's consider two forces equal magnitude but opposite direction acting on a rigid object as shown in the figure. What do you think will happen?

The object will rotate about the CM. The net torque acting on the object about any axis must be 0. $\sum \vec{t} = 0$

For an object to be at its *static equilibrium*, the object should not have linear or angular speed. $v_{CM} = 0$ $w = 0$

More on Conditions for Equilibrium

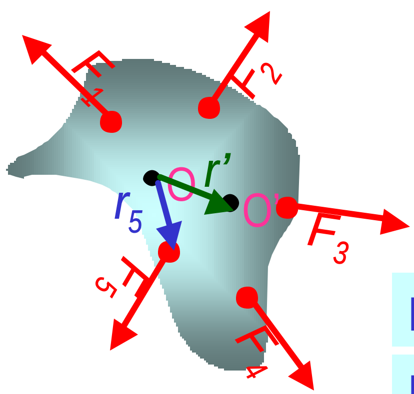
To simplify the problems, we will only deal with forces acting on x-y plane, giving torque only along z-axis. What do you think the conditions for equilibrium be in this case?

The six possible equations from the two vector equations turns to three equations.

$$\sum \vec{F} = 0 \qquad \sum F_x = 0 \qquad \sum F_y = 0$$

$$\sum \vec{\tau} = 0 \qquad \sum \tau_z = 0$$

What happens if there are many forces exerting on the object?



If an object is at its translational static equilibrium, and if the net torque acting on the object is 0 about one axis, the net torque must be 0 about any arbitrary axis.

Net Force exerting on the object $\sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots = 0$

Net torque about O $\sum \vec{\tau}_O = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3 + \cdots = \sum \vec{r}_i \times \vec{F}_i = 0$

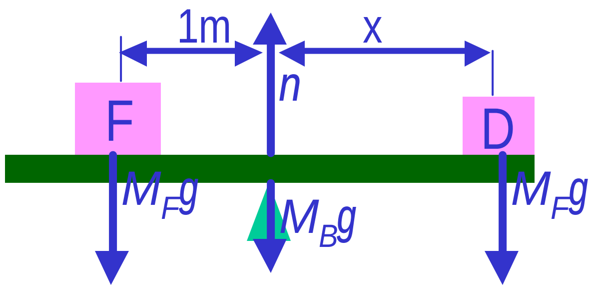
Position of force F_i about O' $\vec{r}_i' = \vec{r}_i - \vec{r}'$

Net torque about O' $\sum \vec{\tau}_{O'} = \vec{r}'_1 \times \vec{F}_1 + \vec{r}'_2 \times \vec{F}_2 + \cdots = (\vec{r}_1 - \vec{r}') \times \vec{F}_1 + (\vec{r}_2 - \vec{r}') \times \vec{F}_2 + \cdots = \sum \vec{r}_i \times \vec{F}_i - \vec{r}' \times \sum \vec{F}_i$

$$\sum \vec{\tau}_{O'} = \sum \vec{r}_i \times \vec{F}_i - \vec{r}' \times 0 = \sum \vec{\tau}_O = 0$$

Example 12.1

A uniform 40.0 N board supports a father and daughter weighing 800 N and 350 N, respectively. If the support (or fulcrum) is under the center of gravity of the board and the father is 1.00 m from CoG, what is the magnitude of normal force *n* exerted on the board by the support?



Since there is no linear motion, this system is in its translational equilibrium

$$\sum F_x = 0$$

$$\sum F_y = M_B g + M_F g + M_D g - n = 0$$

Therefore the magnitude of the normal force

$$n = 40.0 + 800 + 350 = 1190N$$

Determine where the child should sit to balance the system.

The net torque about the fulcrum by the three forces are

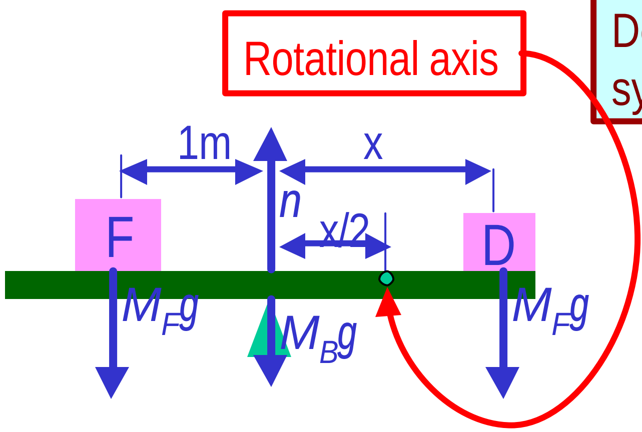
$$t = M_B g \cdot 0 + M_F g \cdot 1.00 - M_D g \cdot x = 0$$

Therefore to balance the system the daughter must sit

$$x = \frac{M_F g}{M_D g} \cdot 1.00m = \frac{800}{350} \cdot 1.00m = 2.29m$$

Example 12.1 Continued

Determine the position of the child to balance the system for different position of axis of rotation.



The net torque about the axis of rotation by all the forces are

$$\tau = M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2) - n \cdot x/2 - M_D g \cdot x/2 = 0$$

Since the normal force is $n = M_B g + M_F g + M_D g$

The net torque can be rewritten

$$\begin{aligned}\tau &= M_B g \cdot x/2 + M_F g \cdot (1.00 + x/2) \\ &\quad - (M_B g + M_F g + M_D g) \cdot x/2 - M_D g \cdot x/2 \\ &= M_F g \cdot 1.00 - M_D g \cdot x = 0\end{aligned}$$

Therefore

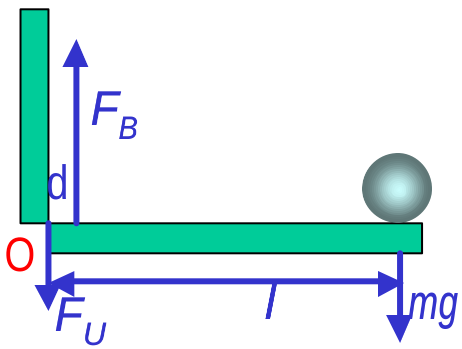
$$x = \frac{M_F g}{M_D g} \cdot 1.00\,m = \frac{800}{350} \cdot 1.00\,m = 2.29\,m$$

What do we learn?

No matter where the rotation axis is, net effect of the torque is identical.

Example 12.2

A person holds a 50.0N sphere in his hand. The forearm is horizontal. The biceps muscle is attached 3.00 cm from the joint, and the sphere is 35.0cm from the joint. Find the upward force exerted by the biceps on the forearm and the downward force exerted by the upper arm on the forearm and acting at the joint. Neglect the weight of forearm.



Since the system is in equilibrium, from the translational equilibrium condition

$$\sum F_x = 0$$

$$\sum F_y = F_B - F_U - mg = 0$$

From the rotational equilibrium condition

$$\sum \tau = F_U \cdot 0 + F_B \cdot d - mg \cdot l = 0$$

Thus, the force exerted by the biceps muscle is

$$F_B \cdot d = mg \cdot l$$

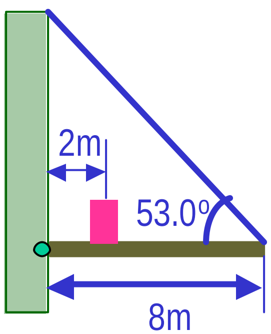
$$F_B = \frac{mg \cdot l}{d} = \frac{50.0 \times 35.0}{3.00} = 583N$$

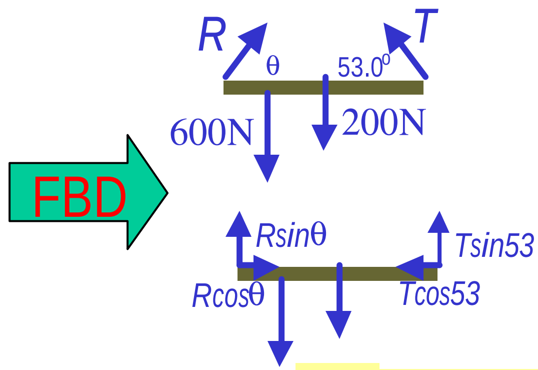
Force exerted by the upper arm is

$$F_U = F_B - mg = 583 - 50.0 = 533N$$

Example 12.3

A uniform horizontal beam with a length of 8.00m and a weight of 200N is attached to a wall by a pin connection. Its far end is supported by a cable that makes an angle of 53.0º with the horizontal. If 600N person stands 2.00m from the wall, find the tension in the cable, as well as the magnitude and direction of the force exerted by the wall on the beam.





First the translational equilibrium, using components

$$\sum F_x = R\cos\theta - T\cos 53.0^\circ = 0$$

$$\sum F_y = R\sin\theta + T\sin 53.0^\circ - 600N - 200N = 0$$

From the rotational equilibrium

$$\sum \tau = T\sin 53.0^\circ \times 8.00 - 600N \times 2.00 - 200N \cdot 4.00m = 0$$

$$T = 313N$$

Using the translational equilibrium

$$R\cos\theta = T\cos 53.0^\circ$$

$$R\sin\theta = -T\sin 53.0^\circ + 600N + 200N$$

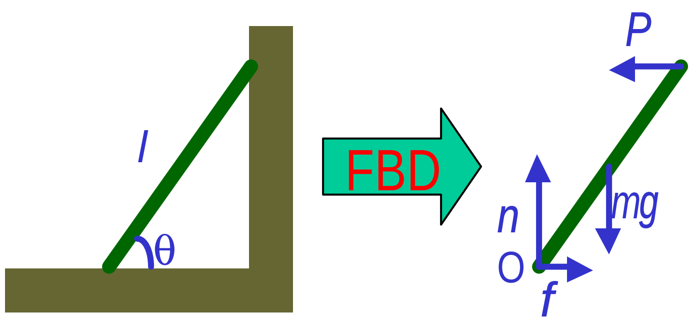
$$\theta = \tan^{-1}\left(\frac{800 - 313\times\sin 53.0^\circ}{313\cos 53.0^\circ}\right) = 71.7^\circ$$

And the magnitude of R is

$$R = \frac{T\cos 53.0^\circ}{\cos\theta} = \frac{313\times\cos 53.0^\circ}{\cos 71.1^\circ} = 582N$$

Example 12.4

A uniform ladder of length *l* and weight *mg*=50 N rests against a smooth, vertical wall. If the coefficient of static friction between the ladder and the ground is μ_s=0.40, find the minimum angle θ_{min} at which the ladder does not slip.



First the translational equilibrium, using components

$$\sum F_x = f - P = 0$$

$$\sum F_y = -mg + n = 0$$

Thus, the normal force is

$$n = mg = 50N$$

The maximum static friction force just before slipping is, therefore,

$$f_s^{\max} = \mu_s n = 0.4 \times 50N = 20N = P$$

From the rotational equilibrium

$$\sum \tau_O = -mg\frac{l}{2}\cos\theta_{\min} + Pl\sin\theta_{\min} = 0$$

$$\theta_{\min} = \tan^{-1}\left(\frac{mg}{2P}\right) = \tan^{-1}\left(\frac{50N}{40N}\right) = 51^\circ$$

How did we solve equilibrium problems?

1. Identify all the forces and their directions and locations
2. Draw a free-body diagram with forces indicated on it
3. Write down vector force equation for each x and y component with proper signs
4. Select a rotational axis for torque calculations ➔ Selecting the axis such that the torque of one of the unknown forces become 0.
5. Write down torque equation with proper signs
6. Solve the equations for unknown quantities

Elastic Properties of Solids

We have been assuming that the objects do not change their shapes when external forces are exerting on it. It this realistic?

No. In reality, the objects get deformed as external forces act on it, though the internal forces resist the deformation as it takes place.

Deformation of solids can be understood in terms of Stress and Strain

Stress: A quantity proportional to the force causing deformation.
Strain: Measure of degree of deformation

It is empirically known that for small stresses, strain is proportional to stress

The constants of proportionality are called Elastic Modulus

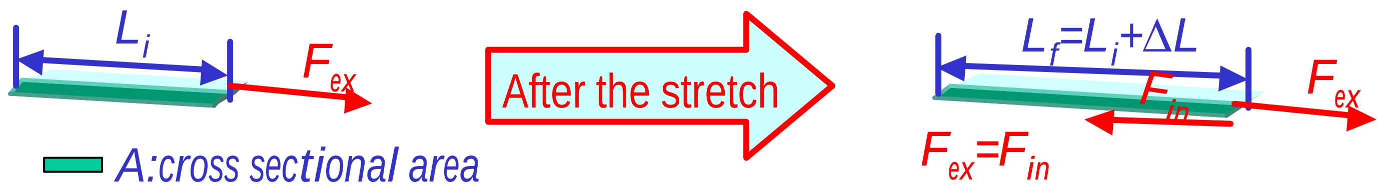
$$\text{Elastic Modulus} \equiv \frac{\text{stress}}{\text{strain}}$$

Three types of Elastic Modulus

1. **Young's modulus**: Measure of the elasticity in length
2. **Shear modulus**: Measure of the elasticity in plane
3. **Bulk modulus**: Measure of the elasticity in volume

Young's Modulus

Let's consider a long bar with cross sectional area A and initial length L_i.



Tensile stress $\text{Tensile Stress} \equiv \dfrac{F_{ex}}{A}$ Tensile strain $\text{Tensile Strain} \equiv \dfrac{\Delta L}{L_i}$

Young's Modulus is defined as

$$Y \equiv \frac{\text{Tensile Stress}}{\text{Tensile Strain}} = \frac{F_{ex}/A}{\Delta L/L_i}$$

Used to characterize a rod or wire stressed under tension or compression

What is the unit of Young's Modulus?

Force per unit area

Experimental Observations

1. For fixed external force, the change in length is proportional to the original length
2. The necessary force to produce a given strain is proportional to the cross sectional area

Elastic limit: Maximum stress that can be applied to the substance before it becomes permanently deformed

Simple Harmonic Motion

What do you think a harmonic motion is?

Motion that occurs by the force that depends on displacement, and the force is always directed toward the system's equilibrium position.

What is a system that has this kind of character? A system consists of a mass and a spring

When a spring is stretched from its equilibrium position by a length x, the force acting on the mass is

$$F = -kx$$

It's negative, because the force resists against the change of length, directed toward the equilibrium position.

From Newton's second law $F = ma = -kx$ we obtain $a = -\frac{k}{m}x$

Condition for simple harmonic motion

This is a second order differential equation that can be solved but it is beyond the scope of this class.

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

What do you observe from this equation?

Acceleration is proportional to displacement from the equilibrium
Acceleration is opposite direction to displacement

Equation of Simple Harmonic Motion

The solution for the 2nd order differential equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$x = A\cos(\omega t + \phi)$$

Amplitude

Phase

Angular Frequency

Phase constant

Generalized expression of a simple harmonic motion

Let's think about the meaning of this equation of motion

What happens when t=0 and ϕ=0?

$$x = A\cos(0+0) = A$$

What is ϕ if x is not A at t=0?

$$x = A\cos(\phi) = x'$$

$$\phi = \cos^{-1}(x')$$

An oscillation is fully characterized by its:

- Amplitude
- Period or frequency
- Phase constant

What are the maximum/minimum possible values of x?

A/-A

More on Equation of Simple Harmonic Motion

What is the time for full cycle of oscillation?

Since after a full cycle the position must be the same

$$x = A\cos\left(w\left(t+T\right)+f\right) = A\cos\left(wt + 2p + f\right)$$

The period

$$T = \frac{2p}{w}$$

One of the properties of an oscillatory motion

How many full cycles of oscillation does this undergo per unit time?

$$f = \frac{1}{T} = \frac{w}{2p}$$

Frequency

What is the unit?

1/s=Hz

Let's now think about the object's speed and acceleration.

$$x = A\cos\left(wt + f\right)$$

Speed at any given time

$$v = \frac{dx}{dt} = -wA\sin\left(wt+f\right)$$

Max speed

$$v_{max} = wA$$

Acceleration at any given time

$$a = \frac{dv}{dt} = -w^2 A\cos\left(wt+f\right) = -w^2 x$$

Max acceleration

$$a_{max} = w^2 A$$

What do we learn about acceleration?

Acceleration is reverse direction to displacement

Acceleration and speed are $\pi/2$ off phase:

When *v* is maximum, *a* is at its minimum

Fig13-10.ip

Simple Block-Spring System

A block attached at the end of a spring on a frictionless surface experiences acceleration when the spring is displaced from an equilibrium position.

$$a = -\frac{k}{m}x$$

This becomes a second order differential equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

If we denote

$$w^2 = \frac{k}{m}$$

The resulting differential equation becomes

$$\frac{d^2x}{dt^2} = -w^2 x$$

Since this satisfies condition for simple harmonic motion, we can take the solution

$$\boxed{x = A\cos(wt + f)}$$

Does this solution satisfy the differential equation?

Let's take derivatives with respect to time

$$\frac{dx}{dt} = A\frac{d}{dt}(\cos(wt + f)) = -wA\sin(wt + f)$$

Now the second order derivative becomes

$$\frac{d^2x}{dt^2} = -wA\frac{d}{dt}(\sin(wt + f)) = -w^2 A\cos(wt + f) = -w^2 x$$

Whenever the force acting on a particle is linearly proportional to the displacement from some equilibrium position and is in the opposite direction, the particle moves in simple harmonic motion.

More Simple Block-Spring System

How do the period and frequency of this harmonic motion look?

Since the angular frequency ω is $w = \sqrt{\frac{k}{m}}$

The period, T, becomes $T = \frac{2p}{w} = 2p\sqrt{\frac{m}{k}}$

So the frequency is $f = \frac{1}{T} = \frac{w}{2p} = \frac{1}{2p}\sqrt{\frac{k}{m}}$

What can we learn from these?

- Frequency and period do not depend on amplitude
- Period is inversely proportional to spring constant and proportional to mass

---

Special case #1

Let's consider that the spring is stretched to distance A and the block is let go from rest, giving 0 initial speed; x_i=A, v_i=0,

$$x = A\cos wt \qquad v = \frac{dx}{dt} = -wA\sin wt \qquad a = \frac{d^2x}{dt^2} = -w^2A\cos wt \qquad a_i = -w^2A = -kA/m$$

This equation of motion satisfies all the conditions. So it isthe solution for this motion.

Special case #2

Suppose block is given non-zero initial velocity v_i to positive x at the instant it is at the equilibrium, x_i=0

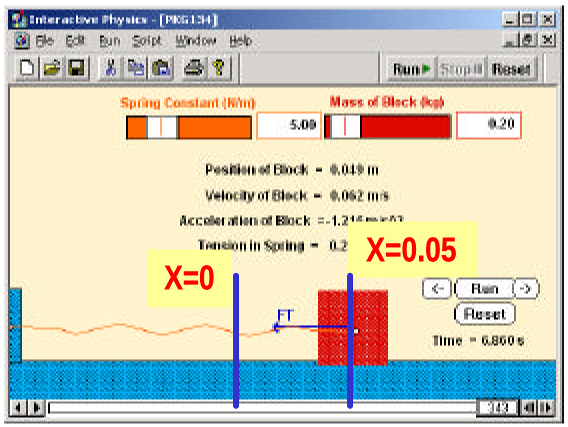
$$f = \tan^{-1}\left(-\frac{v_i}{wx_i}\right) = \tan^{-1}(-\infty) = -\frac{p}{2} \qquad x = A\cos\left(wt - \frac{p}{2}\right) = A\sin(wt)$$

Is this a good solution?

Ex13-03.ip

Example 13.3

A block with a mass of 200g is connected to a light spring for which the force constant is 5.00 N/m and is free to oscillate on a horizontal, frictionless surface. The block is displaced 5.00 cm from equilibrium and released from reset. Find the period of its motion.



From the Hook's law, we obtain

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.00}{0.20}} = 5.00\ s^{-1}$$

As we know, period does not depend on the amplitude or phase constant of the oscillation, therefore the period, T, is simply

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5.00} = 1.26\ s$$

Determine the maximum speed of the block.

From the general expression of the simple harmonic motion, the speed is

$$v_{\max} = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$= \omega A = 5.00 \times 0.05 = 0.25\, m/s$$

Energy of the Simple Harmonic Oscillator

How do you think the mechanical energy of the harmonic oscillator look without friction?

Kinetic energy of a harmonic oscillator is
$$KE=\frac{1}{2}mv^2=\frac{1}{2}mw^2A^2\sin^2(wt+f)$$

The elastic potential energy stored in the spring
$$PE=\frac{1}{2}kx^2=\frac{1}{2}kA^2\cos^2(wt+f)$$

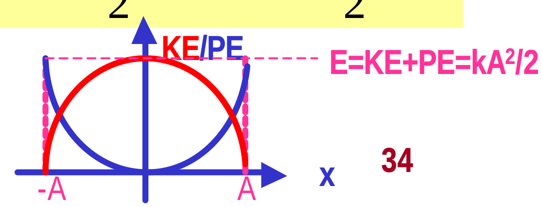
Therefore the total mechanical energy of the harmonic oscillator is
$$E=KE+PE=\frac{1}{2}\left[mw^2A^2\sin^2(wt+f)+kA^2\cos^2(wt+f)\right]$$

Since $w=\sqrt{k/m}$
$$E=KE+PE=\frac{1}{2}\left[kA^2\sin^2(wt+f)+kA^2\cos^2(wt+f)\right]=\frac{1}{2}kA^2$$

Total mechanical energy of a simple harmonic oscillator is a constant of a motion and is proportional to the square of the amplitude

Maximum KE is when PE=0
$$KE_{\max}=\frac{1}{2}mv_{\max}^2=\frac{1}{2}mw^2A^2\sin^2(wt+f)=\frac{1}{2}mw^2A^2=\frac{1}{2}kA^2$$

One can obtain speed
$$E=KE+PE=\frac{1}{2}mv^2+\frac{1}{2}kx^2=\frac{1}{2}kA^2$$
$$v=+\sqrt{k/m\left(A^2-x^2\right)}=+w\sqrt{A^2-x^2}$$



Ex13-04.ip

Example 13.4

A 0.500kg cube connected to a light spring for which the force constant is 20.0 N/m oscillates on a horizontal, frictionless track. a) Calculate the total energy of the system and the maximum speed of the cube if the amplitude of the motion is 3.00 cm.

From the problem statement, A and k are

$$k = 20.0N/m$$
$$A = 3.00cm = 0.03m$$

The total energy of the cube is

$$E = KE + PE = \frac{1}{2}kA^2 = \frac{1}{2}(20.0)\times(0.03)^2 = 9.00\times10^{-3}J$$

Maximum speed occurs when kinetic energy is the same as the total energy

$$KE_{\max} = \frac{1}{2}mv_{\max}^2 = E = \frac{1}{2}kA^2$$

$$v_{\max} = A\sqrt{\frac{k}{m}} = 0.03\sqrt{\frac{20.0}{0.500}} = 0.190m/s$$

b) What is the velocity of the cube when the displacement is 2.00 cm.

velocity at any given displacement is

$$v = \sqrt{k/m\left(A^2 - x^2\right)} = \sqrt{20.0\cdot\left(0.03^2 - 0.02^2\right)/0.500} = 0.141m/s$$

c) Compute the kinetic and potential energies of the system when the displacement is 2.00 cm.

Kinetic energy, KE

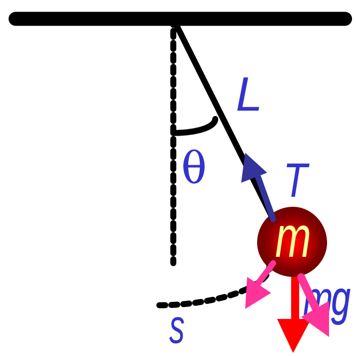
$$KE = \frac{1}{2}mv^2 = \frac{1}{2}0.500\times(0.141)^2 = 4.97\times10^{-3}J$$

Potential energy, PE

$$PE = \frac{1}{2}kx^2 = \frac{1}{2}20.0\times(0.02)^2 = 4.00\times10^{-3}J$$

The Pendulum

A simple pendulum also performs periodic motion.



The net force exerted on the bob is

$$\sum F_r = T - mg\cos\theta_A = 0$$

$$\sum F_t = -mg\sin\theta_A = ma = m\frac{d^2s}{dt^2}$$

Since the arc length, s, is $s = L\theta_A$

$$\frac{d^2s}{dt^2} = L\frac{d^2\theta}{dt^2} = -g\sin\theta$$

results

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Again became a second degree differential equation, satisfying conditions for simple harmonic motion

If θ is very small, sinθ~θ $\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta = -\omega^2\theta$ giving angular frequency $\omega = \sqrt{\frac{g}{L}}$

The period for this motion is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$

The period only depends on the length of the string and the gravitational acceleration

Example 13.5

Christian Huygens (1629-1695), the greatest clock maker in history, suggested that an international unit of length could be defined as the length of a simple pendulum having a period of exactly 1s. How much shorter would out length unit be had this suggestion been followed?

Since the period of a simple pendulum motion is

$$T = \frac{2p}{w} = 2p\sqrt{\frac{L}{g}}$$

The length of the pendulum in terms of T is

$$L = \frac{T^2 g}{4p^2}$$

Thus the length of the pendulum when T=1s is

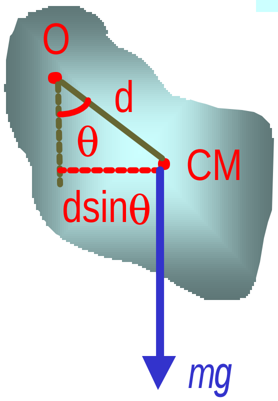
$$L = \frac{T^2 g}{4p^2} = \frac{1\times 9.8}{4p^2} = 0.248\ m$$

Therefore the difference in length with respect to the current definition of 1m is

$$\Delta L = 1 - L = 1 - 0.248 = 0.752\ m$$

Physical Pendulum

Physical pendulum is an object that oscillates about a fixed axis which does not go through the object's center of mass.



Consider a rigid body pivoted at a point O that is a distance d from the CM.

The magnitude of the net torque provided by the gravity is

$$\sum \tau = -mgd \sin\theta$$

Then

$$\sum \tau = I\alpha = I\frac{d^2\theta}{dt^2} = -mgd \sin\theta$$

Therefore, one can rewrite

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\sin\theta \approx -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta$$

Thus, the angular frequency ω is

$$\omega = \sqrt{\frac{mgd}{I}}$$

And the period for this motion is

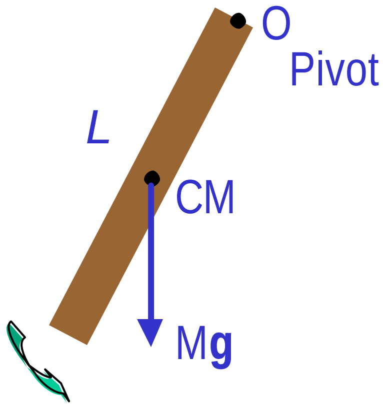
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

By measuring the period of physical pendulum, one can measure moment of inertia.

Does this work for simple pendulum?

Example 13.6

A uniform rod of mass M and length L is pivoted about one end and oscillates in a vertical plane. Find the period of oscillation if the amplitude of the motion is small.



Moment of inertia of a uniform rod, rotating about the axis at one end is $I = \frac{1}{3}ML^2$

The distance d from the pivot to the CM is L/2, therefore the period of this physical pendulum is

$$T = \frac{2p}{w} = 2p\sqrt{\frac{I}{Mgd}} = 2p\sqrt{\frac{2ML^2}{3MgL}} = 2p\sqrt{\frac{2L}{3g}}$$

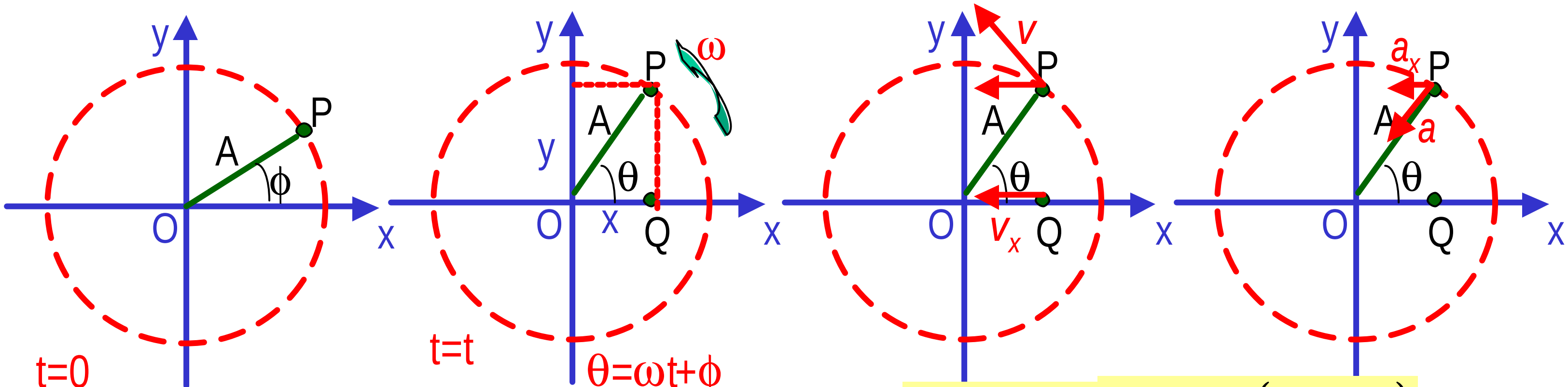
Calculate the period of a meter stick that is pivot about one end and is oscillating in a vertical plane.

Since L=1m, the period is $T = 2p\sqrt{\frac{2L}{3g}} = 2p\sqrt{\frac{2}{3 \cdot 9.8}} = 1.64s$

So the frequency is $f = \frac{1}{T} = 0.61s^{-1}$

Simple Harmonic and Uniform Circular Motions

Uniform circular motion can be understood as a superposition of two simple harmonic motions in x and y axis.



When the particle rotates at a uniform angular speed ω, x and y coordinate position become

$$x = A\cos\theta = A\cos(\omega t + \phi)$$
$$y = A\sin\theta = A\sin(\omega t + \phi)$$

Since the linear velocity in a uniform circular motion is Aω, the velocity components are

$$v_x = -v\sin\theta = -A\omega\sin(\omega t + \phi)$$
$$v_y = +v\cos\theta = A\omega\cos(\omega t + \phi)$$

Since the radial acceleration in a uniform circular motion is $v^2/A=\omega^2 A$, the components are

$$a_x = -a\cos\theta = -A\omega^2\cos(\omega t + \phi)$$
$$a_y = -a\sin\theta = -A\omega^2\sin(\omega t + \phi)$$

Example 13.7

A particle rotates counterclockwise in a circle of radius 3.00m with a constant angular speed of 8.00 rad/s. At t=0, the particle has an x coordinate of 2.00m and is moving to the right. A) Determine the x coordinate as a function of time.

Since the radius is 3.00m, the amplitude of oscillation in x direction is 3.00m. And the angular frequency is 8.00rad/s. Therefore the equation of motion in x direction is

$$X = A\cos q = (3.00m)\cos(8.00t + f)$$

Since x=2.00, when t=0 $\quad 2.00 = (3.00\,m)\cos f \quad f = \cos^{-1}\left(\frac{2.00}{3.00}\right) = 48.2^\circ$

However, since the particle was moving to the right ϕ=-48.2^o,

$$x = (3.00\,m)\cos\left(8.00t - 48.2^\circ\right)$$

Find the x components of the particle's velocity and acceleration at any time t.

Using the displcement

$$v_x = \frac{dx}{dt} = -(3.00\cdot 8.00)\sin(8.00t - 48.2) = (-24.0m/s)\sin\left(8.00t - 48.2^\circ\right)$$

Likewise, from velocity

$$a_x = \frac{dv}{dt} = (-24.0\cdot 8.00)\cos(8.00t - 48.2) = \left(-192m/s^2\right)\cos\left(8.00t - 48.2^\circ\right)$$

Newton's Law of Universal Gravitation

People have been very curious about the stars in the sky, making observations for a long time. But the data people collected have not been explained until Newton has discovered the law of gravitation.

Every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

How would you write this principle mathematically?

$$F_g \propto \frac{m_1 m_2}{r_{12}^2}$$

With G

$$F_g = G\frac{m_1 m_2}{r_{12}^2}$$

G is the universal gravitational constant, and its value is

$$G = 6.673 \times 10^{-11}$$

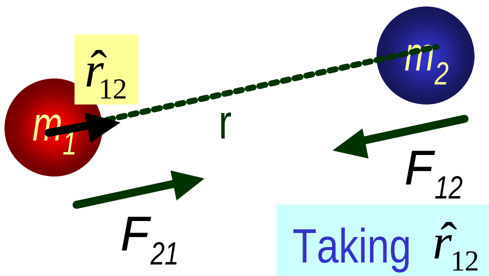
Unit? $N \cdot m^2 / kg^2$

This constant is not given by the theory but must be measured by experiment.

This form of forces is known as an inverse-square law, because the magnitude of the force is inversely proportional to the square of the distances between the objects.

More on Law of Universal Gravitation

Consider two particles exerting gravitational forces to each other.



Two objects exert gravitational force on each other following Newton's 3rd law.

Taking $\hat{r}_{12}$ as the unit vector, we can write the force m_2 experiences as

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r^2}\hat{r}_{12}$$

What do you think the negative sign mean?

It means that the force exerted on the particle 2 by particle 1 is attractive force, pulling #2 toward #1.

Gravitational force is a field force: Forces act on object without physical contact between the objects at all times, independent of medium between them.

The gravitational force exerted by a finite size, spherically symmetric mass distribution on a particle outside the distribution is the same as if the entire mass of the distributions was concentrated at the center.

How do you think the gravitational force on the surface of the earth look?

$$F_g = G\frac{M_E m}{R_E^2}$$

Free Fall Acceleration & Gravitational Force

Weight of an object with mass *m* is *mg*. Using the force exerting on a particle of mass *m* on the surface of the Earth, one can get

$$mg = G\frac{M_E m}{R_E^2}$$

$$g = G\frac{M_E}{R_E^2}$$

What would the gravitational acceleration be if the object is at an altitude *h* above the surface of the Earth?

$$F_g = mg' = G\frac{M_E m}{r^2} = G\frac{M_E m}{(R_E + h)^2}$$

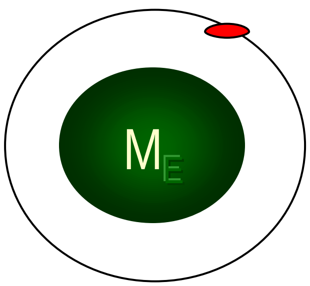
$$g' = G\frac{M_E}{(R_E + h)^2}$$

What do these tell us about the gravitational acceleration?

- The gravitational acceleration is independent of the mass of the object
- The gravitational acceleration decreases as the altitude increases
- If the distance from the surface of the Earth gets infinitely large, the weight of the object approaches 0.

Example 14.2

The international space station is designed to operate at an altitude of 350km. When completed, it will have a weight (measured on the surface of the Earth) of 4.22×10^6N. What is its weight when in its orbit?



The total weight of the station on the surface of the Earth is

$$F_{GE} = mg = G\frac{M_E m}{R_E^2} = 4.22 \times 10^6\,N$$

Since the orbit is at 350km above the surface of the Earth, the gravitational force at that height is

$$F_O = mg = G\frac{M_E m}{(R_E + h)^2} = \frac{R_E^2}{(R_E + h)^2} F_{GE}$$

Therefore the weight in the orbit is

$$F_O = \frac{R_E^2}{(R_E + h)^2} F_{GE} = \frac{\left(6.37 \times 10^6\right)^2}{\left(6.37 \times 10^6 + 3.50 \times 10^5\right)^2} \times 4.22 \times 10^6 = 3.80 \times 10^6\,N$$

Example 14.3

Using the fact that g=9.80m/s^2 at the Earth's surface, find the average density of the Earth.

Since the gravitational acceleration is

$$g = G\frac{M_E}{R_E^{\ 2}} = 6.67\times10^{-11}\frac{M_E}{R_E^{\ 2}}$$

So the mass of the Earth is

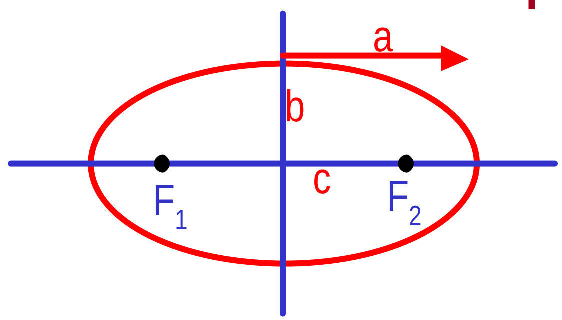
$$M_E = \frac{R_E^{\ 2}g}{G}$$

Therefore the density of the Earth is

$$\rho = \frac{M_E}{V_E} = \frac{\frac{R_E^{\ 2}g}{G}}{\frac{4\pi}{3}R_E^{\ 3}} = \frac{3g}{4\pi G R_E}$$

$$= \frac{3\times9.80}{4\pi\times6.67\times10^{-11}\times6.37\times10^{6}} = 5.50\times10^{3}kg/m^{3}$$

Kepler's Laws & Ellipse



Ellipses have two different axis, major (long) and minor (short) axis, and two focal points, F_1 & F_2

a is the length of a semi-major axis

b is the length of a semi-minor axis

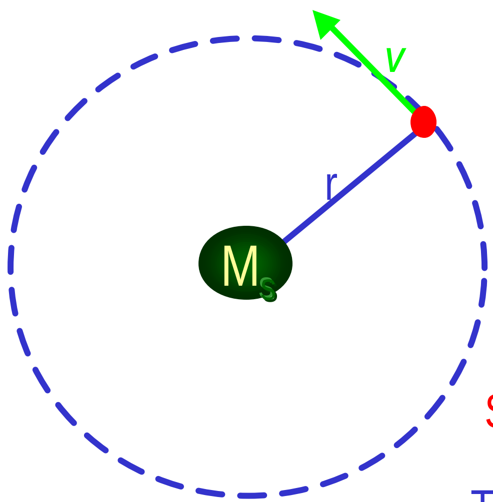
Kepler lived in Germany and discovered the law's governing planets' movement some 70 years before Newton, by analyzing data.

- All planets move in elliptical orbits with the Sun at one focal point.
- The radius vector drawn from the Sun to a planet sweeps out equal area in equal time intervals. *(Angular momentum conservation)*
- The square of the orbital period of any planet is proportional to the cube of the semi-major axis of the elliptical orbit.

Newton's laws explain the cause of the above laws. Kepler's third law is the direct consequence of law of gravitation being inverse square law.

Kepler's Third Law

It is crucial to show that Keper's third law can be predicted from the inverse square law for circular orbits.



Since the gravitational force exerted by the Sun is radially directed toward the Sun to keep the planet circle, we can apply Newton's second law

$$\frac{GM_sM_P}{r^2} = \frac{M_pv^2}{r}$$

Since the orbital speed, v, of the planet with period T is $v = \frac{2\pi r}{T}$

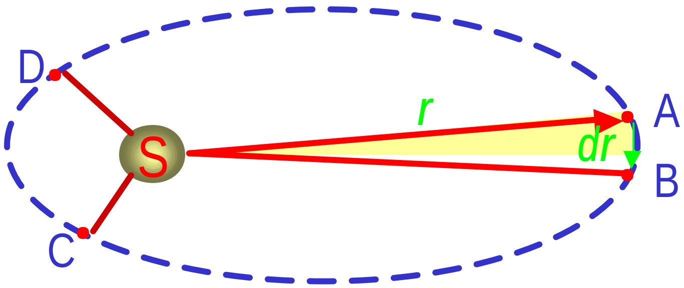
The above can be written $\frac{GM_sM_P}{r^2} = \frac{M_p(2\pi r/T)^2}{r}$

Solving for T one can obtain $T^2 = \left(\frac{4\pi^2}{GM_s}\right)r^3 = K_s r^3$ and $K_s = \left(\frac{4\pi^2}{GM_s}\right) = 2.97\times10^{-19}\ s^2/m^3$

This is Keper's third law. It's also valid for ellipse for r being the length of the semi-major axis. The constant K_s is independent of mass of the planet.

Kepler's Second Law and Angular Momentum Conservation

Consider a planet of mass M_p moving around the Sun in an elliptical orbit.



Since the gravitational force acting on the planet is always toward radial direction, it is a *central force*

Therefore the torque acting on the planet by this force is always 0.

$$\vec{t} = \vec{r}\times\vec{F} = \vec{r}\times F\hat{r} = 0$$

Since torque is the time rate change of angular momentum ***L***, the angular momentum is constant.

$$\vec{t} = \frac{d\vec{L}}{dt} = 0 \qquad \vec{L} = const$$

Because the gravitational force exerted on a planet by the Sun results in no torque, the angular momentum L of the planet is constant.

$$\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times M_p\vec{v} = M_p\vec{r}\times\vec{v} = const$$

Since the area swept by the motion of the planet is

$$dA = \frac{1}{2}\left|\vec{r}\times d\vec{r}\right| = \frac{1}{2}\left|\vec{r}\times\vec{v}dt\right| = \frac{L}{2M_p}dt \Rightarrow \frac{dA}{dt} = \frac{L}{2M_p} = const$$

This is Keper's second law which states that the radius vector from the Sun to a planet sweeps our equal areas in equal time intervals.

The Gravitational Potential Energy

What is the potential energy of an object at the height y from the surface of the Earth?

$$U = mgy$$

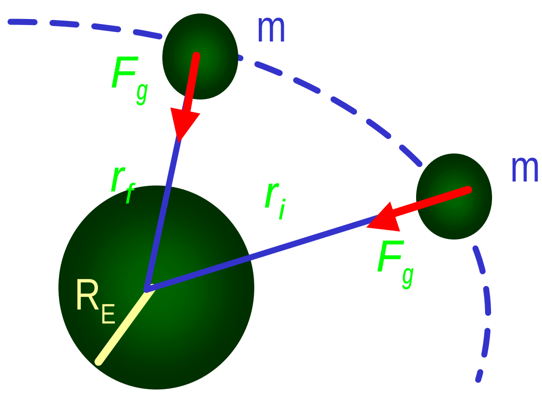
Do you think this would work in general cases?

No, it would not.

Why not?

Because this formula is only valid for the case where the gravitational force is constant, near the surface of the Earth and the generalized gravitational force is inversely proportional to the square of the distance.

OK. Then how would we generalize the potential energy in the gravitational field?



Because gravitational force is a central force, and a central force is a conservative force, the work done by the gravitational force is independent of the path.

The path can be looked at as consisting of many tangential and radial motions.
Tangential motions do not contribute to work!!!

More on The Gravitational Potential Energy

Since the gravitational force is a radial force, it only performed work while the path was radial direction only. Therefore, the work performed by the gravitational force that depends on the position becomes

$$dW = \vec{F} \cdot d\vec{r} = F(r)dr \xrightarrow{\text{For the whole path}} W = \int_{r_i}^{r_f} F(r)dr$$

Potential energy is the negative change of work in the path

$$\Delta U = U_f - U_i = -\int_{r_i}^{r_f} F(r)dr$$

Since the Earth's gravitational force is

$$F(r) = -\frac{GM_E m}{r^2}$$

So the potential energy function becomes

$$U_f - U_i = \int_{r_i}^{r_f} \frac{GM_E m}{r^2} dr = -GM_E m\left[\frac{1}{r_f} - \frac{1}{r_i}\right]$$

Since potential energy only matters for differences, by taking the infinite distance as the initial point of the potential energy, we get

$$U = -\frac{GM_E m}{r}$$

For any two particles?

$$U = -\frac{Gm_1 m_2}{r}$$

The energy needed to take the particles infinitely apart.

For many particles?

$$U = \sum_{i,j} U_{i,j}$$

Example 14.6

A particle of mass m is displaced through a small vertical distance Δy near the Earth's surface. Show that in this situation the general expression for the change in gravitational potential energy is reduced to the $\Delta U = mg\Delta y$.

Taking the general expression of gravitational potential energy

$$\Delta U = -GM_E m\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

The above formula becomes

$$\Delta U = -GM_E m\frac{(r_f - r_i)}{r_f r_i} = -GM_E m\frac{\Delta y}{r_f r_i}$$

Since the situation is close to the surface of the Earth

$$r_i \approx R_E \quad \text{and} \quad r_f \approx R_E$$

Therefore, ΔU becomes

$$\Delta U = -GM_E m\frac{\Delta y}{R_E^2}$$

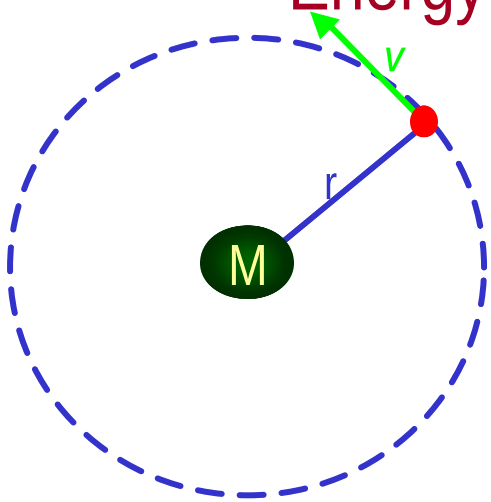
Since on the surface of the Earth the gravitational field is

$$g = \frac{GM_E}{R_E^2}$$

The potential energy becomes

$$\Delta U = -mg\Delta y$$

Energy in Planetary and Satellite Motions



Consider an object of mass m moving at a speed *v* near a massive object of mass M (M>>m).

What's the total energy?

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Systems like the Sun and the Earth or the Earth and the Moon whose motions are contained within a closed orbit is called *Bound Systems.*

For a system to be bound, the total energy must be negative.

Assuming a circular orbit, in order for the object to be kept in the orbit the gravitational force must provide the radial acceleration. Therefore from Newton's second law of motion

$$\frac{GM_E m}{r^2} = ma = m\frac{v^2}{r}$$

The kinetic energy for this system is

$$\frac{1}{2}mv^2 = \frac{GM_E m}{2r}$$

Therefore the total mechanical energy of the system is

$$E = K + U = -\frac{GMm}{2r}$$

Since the gravitational force is conservative, the total mechanical energy of the system is conserved.

Example 14.7

The space shuttle releases a 470kg communication satellite while in an orbit that is 280km above the surface of the Earth. A rocket engine on the satellite boosts it into a geosynchronous orbit, which is an orbit in which the satellite stays directly over a single location on the Earth, How much energy did the engine have to provide?

What is the radius of the geosynchronous orbit?

$$T = 1day = 8.64 \times 10^4 s$$

From Kepler's 3rd law $T^2 = K_E r_{GS}^3$ Where K_E is $K_E = \frac{4p^2}{GM_E} = 9.89 \times 10^{-14} s^2/m^3$

Therefore the geosynchronous radius is

$$r_{GS} = \sqrt[3]{\frac{T^2}{K_E}} = \sqrt[3]{\frac{\left(8.64 \times 10^4\right)^2}{9.89 \times 10^{-14}}} = \sqrt[3]{\frac{\left(8.64 \times 10^4\right)^2}{9.89 \times 10^{-14}}} = 4.23 \times 10^7 m$$

Because the initial position before the boost is 280km

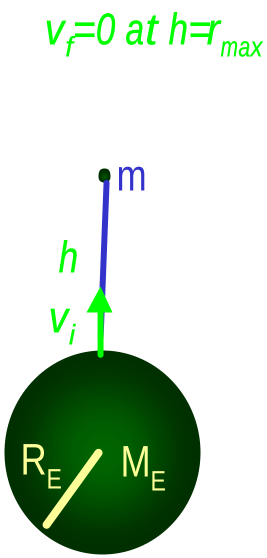
$$r_i = R_E + 2.80 \times 10^5 m = 6.65 \times 10^6 m$$

The total energy needed to boost the satellite at the geosynchronous radius is the difference of the total energy before and after the boost

$$\Delta E = -\frac{GM_E m_s}{2}\left(\frac{1}{r_{GS}} - \frac{1}{r_i}\right)$$

$$= -\frac{6.67\times10^{-11}\times5.98\times10^{24}\times470}{2}\left(\frac{1}{4.23\times10^7} - \frac{1}{6.65\times10^6}\right) = 1.19\times10^{10} J$$

Escape Speed



Consider an object of mass m is projected vertically from the surface of the Earth with an initial speed v_i and eventually comes to stop v_f=0 at the distance r_{max}.

Because the total energy is conserved

$$E = K + U = \frac{1}{2}mv_i^2 - \frac{GM_E m}{R_E} = -\frac{GM_E m}{r_{\max}}$$

Solving the above equation for v_i, one obtains

$$v_i = \sqrt{2GM_E\left(\frac{1}{R_E} - \frac{1}{r_{\max}}\right)}$$

Therefore if the initial speed v_i is known, one can use this formula to compute the final height h of the object.

$$h = r_{\max} - R_E = \frac{v_i^2 R_E^2}{2GM_E - v_i^2 R_E}$$

In order for the object to escape Earth's gravitational field completely, the initial speed needs to be

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{6.37 \times 10^6}}$$

$$= 1.12 \times 10^4 m/s = 11.2 km/s$$

This is called the escape speed. This formula is valid for any planet or large mass objects.

How does this depend on the mass of the escaping object?

Independent of the mass of the escaping object

Fluid and Pressure

What are the three states of matter?

Solid, Liquid, and Gas

How do you distinguish them?

By the time it takes for a particular substance to change its shape in reaction to external forces.

What is a fluid?

A collection of molecules that are randomly arranged and loosely bound by forces between them or by the external container.

We will first learn about mechanics of fluid at rest, *fluid statics*.

In what way do you think fluid exerts stress on the object submerged in it?

Fluid cannot exert shearing or tensile stress. Thus, the only force the fluid exerts on an object immersed in it is the forces perpendicular to the surfaces of the object.

This force by the fluid on an object usually is expressed in the form of the force on a unit area at the given depth, the pressure, defined as

$$P \equiv \frac{F}{A}$$

Expression of pressure for an infinitesimal area dA by the force dF is

$$P = \frac{dF}{dA}$$

Note that pressure is a scalar quantity because it's the magnitude of the force on a surface area A.

What is the unit and dimension of pressure?

Unit: N/m^2

Dim.: $[M][L^{-1}][T^{-2}]$

Special SI unit for pressure is Pascal

$$1Pa \equiv 1N/m^2$$

Example 15.1

The mattress of a water bed is 2.00m long by 2.00m wide and 30.0cm deep.
a) Find the weight of the water in the mattress.

The volume density of water at the normal condition (0^oC and 1 atm) is $1000kg/m^3$. So the total mass of the water in the mattress is

$$m = r_W V_M = 1000 \times 2.00 \times 2.00 \times 0.300 = 1.20 \times 10^3 kg$$

Therefore the weight of the water in the mattress is

$$W = mg = 1.20 \times 10^3 \times 9.8 = 1.18 \times 10^4 N$$

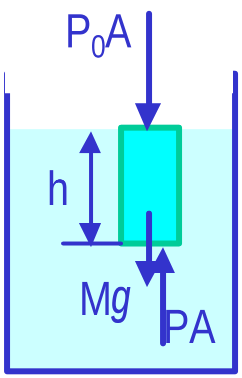
b) Find the pressure exerted by the water on the floor when the bed rests in its normal position, assuming the entire lower surface of the mattress makes contact with the floor.

Since the surface area of the mattress is 4.00 m^2, the pressure exerted on the floor is

$$P = \frac{F}{A} = \frac{mg}{A} = \frac{1.18 \times 10^4}{4.00} = 2.95 \times 10^3$$

Variation of Pressure and Depth

Water pressure increases as a function of depth, and the air pressure decreases as a function of altitude.   Why?



It seems that the pressure has a lot to do with the total mass of the fluid above the object that puts weight on the object.

Let's consider a liquid contained in a cylinder with height h and cross sectional area A immersed in a fluid of density ρ at rest, as shown in the figure, and the system is in its equilibrium.

If the liquid in the cylinder is the same substance as the fluid, the mass of the liquid in the cylinder is $M = \rho V = \rho A h$

Since the system is in its equilibrium

$$PA - P_0A - Mg = PA - P_0A - \rho Ahg = 0$$

Therefore, we obtain $P = P_0 + \rho g h$

The pressure at the depth *h* below the surface of a fluid open to the atmosphere is greater than atmospheric pressure by ρgh.

Atmospheric pressure P_0 is

$$1.00 atm = 1.013 \times 10^5 Pa$$

What else can you learn from this?

Pascal's Law and Hydraulics

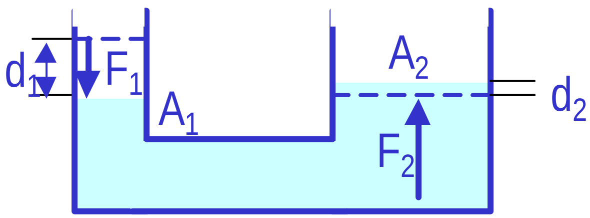
A change in the pressure applied to a fluid is transmitted undiminished to every point of the fluid and to the walls of the container.

$P = P_0 + \rho g h$

What happens if P_0 is changed?

The resultant pressure P at any given depth h increases as much as the change in P_0.

This is the principle behind hydraulic pressure. How?



Since the pressure change caused by the the force F_1 applied on to the area A_1 is transmitted to the F_2 on an area A_2.

$$P = \frac{F_1}{A_1} = \frac{F_2}{A_2}$$

Therefore, the resultant force F_2 is

$$F_2 = \frac{A_2}{A_1} F_1$$

In other words, the force get multiplied by the ratio of the areas A_2/A_1 is transmitted to the F_2 on an area.

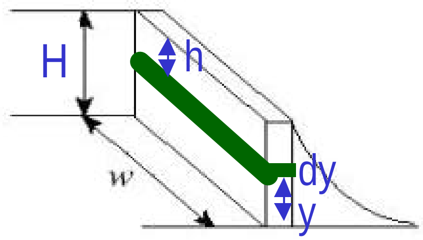
This seems to violate some kind of conservation law, doesn't it?

No, the actual displaced volume of the fluid is the same. And the work done by the forces are still the same.

$$F_2 = \frac{d_1}{d_2} F_1$$

Example 15.4

Water is filled to a height H behind a dam of width w. Determine the resultant force exerted by the water on the dam.



Since the water pressure varies as a function of depth, we will have to do some calculus to figure out the total force.

The pressure at the depth h is

$$P = \rho g h = \rho g (H - y)$$

The infinitesimal force dF exerting on a small strip of dam dy is

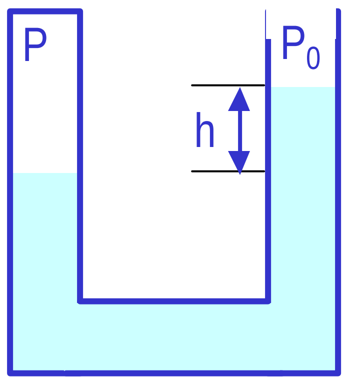
$$dF = PdA = \rho g (H - y) w dy$$

Therefore the total force exerted by the water on the dam is

$$F = \int_{y=0}^{y=H} \rho g (H - y) w dy = \rho g \left[Hy - \frac{1}{2} y^2 \right]_{y=0}^{y=H} = \frac{1}{2} \rho g H^2$$

Absolute and Relative Pressure

How can one measure pressure?



One can measure pressure using an open-tube manometer, where one end is connected to the system with unknown pressure P and the other open to air with pressure P_0.

The measured pressure of the system is $P = P_0 + rgh$

This is called the absolute pressure, because it is the actual value of the system's pressure.

In many cases we measure pressure difference with respect to atmospheric pressure due to changes in P_0 depending on the environment. This is called gauge or relative pressure.

$$P - P_0 = rgh$$

The common barometer which consists of a mercury column with one end closed at vacuum and the other open to the atmosphere was invented by Evangelista Torricelli.

Since the closed end is at vacuum, it does not exert any force. 1 atm is

$$P_0 = rgh = (13.595 \times 10^3 kg/m^3)(9.80665\ m/s^2)(0.7600\ m)$$

$$= 1.013 \times 10^5 Pa = 1 atm$$

Buoyant Forces and Archimedes' Principle

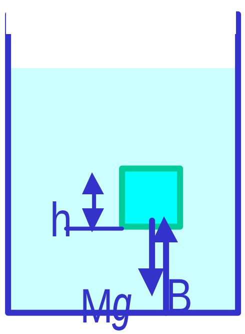
Why is it so hard to put a beach ball under water while a piece of small steel sinks in the water?

The water exerts force on an object immersed in the water.
This force is called Buoyant force.

How does the Buoyant force work?

The magnitude of the buoyant force always equals the weight of the fluid in the volume displaced by the submerged object.

This is called, Archimedes' principle. What does this mean?



Let's consider a cube whose height is h and is filled with fluid and at its equilibrium. Then the weight Mg is balanced by the buoyant force B.

$$B = F_g = Mg$$

And the pressure at the bottom of the cube is larger than the top by ρgh.

Therefore,

$$\Delta P = B / A = rgh$$

$$B = \Delta P A = rghA = rVg$$

$$B = F_g = rVg = Mg$$

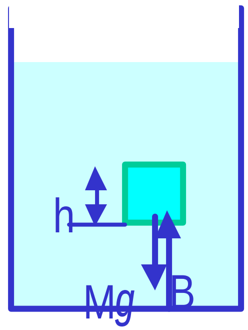
Where Mg is the weight of the fluid.

More Archimedes' Principle

Let's consider buoyant forces in two special cases.

Case 1: Totally submerged object

Let's consider an object of mass M, with density ρ_0, is immersed in the fluid with density ρ_f .



The magnitude of the buoyant force is $B = r_f Vg$

The weight of the object is $F_g = Mg = r_0 Vg$

Therefore total force of the system is $F = B - F_g = (r_f - r_0)Vg$

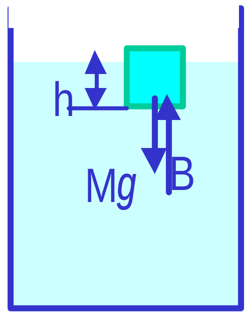
What does this tell you?

The total force applies to different directions, depending on the difference of the density between the object and the fluid.

1. If the density of the object is smaller than the density of the fluid, the buoyant force will push the object up to the surface.
2. If the density of the object is larger that the fluid's, the object will sink to the bottom of the fluid.

More Archimedes' Principle

Case 2: Floating object



Let's consider an object of mass M, with density ρ_0, is in static equilibrium floating on the surface of the fluid with density ρ_f , and the volume submerged in the fluid is V_f.

The magnitude of the buoyant force is $B = \rho_f V_f g$

The weight of the object is $F_g = Mg = \rho_0 V_0 g$

Therefore total force of the system is $F = B - F_g = \rho_f V_f g - \rho_0 V_0 g = 0$

Since the system is in static equilibrium

$$\rho_f V_f g = \rho_0 V_0 g$$

$$\frac{\rho_0}{\rho_f} = \frac{V_f}{V_0}$$

What does this tell you?

Since the object is floating its density is always smaller than that of the fluid.

The ratio of the densities between the fluid and the object determines the submerged volume under the surface.

Example 15.5

Archimedes was asked to determine the purity of the gold used in the crown. The legend says that he solved this problem by weighing the crown in air and in water. Suppose the scale read 7.84N in air and 6.86N in water. What should he have to tell the king about the purity of the gold in the crown?

In the air the tension exerted by the scale on the object is the weight of the crown

$$T_{air} = mg = 7.84N$$

In the water the tension exerted by the scale on the object is

$$T_{water} = mg - B = 6.86N$$

Therefore the buoyant force B is

$$B = T_{air} - T_{water} = 0.98N$$

Since the buoyant force B is

$$B = \rho_w V_w g = \rho_w V_c g = 0.98N$$

The volume of the displaced water by the crown is

$$V_c = V_w = \frac{0.98N}{\rho_w g} = \frac{0.98}{1000 \times 9.8} = 1.0 \times 10^{-4} m^3$$

Therefore the density of the crown is

$$\rho_c = \frac{m_c}{V_c} = \frac{m_c g}{V_c g} = \frac{7.84}{V_c g} = \frac{7.84}{1.0 \times 10^{-4} \times 9.8} = 8.3 \times 10^3 kg/m^3$$

Since the density of pure gold is $19.3 \times 10^3 kg/m^3$, this crown is either not made of pure gold or hollow.

Example 15.6

What fraction of an iceberg is submerged in the sea water?

Let's assume that the total volume of the iceberg is V_i. Then the weight of the iceberg F_{gi} is

$$F_{gi} = r_i V_i g$$

Let's then assume that the volume of the iceberg submerged in the sea water is V_w. The buoyant force B caused by the displaced water becomes

$$B = r_w V_w g$$

Since the whole system is at its static equilibrium, we obtain

$$r_i V_i g = r_w V_w g$$

Therefore the fraction of the volume of the iceberg submerged under the surface of the sea water is

$$\frac{V_w}{V_i} = \frac{r_i}{r_w} = \frac{917\,kg/m^3}{1030\,kg/m^3} = 0.890$$

About 90% of the entire iceberg is submerged in the water!!!

Congratulations!!!!

You all have done very well!!!

- *Good luck with your exams!!!*

Happy Holidays!!

Enjoy the winter break!!!