

PHYS 1443 – Section 003

Lecture #11

Monday, Oct. 6, 2003

Dr. Jaehoon Yu

- Newton's Law of Gravitation
- Kepler's Laws
- Work Done by Constant Force
- Work Done by Varying Force

Deadline for Homework #5 is noon, Wednesday, Oct. 8!!

Deadline for Homework #6 is noon, Wednesday, Oct. 15!!

I will be out of town this Wednesday, Oct. 8 → Dr. Sosebee will give the lecture.

Monday, Oct. 6, 2003



PHYS 1443-003, Fall 2003
Dr. Jaehoon Yu

Free Fall Acceleration & Gravitational Force

Weight of an object with mass m is mg . Using the force exerting on a particle of mass m on the surface of the Earth, one can get

$$mg = G \frac{M_E m}{R_E^2}$$

$$g = G \frac{M_E}{R_E^2}$$

What would the gravitational acceleration be if the object is at an altitude h above the surface of the Earth?

$$F_g = mg' = G \frac{M_E m}{r^2} = G \frac{M_E m}{(R_E + h)^2}$$

$$g' = G \frac{M_E}{(R_E + h)^2}$$

What do these tell us about the gravitational acceleration?

- The gravitational acceleration is independent of the mass of the object
- The gravitational acceleration decreases as the altitude increases
- If the distance from the surface of the Earth gets infinitely large, the weight of the object approaches 0.



Example for Gravitation

Using the fact that $g=9.80\text{m/s}^2$ at the Earth's surface, find the average density of the Earth.

Since the gravitational acceleration is

$$g = G \frac{M_E}{R_E^2} = 6.67 \times 10^{-11} \frac{M_E}{R_E^2}$$

So the mass of the Earth is

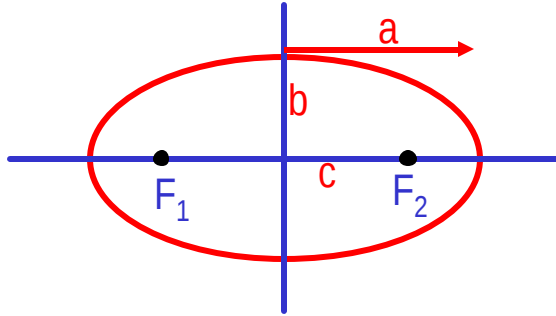
$$M_E = \frac{R_E^2 g}{G}$$

Therefore the density of the Earth is

$$\begin{aligned} \rho &= \frac{M_E}{V_E} = \frac{\frac{R_E^2 g}{G}}{\frac{4\pi}{3} R_E^3} = \frac{3g}{4\pi G R_E} \\ &= \frac{3 \times 9.80}{4\pi \times 6.67 \times 10^{-11} \times 6.37 \times 10^6} = 5.50 \times 10^3 \text{ kg/m}^3 \end{aligned}$$



Kepler's Laws & Ellipse



Ellipses have two different axis, major (long) and minor (short) axis, and two focal points, F_1 & F_2

a is the length of a semi-major axis

b is the length of a semi-minor axis

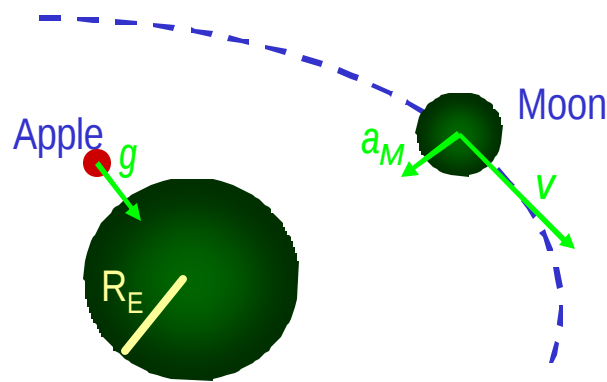
Kepler lived in Germany and discovered the law's governing planets' movement some 70 years before Newton, by analyzing data.

1. All planets move in elliptical orbits with the Sun at one focal point.
2. The radius vector drawn from the Sun to a planet sweeps out equal area in equal time intervals. (*Angular momentum conservation*)
3. The square of the orbital period of any planet is proportional to the cube of the semi-major axis of the elliptical orbit.

Newton's laws explain the cause of the above laws. Kepler's third law is the direct consequence of law of gravitation being inverse square law.

The Law of Gravity and Motions of Planets

- Newton assumed that the law of gravitation applies the same whether it is on the Moon or the apple on the surface of the Earth.
- The interacting bodies are assumed to be point like particles.



Newton predicted that the ratio of the Moon's acceleration a_M to the apple's acceleration g would be

$$\frac{a_M}{g} = \frac{(1/r_M)^2}{(1/R_E)^2} = \left(\frac{R_E}{r_M}\right)^2 = \left(\frac{6.37 \times 10^6}{3.84 \times 10^8}\right)^2 = 2.75 \times 10^{-4}$$

Therefore the centripetal acceleration of the Moon, a_M , is

$$a_M = 2.75 \times 10^{-4} \times 9.80 = 2.70 \times 10^{-3} \text{ m/s}^2$$

Newton also calculated the Moon's orbital acceleration a_M from the knowledge of its distance from the Earth and its orbital period, $T=27.32 \text{ days}=2.36 \times 10^6 \text{ s}$

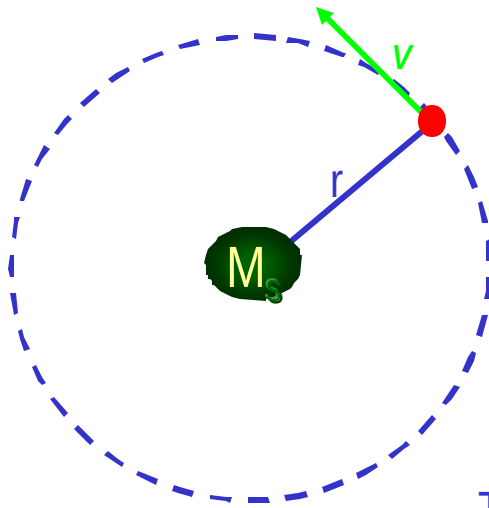
$$a_M = \frac{v^2}{r_M} = \frac{(2\pi r_M / T)^2}{r_M} = \frac{4\pi^2 r_M}{T^2} = \frac{4\pi^2 \times 3.84 \times 10^8}{(2.36 \times 10^6)^2} = 2.72 \times 10^{-3} \text{ m/s}^2 \approx \frac{9.80}{(60)^2}$$

This means that the Moon's distance is about 60 times that of the Earth's radius, its acceleration is reduced by the square of the ratio. This proves that the inverse square law is valid.



Kepler's Third Law

It is crucial to show that Kepler's third law can be predicted from the inverse square law for circular orbits.



Since the gravitational force exerted by the Sun is radially directed toward the Sun to keep the planet circle, we can apply Newton's second law

$$\frac{GM_s M_p}{r^2} = \frac{M_p v^2}{r}$$

Since the orbital speed, v , of the planet with period T is $v = \frac{2\pi r}{T}$

The above can be written $\frac{GM_s M_p}{r^2} = \frac{M_p (2\pi r / T)^2}{r}$

Solving for T one can obtain $T^2 = \left(\frac{4\pi^2}{GM_s} \right) r^3 = K_s r^3$ and $K_s = \left(\frac{4\pi^2}{GM_s} \right) = 2.97 \times 10^{-19} \text{ s}^2 / \text{m}^3$

This is Kepler's third law. It's also valid for ellipse for r being the length of the semi-major axis. The constant K_s is independent of mass of the planet.

Example of Kepler's Third Law

Calculate the mass of the Sun using the fact that the period of the Earth's orbit around the Sun is 3.16×10^7 s, and its distance from the Sun is 1.496×10^{11} m.

Using Kepler's third law.

$$T^2 = \left(\frac{4\pi^2}{GM_s} \right) r^3 = K_s r^3$$

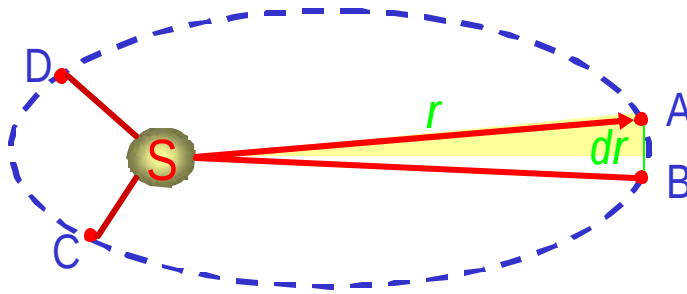
The mass of the Sun, M_s , is

$$\begin{aligned} M_s &= \left(\frac{4\pi^2}{GT} \right) r^3 \\ &= \left(\frac{4\pi^2}{6.67 \times 10^{-11} \times 3.16 \times 10^7} \right) \times (1.496 \times 10^{11})^3 \\ &= 1.99 \times 10^{30} \text{ kg} \end{aligned}$$



Kepler's Second Law and Angular Momentum Conservation

Consider a planet of mass M_p moving around the Sun in an elliptical orbit.



Since the gravitational force acting on the planet is always toward radial direction, it is a *central force*. Therefore the torque acting on the planet by this force is always 0.

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times F\hat{r} = 0$$

Since torque is the time rate change of angular momentum $\vec{L}$, the angular momentum is constant.

$$\frac{d\vec{L}}{dt} = 0 \quad \vec{L} = \text{const}$$

Because the gravitational force exerted on a planet by the Sun results in no torque, the angular momentum L of the planet is constant.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times M_p \vec{v} = M_p \vec{r} \times \vec{v} = \text{const}$$

Since the area swept by the motion of the planet is

$$dA = \frac{1}{2} |\vec{r} \times d\vec{r}| = \frac{1}{2} |\vec{r} \times \vec{v} dt| = \frac{L}{2M_p} dt \quad \Rightarrow \quad \frac{dA}{dt} = \frac{L}{2M_p} = \text{const}$$

This is Kepler's second law which states that the radius vector from the Sun to a planet sweeps out equal areas in equal time intervals.

Work Done by a Constant Force

Work in physics is done only when a sum of forces exerted on an object made a motion to the object.



Which force did the work? Force $\vec{F}$

How much work did it do? $W = \left(\sum \vec{F} \right) \cdot \vec{d} = Fd \cos \theta$ Unit?

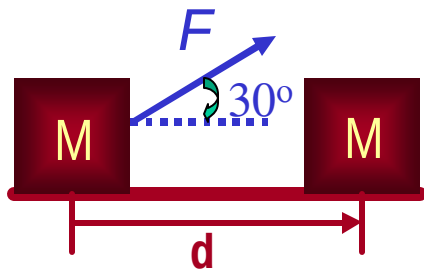
$N \cdot m$
 $= J$ (for Joule)

What does this mean?

Physical work is done only by the component of the force along the movement of the object.

Example of Work w/ Constant Force

A man cleaning a floor pulls a vacuum cleaner with a force of magnitude $F=50.0\text{N}$ at an angle of 30.0° with East. Calculate the work done by the force on the vacuum cleaner as the vacuum cleaner is displaced by 3.00m to East.



$$W = \left(\sum \vec{F} \right) \cdot \vec{d} = \left| \left(\sum \vec{F} \right) \right| |\vec{d}| \cos q$$

$$W = 50.0 \times 3.00 \times \cos 30^\circ = 130\text{J}$$

Does work depend on mass of the object being worked on?

Yes

Why don't I see the mass term in the work at all then?

It is reflected in the force. If the object has smaller mass, it would take less force to move it the same distance as the heavier object. So it would take less work. Which makes perfect sense, doesn't it?

Scalar Product of Two Vectors

- Product of magnitude of the two vectors and the cosine of the angle between them

$$\vec{A} \bullet \vec{B} \equiv |\vec{A}| |\vec{B}| \cos q$$

- Operation is commutative $\vec{A} \bullet \vec{B} = |\vec{A}| |\vec{B}| \cos q = |\vec{B}| |\vec{A}| \cos q = \vec{B} \bullet \vec{A}$

- Operation follows distribution law of multiplication $\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$

- Scalar products of Unit Vectors $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \quad \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

- How does scalar product look in terms of components?

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \quad \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \quad \vec{C} = C_x \hat{i} + C_y \hat{j} + C_z \hat{k}$$

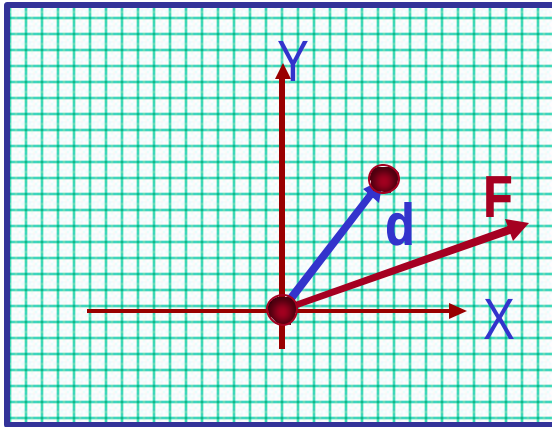
$$\vec{A} \cdot \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) = (A_x B_x \hat{i} \cdot \hat{i} + A_y B_y \hat{j} \cdot \hat{j} + A_z B_z \hat{k} \cdot \hat{k}) + \text{cross terms}$$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$



Example of Work by Scalar Product

A particle moving in the xy plane undergoes a displacement $\mathbf{d}=(2.0\mathbf{i}+3.0\mathbf{j})\text{m}$ as a constant force $\mathbf{F}=(5.0\mathbf{i}+2.0\mathbf{j})\text{ N}$ acts on the particle.



a) Calculate the magnitude of the displacement and that of the force.

$$|\vec{d}| = \sqrt{d_x^2 + d_y^2} = \sqrt{(2.0)^2 + (3.0)^2} = 3.6\text{m}$$

$$|\vec{F}| = \sqrt{F_x^2 + F_y^2} = \sqrt{(5.0)^2 + (2.0)^2} = 5.4\text{N}$$

b) Calculate the work done by the force $\mathbf{F}$.

$$W = \vec{F} \cdot \vec{d} = (2.0\hat{i} + 3.0\hat{j}) \cdot (5.0\hat{i} + 2.0\hat{j}) = 2.0 \times 5.0 \hat{i} \cdot \hat{i} + 3.0 \times 2.0 \hat{j} \cdot \hat{j} = 10 + 6 = 16(J)$$

Can you do this using the magnitudes and the angle between $\mathbf{d}$ and $\mathbf{F}$?

$$W = \vec{F} \cdot \vec{d} = |\vec{F}| |\vec{d}| \cos q$$