

PHYS 1443 – Section 003

Lecture #16

Monday, Oct. 27, 2002

Dr. Jaehoon Yu

1. Center of Mass
2. Motion of a group of particles
3. Rotational Motion
4. Rotational Kinematics
5. Rotational Energy
6. Moment of Inertia

Remember the 2nd term exam (ch 6 – 11), Monday, Nov. 3!



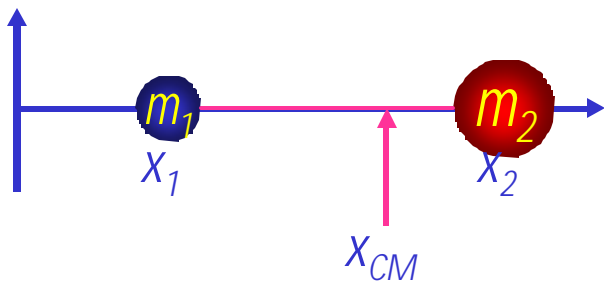
Center of Mass

We've been solving physical problems treating objects as sizeless points with masses, but in realistic situation objects have shapes with masses distributed throughout the body.

Center of mass of a system is the average position of the system's mass and represents the motion of the system as if all the mass is on the point.

What does above statement tell you concerning forces being exerted on the system?

The total external force exerted on the system of total mass M causes the center of mass to move at an acceleration given by $\vec{a} = \sum \vec{F} / M$ as if all the mass of the system is concentrated on the center of mass.



Consider a massless rod with two balls attached at either end.
The position of the center of mass of this system is the mass averaged position of the system

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

CM is closer to the heavier object

Center of Mass of a Rigid Object

The formula for CM can be expanded to Rigid Object or a system of many particles

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

$$y_{CM} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

$$z_{CM} = \frac{\sum_i m_i z_i}{\sum_i m_i}$$

The position vector of the center of mass of a many particle system is

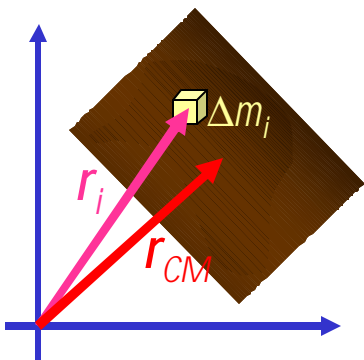
$$\vec{r}_{CM} = x_{CM} \vec{i} + y_{CM} \vec{j} + z_{CM} \vec{k} = \frac{\sum_i m_i x_i \vec{i} + \sum_i m_i y_i \vec{j} + \sum_i m_i z_i \vec{k}}{\sum_i m_i}$$

$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{M}$$

$$x_{CM} \approx \frac{\sum_i \Delta m_i x_i}{M}$$

$$x_{CM} = \lim_{\Delta m_i \rightarrow 0} \frac{\sum_i \Delta m_i x_i}{M} = \frac{1}{M} \int x dm$$

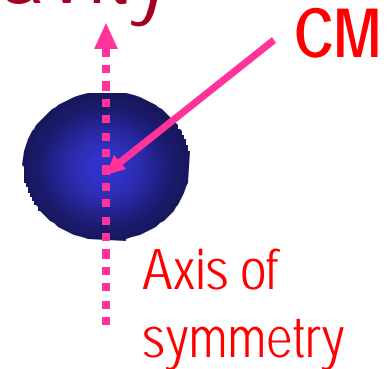
$$\vec{r}_{CM} = \frac{1}{M} \int \vec{r} dm$$



A rigid body – an object with shape and size with mass spread throughout the body, ordinary objects – can be considered as a group of particles with mass m_i densely spread throughout the given shape of the object

Center of Mass and Center of Gravity

The center of mass of any symmetric object lies on an axis of symmetry and on any plane of symmetry, if object's mass is evenly distributed throughout the body.

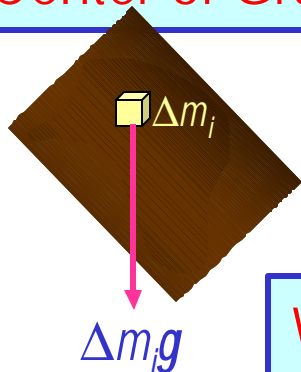


How do you think you can determine the CM of objects that are not symmetric?

One can use gravity to locate CM.

1. Hang the object by one point and draw a vertical line following a plum-bob.
2. Hang the object by another point and do the same.
3. The point where the two lines meet is the CM.

Center of Gravity



Since a rigid object can be considered as **collection of small masses**, one can see the total gravitational force exerted on the object as

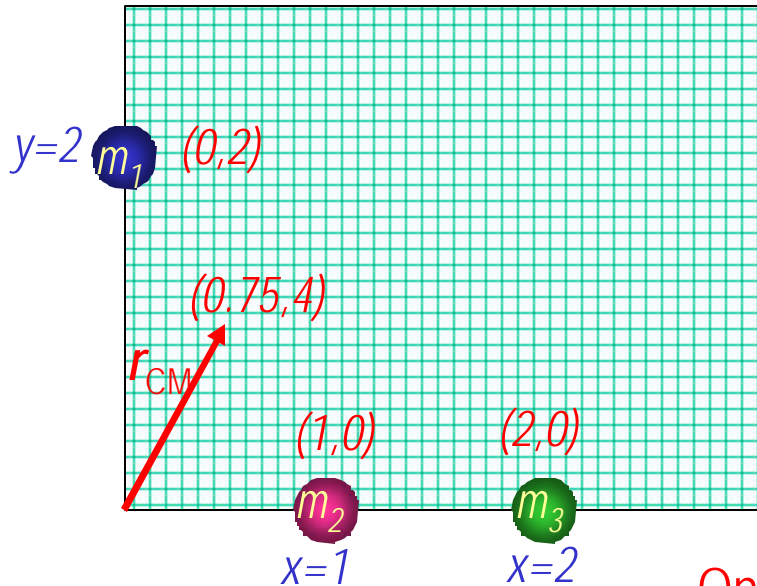
$$\vec{F}_g = \sum_i \vec{F}_i = \sum_i \Delta m_i \vec{g} = M \vec{g}$$

What does this equation tell you?

The net effect of these small gravitational forces is equivalent to a single force acting on a point (**Center of Gravity**) with mass M.

Example for Center of Mass

A system consists of three particles as shown in the figure. Find the position of the center of mass of this system.



Using the formula for CM for each position vector component

$$x_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i} \quad y_{CM} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

One obtains $\vec{r}_{CM} = x_{CM} \vec{i} + y_{CM} \vec{j} = \frac{(m_2 + 2m_3) \vec{i} + 2m_1 \vec{j}}{m_1 + m_2 + m_3}$

$$x_{CM} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{m_2 + 2m_3}{m_1 + m_2 + m_3}$$

$$y_{CM} = \frac{\sum_i m_i y_i}{\sum_i m_i} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{2m_1}{m_1 + m_2 + m_3}$$

If $m_1 = 2\text{kg}; m_2 = m_3 = 1\text{kg}$

$$\vec{r}_{CM} = \frac{3\vec{i} + 4\vec{j}}{4} = 0.75\vec{i} + \vec{j}$$

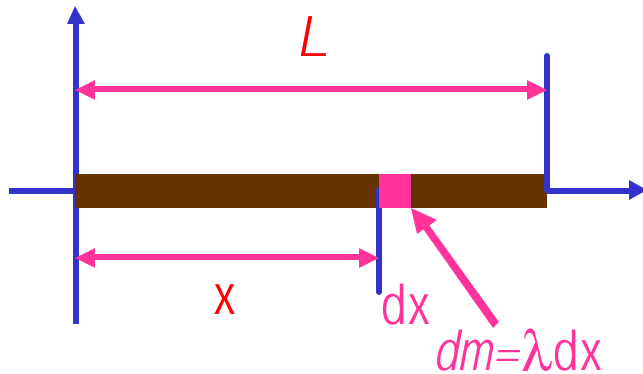
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Example of Center of Mass; Rigid Body

Show that the center of mass of a rod of mass M and length L lies in midway between its ends, assuming the rod has a uniform mass per unit length.



The formula for CM of a continuous object is

$$x_{CM} = \frac{1}{M} \int_{x=0}^{x=L} x dm$$

Since the density of the rod (λ) is constant; $I = M / L$

The mass of a small segment $dm = I dx$

Therefore
$$x_{CM} = \frac{1}{M} \int_{x=0}^{x=L} I x dx = \frac{1}{M} \left[\frac{1}{2} I x^2 \right]_{x=0}^{x=L} = \frac{1}{M} \left(\frac{1}{2} I L^2 \right) = \frac{1}{M} \left(\frac{1}{2} M L \right) = \frac{L}{2}$$

Find the CM when the density of the rod non-uniform but varies linearly as a function of x , $\lambda = \alpha x$

$$\begin{aligned} M &= \int_{x=0}^{x=L} I dx = \int_{x=0}^{x=L} \alpha x dx \\ &= \left[\frac{1}{2} \alpha x^2 \right]_{x=0}^{x=L} = \frac{1}{2} \alpha L^2 \end{aligned}$$

$$x_{CM} = \frac{1}{M} \int_{x=0}^{x=L} I x dx = \frac{1}{M} \int_{x=0}^{x=L} \alpha x^2 dx = \frac{1}{M} \left[\frac{1}{3} \alpha x^3 \right]_{x=0}^{x=L}$$

$$x_{CM} = \frac{1}{M} \left(\frac{1}{3} \alpha L^3 \right) = \frac{1}{M} \left(\frac{2}{3} M L \right) = \frac{2L}{3}$$

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Motion of a Group of Particles

We've learned that the CM of a system can represent the motion of a system. Therefore, for an isolated system of many particles in which the total mass M is preserved, the velocity, total momentum, acceleration of the system are

Velocity of the system

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{d}{dt} \left(\frac{1}{M} \sum m_i \vec{r}_i \right) = \frac{1}{M} \sum m_i \frac{d\vec{r}_i}{dt} = \frac{\sum m_i \vec{v}_i}{M}$$

Total Momentum of the system

$$\vec{p}_{CM} = M \vec{v}_{CM} = M \frac{\sum m_i \vec{v}_i}{M} = \sum m_i \vec{v}_i = \sum \vec{p}_i = \vec{p}_{tot}$$

Acceleration of the system

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{d}{dt} \left(\frac{1}{M} \sum m_i \vec{v}_i \right) = \frac{1}{M} \sum m_i \frac{d\vec{v}_i}{dt} = \frac{\sum m_i \vec{a}_i}{M}$$

External force exerting on the system

$$\sum \vec{F}_{ext} = M \vec{a}_{CM} = \sum m_i \vec{a}_i = \frac{d\vec{p}_{tot}}{dt}$$

What about the internal forces?

If net external force is 0

$$\sum \vec{F}_{ext} = 0 = \frac{d\vec{p}_{tot}}{dt}$$

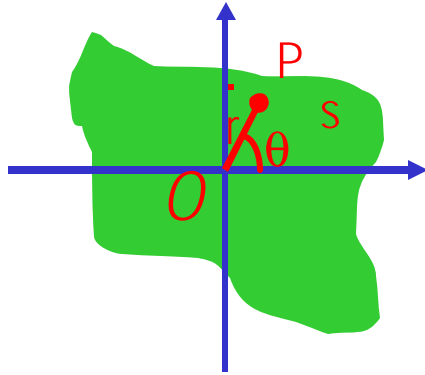
$$\vec{p}_{tot} = \text{const}$$

System's momentum is conserved.

Fundamentals on Rotation

Linear motions can be described as the motion of the center of mass with all the mass of the object concentrated on it.

Is this still true for rotational motions?



No, because different parts of the object have different linear velocities and accelerations.

Consider a motion of a rigid body – an object that does not change its shape – rotating about the axis protruding out of the slide.

The arc length, or sergita, is $s = r\theta$

Therefore the angle, θ , is $\theta = \frac{s}{r}$. And the unit of the angle is in radian.

One radian is the angle swept by an arc length equal to the radius of the arc.

Since the circumference of a circle is $2\pi r$, $360^\circ = 2\pi r / r = 2\pi$

The relationship between radian and degrees is $1 \text{ rad} = 360^\circ / 2\pi = 180^\circ / \pi$

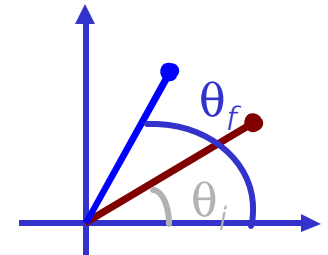
Angular Displacement, Velocity, and Acceleration

Using what we have learned in the previous slide, how would you define the angular displacement?

$$\Delta \mathbf{q} = \mathbf{q}_f - \mathbf{q}_i$$

How about the average angular speed?

$$\overline{\mathbf{w}} \equiv \frac{\mathbf{q}_f - \mathbf{q}_i}{t_f - t_i} = \frac{\Delta \mathbf{q}}{\Delta t}$$



And the instantaneous angular speed?

$$\mathbf{w} \equiv \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{q}}{\Delta t} = \frac{d\mathbf{q}}{dt}$$

By the same token, the average angular acceleration

$$\overline{\mathbf{a}} \equiv \frac{\mathbf{w}_f - \mathbf{w}_i}{t_f - t_i} = \frac{\Delta \mathbf{w}}{\Delta t}$$

And the instantaneous angular acceleration?

$$\mathbf{a} \equiv \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{w}}{\Delta t} = \frac{d\mathbf{w}}{dt}$$

When rotating about a fixed axis, every particle on a rigid object rotates through the same angle and has the same angular speed and angular acceleration.

Rotational Kinematics

The first type of motion we have learned in linear kinematics was under a constant acceleration. We will learn about the rotational motion under constant angular acceleration, because these are the simplest motions in both cases.

Just like the case in linear motion, one can obtain

Angular Speed under constant angular acceleration:

$$\mathbf{w}_f = \mathbf{w}_i + \mathbf{a}t$$

Angular displacement under constant angular acceleration:

$$\mathbf{q}_f = \mathbf{q}_i + \mathbf{w}_i t + \frac{1}{2} \mathbf{a} t^2$$

One can also obtain

$$\mathbf{w}_f^2 = \mathbf{w}_i^2 + 2\mathbf{a}(\mathbf{q}_f - \mathbf{q}_i)$$

Example for Rotational Kinematics

A wheel rotates with a constant angular acceleration of 3.50 rad/s^2 . If the angular speed of the wheel is 2.00 rad/s at $t_i=0$, a) through what angle does the wheel rotate in 2.00s ?

Using the angular displacement formula in the previous slide, one gets

$$\begin{aligned}\mathbf{q}_f - \mathbf{q}_i &= \mathbf{w}t + \frac{1}{2}\mathbf{a}t^2 \\ &= 2.00 \times 2.00 + \frac{1}{2}3.50 \times (2.00)^2 = 11.0 \text{ rad} \\ &= \frac{11.0}{2\pi} \text{ rev} = 1.75 \text{ rev}.\end{aligned}$$

What is the angular speed at $t=2.00\text{s}$?

Using the angular speed and acceleration relationship

$$\begin{aligned}\mathbf{w}_f &= \mathbf{w}_i + \mathbf{a}t \\ &= 2.00 + 3.50 \times 2.00 = 9.00 \text{ rad/s}\end{aligned}$$

Find the angle through which the wheel rotates between $t=2.00 \text{ s}$ and $t=3.00 \text{ s}$.

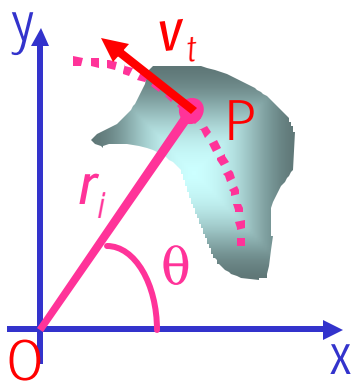
$$\mathbf{q}_2 = 11.0 \text{ rad} \quad \mathbf{q}_3 = 2.00 \times 3.00 + \frac{1}{2}3.50 \times (3.00)^2 = 21.8 \text{ rad}$$

$$\Delta \mathbf{q} = \mathbf{q}_3 - \mathbf{q}_2 = 10.8 \text{ rad} = \frac{10.8}{2\pi} \text{ rev} = 1.72 \text{ rev}.$$

Relationship Between Angular and Linear Quantities

What do we know about a rigid object that rotates about a fixed axis of rotation?

Every particle (or masslet) in the object moves in a circle centered at the axis of rotation.



When a point rotates, it has both the linear and angular motion components in its motion.

What is the linear component of the motion you see?

Linear velocity along the tangential direction.

How do we relate this linear component of the motion with angular component?

The direction of ω follows a right-hand rule.

The arc-length is $s = r\theta$ So the tangential speed v_t is $v_t = \frac{ds}{dt} = \frac{d}{dt}(r\theta) = r \frac{d\theta}{dt} = r\omega$

What does this relationship tell you about the tangential speed of the points in the object and their angular speed?

Although every particle in the object has the same angular speed, its tangential speed differs proportional to its distance from the axis of rotation.

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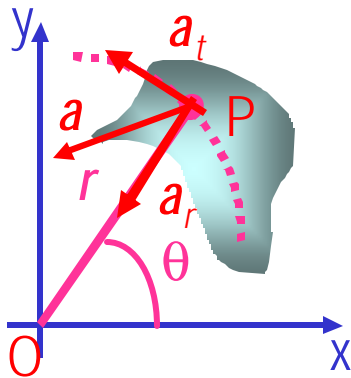
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The farther away the particle is from the center of rotation, the higher the tangential speed.

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How about the Accelerations?

How many different linear accelerations do you see in a circular motion and what are they? **Two**



Tangential, a_t , and the radial acceleration, a_r .

Since the tangential speed v_t is $v_t = r\omega$

The magnitude of tangential acceleration a_t is $a_t = \frac{dv_t}{dt} = \frac{d}{dt}(r\omega) = r \frac{d\omega}{dt} = r\alpha$

What does this relationship tell you?

Although every particle in the object has the same angular acceleration, its tangential acceleration differs proportional to its distance from the axis of rotation.

The radial or centripetal acceleration a_r is $a_r = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$

What does this tell you?

The farther away the particle from the rotation axis the more radial acceleration it receives. In other words, it receives more centripetal force.

Total linear acceleration is

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{(r\alpha)^2 + (r\omega^2)^2} = r\sqrt{\alpha^2 + \omega^4}$$

Example for Rotational Motion

Audio information on compact discs are transmitted digitally through the readout system consisting of laser and lenses. The digital information on the disc are stored by the pits and flat areas on the track. Since the speed of readout system is constant, it reads out the same number of pits and flats in the same time interval. In other words, the linear speed is the same no matter which track is played. a) Assuming the linear speed is 1.3 m/s, find the angular speed of the disc in revolutions per minute when the inner most ($r=23\text{mm}$) and outer most tracks ($r=58\text{mm}$) are read.

Using the relationship between angular and tangential speed $v=r\omega$

$$r = 23\text{mm} \quad \omega = \frac{v}{r} = \frac{1.3\text{m/s}}{23\text{mm}} = \frac{1.3}{23 \times 10^{-3}} = 56.5\text{rad/s} \\ = 9.00\text{rev/s} = 5.4 \times 10^2 \text{rev/min}$$

$$r = 58\text{mm} \quad \omega = \frac{1.3\text{m/s}}{58\text{mm}} = \frac{1.3}{58 \times 10^{-3}} = 22.4\text{rad/s} = 2.1 \times 10^2 \text{rev/min}$$

$$\bar{\omega} = \frac{(\omega_i + \omega_f)}{2} = \frac{(540 + 210)\text{rev/min}}{2} = 375\text{rev/min}$$

$$\theta_f = \theta_i + \bar{\omega}t = 0 + \frac{375}{60}\text{rev/s} \times 4473 = 2.8 \times 10^4 \text{rev}$$

b) The maximum playing time of a standard music CD is 74 minutes and 33 seconds. How many revolutions does the disk make during that time?

c) What is the total length of the track past through the readout mechanism?

$$l = v_t \Delta t = 1.3\text{m/s} \times 4473\text{s} \\ = 5.8 \times 10^3 \text{m}$$

d) What is the angular acceleration of the CD over the 4473s time interval, assuming constant α ?

$$\alpha = \frac{(\omega_f - \omega_i)}{\Delta t} = \frac{(22.4 - 56.5)\text{rad/s}}{4473\text{s}}$$

$$= 7.6 \times 10^{-3} \text{rad/s}^2 \quad 14$$

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