

PHYS 1443 – Section 003

Lecture #17

Wednesday, Oct. 29, 2002

Dr. Jaehoon Yu

1. Rolling Motion of a Rigid Body
2. Torque
3. Moment of Inertia
4. Rotational Kinetic Energy
5. Torque and Vector Products

Remember the 2nd term exam (ch 6 – 11), Monday, Nov. 3!



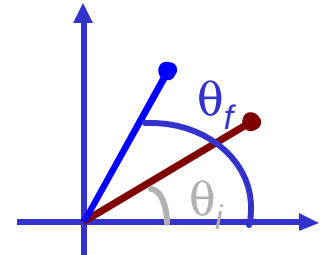
Angular Displacement, Velocity, and Acceleration

Using what we have learned in the previous slide, how would you define the angular displacement?

$$\Delta q = q_f - q_i$$

How about the average angular speed?

$$\bar{w} \equiv \frac{q_f - q_i}{t_f - t_i} = \frac{\Delta q}{\Delta t}$$



And the instantaneous angular speed?

$$w \equiv \lim_{\Delta t \rightarrow 0} \frac{\Delta q}{\Delta t} = \frac{dq}{dt}$$

By the same token, the average angular acceleration

$$\bar{a} \equiv \frac{w_f - w_i}{t_f - t_i} = \frac{\Delta w}{\Delta t}$$

And the instantaneous angular acceleration?

$$a \equiv \lim_{\Delta t \rightarrow 0} \frac{\Delta w}{\Delta t} = \frac{dw}{dt}$$

When rotating about a fixed axis, every particle on a rigid object rotates through the same angle and has the same angular speed and angular acceleration.

Rotational Kinematics

The first type of motion we have learned in linear kinematics was under a constant acceleration. We will learn about the rotational motion under constant angular acceleration, because these are the simplest motions in both cases.

Just like the case in linear motion, one can obtain

Angular Speed under constant angular acceleration:

$$\omega_f = \omega_i + at$$

Angular displacement under constant angular acceleration:

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2} at^2$$

One can also obtain

$$\omega_f^2 = \omega_i^2 + 2a(\theta_f - \theta_i)$$

Rolling Motion of a Rigid Body

What is a rolling motion?

A more generalized case of a motion where the rotational axis moves together with the object

A rotational motion about the moving axis

To simplify the discussion, let's make a few assumptions

1. Limit our discussion on very symmetric objects, such as cylinders, spheres, etc
2. The object rolls on a flat surface

Let's consider a cylinder rolling without slipping on a flat surface

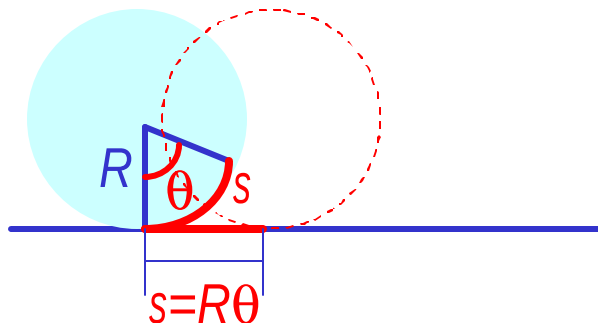
Under what condition does this “Pure Rolling” happen?

The total linear distance the CM of the cylinder moved is

$$s = Rq$$

Thus the linear speed of the CM is

$$v_{CM} = \frac{ds}{dt} = R \frac{dq}{dt} = R\omega$$



Wednesday, Oct. 29, 2003



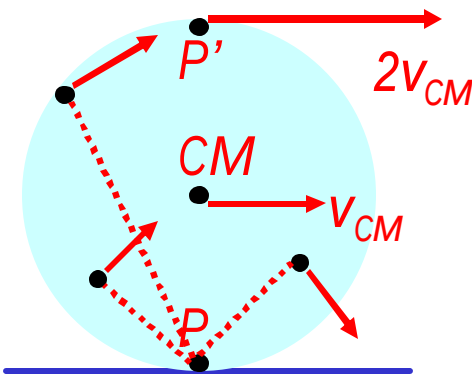
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Condition for “Pure Rolling”

More Rolling Motion of a Rigid Body

The magnitude of the linear acceleration of the CM is

$$a_{CM} = \frac{dv_{CM}}{dt} = R \frac{d\omega}{dt} = Ra$$

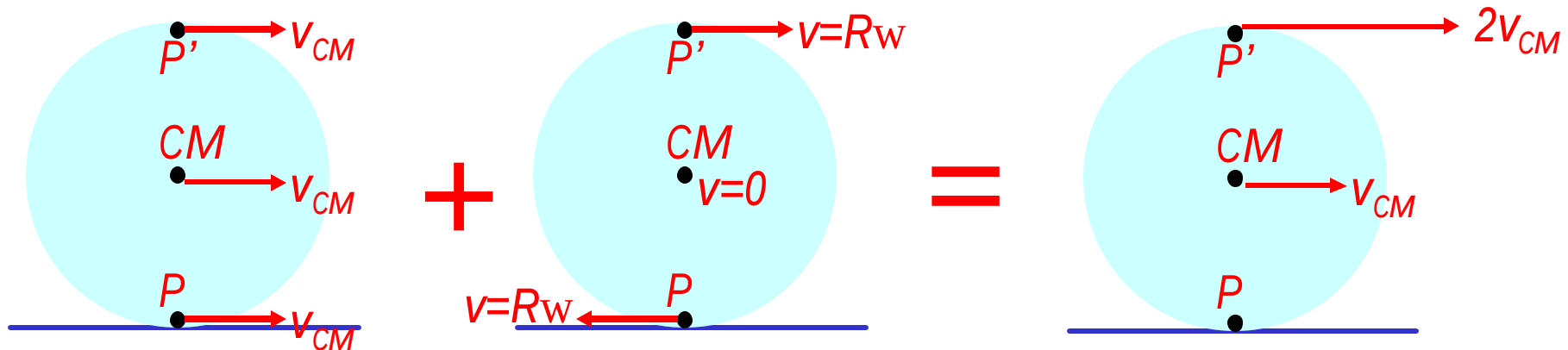


As we learned in the rotational motion, all points in a rigid body moves at the same angular speed but at a different linear speed.

At any given time the point that comes to P has 0 linear speed while the point at P' has twice the speed of CM
CM is moving at the same speed at all times.

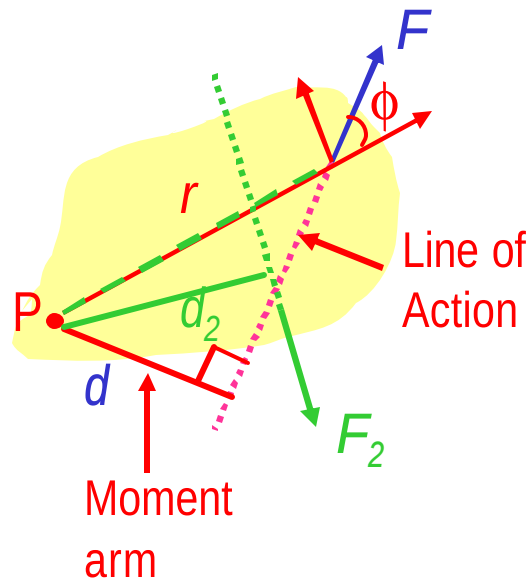
Why??

A rolling motion can be interpreted as the sum of Translation and Rotation



Torque

Torque is the tendency of a force to rotate an object about an axis.
Torque, τ , is a vector quantity.



Consider an object pivoting about the point **P** by the force **F** being exerted at a distance **r**.

The line that extends out of the tail of the force vector is called the **line of action**.

The perpendicular distance from the pivoting point **P** to the line of action is called **Moment arm**.

Magnitude of torque is defined as the product of the force exerted on the object to rotate it and the moment arm.

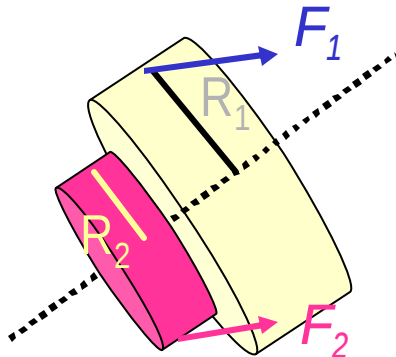
$$\tau \equiv rF \sin \phi = Fd$$

When there are more than one force being exerted on certain points of the object, one can sum up the torque generated by each force vectorially. The convention for sign of the torque is positive if rotation is in counter-clockwise and negative if clockwise.

$$\begin{aligned} \sum \tau &= \tau_1 + \tau_2 \\ &= F_1 d_1 - F_2 d_2 \end{aligned}$$

Example for Torque

A one piece cylinder is shaped as in the figure with core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the picture. A rope wrapped around the drum whose radius is R_1 exerts force F_1 to the right on the cylinder, and another force exerts F_2 on the core whose radius is R_2 downward on the cylinder. A) What is the net torque acting on the cylinder about the rotation axis?



The torque due to F_1 $t_1 = -R_1 F_1$ and due to F_2 $t_2 = R_2 F_2$

So the total torque acting on the system by the forces is

$$\sum t = t_1 + t_2 = -R_1 F_1 + R_2 F_2$$

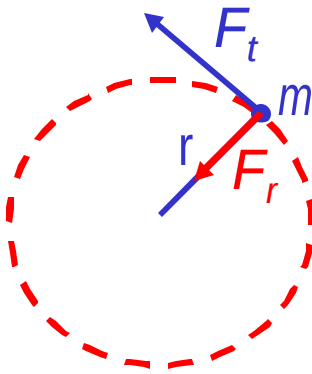
Suppose $F_1 = 5.0 \text{ N}$, $R_1 = 1.0 \text{ m}$, $F_2 = 15.0 \text{ N}$, and $R_2 = 0.50 \text{ m}$. What is the net torque about the rotation axis and which way does the cylinder rotate from the rest?

Using the above result

$$\begin{aligned} \sum t &= -R_1 F_1 + R_2 F_2 \\ &= -5.0 \times 1.0 + 15.0 \times 0.50 = 2.5 \text{ N} \cdot \text{m} \end{aligned}$$

The cylinder rotates in counter-clockwise.

Torque & Angular Acceleration



Let's consider a point object with mass m rotating on a circle.

What forces do you see in this motion?

The tangential force F_t and radial force F_r

The tangential force F_t is

$$F_t = ma_t = mra$$

The torque due to tangential force F_t is

$$\tau = F_t r = ma_t r = mr^2 a = Ia$$

What do you see from the above relationship?

$$\tau = Ia$$

What does this mean?

Torque acting on a particle is proportional to the angular acceleration.

What law do you see from this relationship?

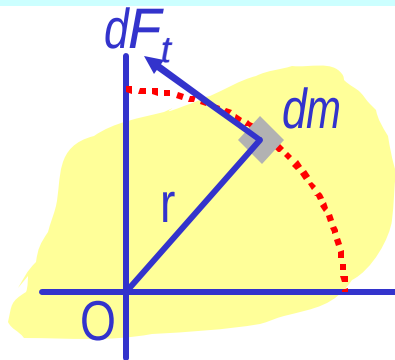
Analogous to Newton's 2nd law of motion in rotation.

How about a rigid object?

The external tangential force dF_t is $dF_t = dma_t = dmra$

The torque due to tangential force F_t is $d\tau = dF_t r = (r^2 dm)a$

The total torque is $\sum \tau = a \int r^2 dm = Ia$



What is the contribution due to radial force and why?

Contribution from radial force is 0, because its line of action passes through the pivoting point, making the moment arm 0.

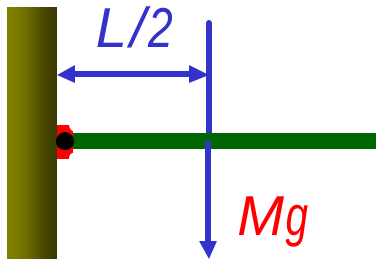
Wednesday, Oct. 29, 2002



Dr. Jaehoon Yu

Example for Torque and Angular Acceleration

A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane. The rod is released from rest in the horizontal position. What are the initial angular acceleration of the rod and the initial linear acceleration of its right end?



The only force generating torque is the gravitational force Mg

$$\tau = Fd = F \frac{L}{2} = Mg \frac{L}{2} = I\alpha$$

Since the moment of inertia of the rod when it rotates about one end

$$I = \int_0^L r^2 dm = \int_0^L x^2 l dx = \left(\frac{M}{L} \right) \left[\frac{x^3}{3} \right]_0^L = \frac{ML^2}{3}$$

We obtain

$$\alpha = \frac{MgL}{2I} = \frac{MgL}{\frac{2ML^2}{3}} = \frac{3g}{2L}$$

Using the relationship between tangential and angular acceleration

$$a_t = L\alpha = \frac{3g}{2}$$

What does this mean?

The tip of the rod falls faster than an object undergoing a free fall.

Moment of Inertia

Rotational Inertia:

Measure of resistance of an object to changes in its rotational motion.
Equivalent to mass in linear motion.

For a group
of particles

$$I \equiv \sum_i m_i r_i^2$$

For a rigid
body

$$I \equiv \int r^2 dm$$

What are the dimension and
unit of Moment of Inertia?

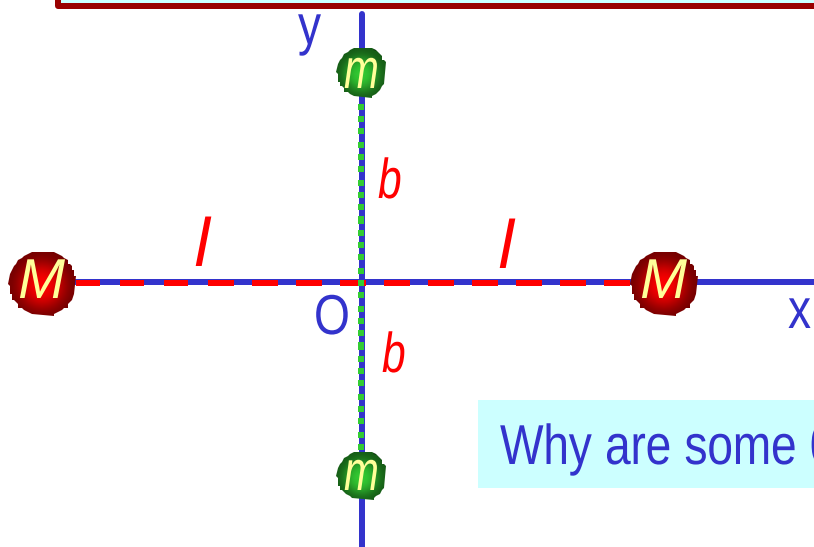
$$[ML^2]$$

$$kg \cdot m^2$$

Determining Moment of Inertia is extremely important for computing equilibrium of a rigid body, such as a building.

Example for Moment of Inertia

In a system consists of four small spheres as shown in the figure, assuming the radii are negligible and the rods connecting the particles are massless, compute the moment of inertia and the rotational kinetic energy when the system rotates about the y-axis at ω .



Since the rotation is about y axis, the moment of inertia about y axis, I_y , is

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + m \cdot 0^2 + m \cdot 0^2 = 2Ml^2$$

Why are some 0s?

This is because the rotation is done about y axis, and the radii of the spheres are negligible.

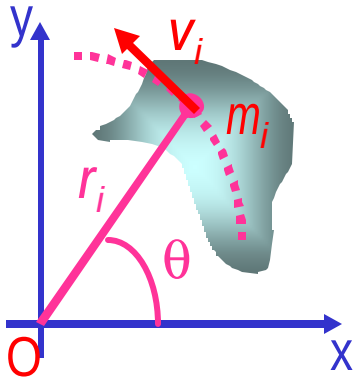
Thus, the rotational kinetic energy is

$$K_R = \frac{1}{2} I \omega^2 = \frac{1}{2} (2Ml^2) \omega^2 = Ml^2 \omega^2$$

Find the moment of inertia and rotational kinetic energy when the system rotates on the x-y plane about the z-axis that goes through the origin O.

$$I = \sum_i m_i r_i^2 = Ml^2 + Ml^2 + mb^2 + mb^2 = 2(Ml^2 + mb^2) \quad K_R = \frac{1}{2} I \omega^2 = \frac{1}{2} (2Ml^2 + 2mb^2) \omega^2 = (Ml^2 + mb^2) \omega^2$$

Rotational Kinetic Energy



What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, m_i , moving at a tangential speed, v_i , is

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 \omega^2$$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

$$K_R = \sum_i K_i = \frac{1}{2} \sum_i m_i r_i^2 \omega^2 = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2$$

Since moment of Inertia, I , is defined as

$$I = \sum_i m_i r_i^2$$

The above expression is simplified as

$$K_R = \frac{1}{2} I \omega^2$$

Calculation of Moments of Inertia

Moments of inertia for large objects can be computed, if we assume the object consists of small volume elements with mass, Δm_i .

The moment of inertia for the large rigid object is
$$I = \lim_{\Delta m_i \rightarrow 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$

It is sometimes easier to compute moments of inertia in terms of volume of the elements rather than their mass

How can we do this?

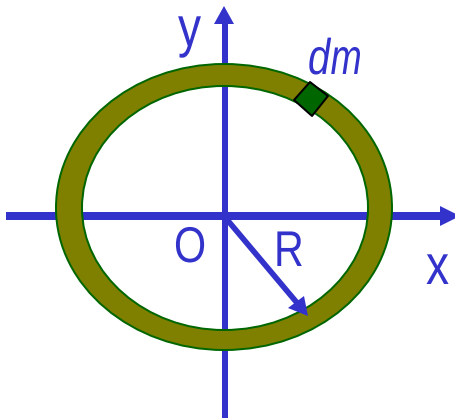
Using the volume density, ρ , replace dm in the above equation with dV .

$$r = \frac{dm}{dV} \quad dm = r dV$$

The moments of inertia becomes

$$I = \int r r^2 dV$$

Example: Find the moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center.



The moment of inertia is

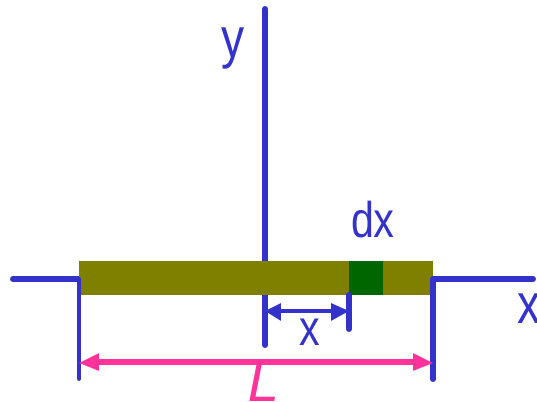
$$I = \int r^2 dm = R^2 \int dm = MR^2$$

What do you notice from this result?

The moment of inertia for this object is the same as that of a point of mass M at the distance R .

Example for Rigid Body Moment of Inertia

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod and passing through its center of mass.



The line density of the rod is $\lambda = \frac{M}{L}$

so the masslet is $dm = \lambda dx = \frac{M}{L} dx$

The moment of inertia is

$$I = \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L} \left[\frac{1}{3} x^3 \right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L} \left[\left(\frac{L}{2} \right)^3 - \left(-\frac{L}{2} \right)^3 \right] = \frac{M}{3L} \left(\frac{L^3}{4} \right) = \frac{ML^2}{12}$$

What is the moment of inertia when the rotational axis is at one end of the rod.

$$I = \int r^2 dm = \int_0^L \frac{x^2 M}{L} dx = \frac{M}{L} \left[\frac{1}{3} x^3 \right]_0^L$$

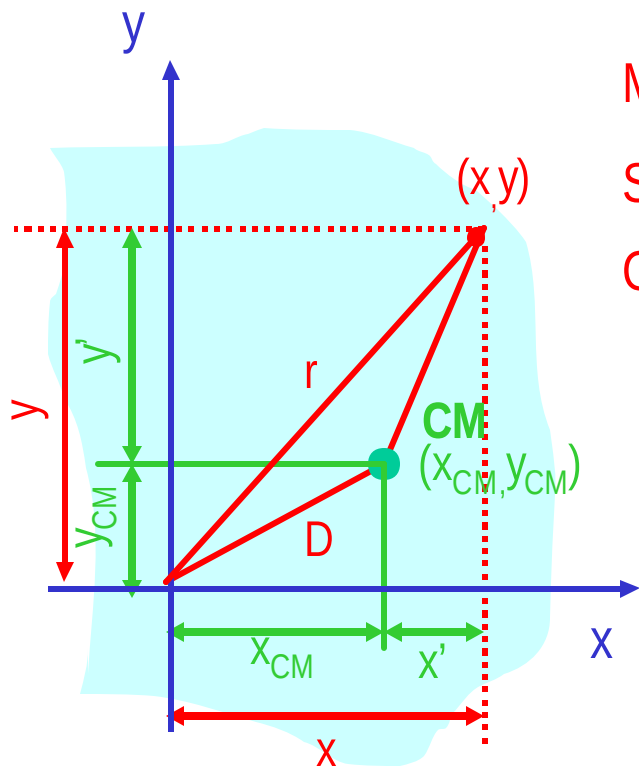
$$= \frac{M}{3L} [(L)^3 - 0] = \frac{M}{3L} (L^3) = \frac{ML^2}{3}$$

Will this be the same as the above.
Why or why not?

Since the moment of inertia is resistance to motion, it makes perfect sense for it to be harder to move when it is rotating about the axis at one end.

Parallel Axis Theorem

Moments of inertia for highly symmetric object is easy to compute if the rotational axis is the same as the axis of symmetry. However if the axis of rotation does not coincide with axis of symmetry, the calculation can still be done in simple manner using **parallel-axis theorem**. $I = I_{CM} + MD^2$



Moment of inertia is defined $I = \int r^2 dm = \int \sqrt{(x^2 + y^2)} dm$ (1)

Since x and y are $x = x_{CM} + x'$ $y = y_{CM} + y'$

One can substitute x and y in Eq. 1 to obtain

$$I = \int [(x_{CM} + x')^2 + (y_{CM} + y')^2] dm$$

$$= (x_{CM}^2 + y_{CM}^2) \int dm + 2x_{CM} \int x' dm + 2y_{CM} \int y' dm + \int (x'^2 + y'^2) dm$$

Since the x' and y' are the distance from CM, by definition $\int x' dm = 0$ $\int y' dm = 0$

Therefore, the parallel-axis theorem

$$I = (x_{CM}^2 + y_{CM}^2) \int dm + \int (x'^2 + y'^2) dm = MD^2 + I_{CM}$$

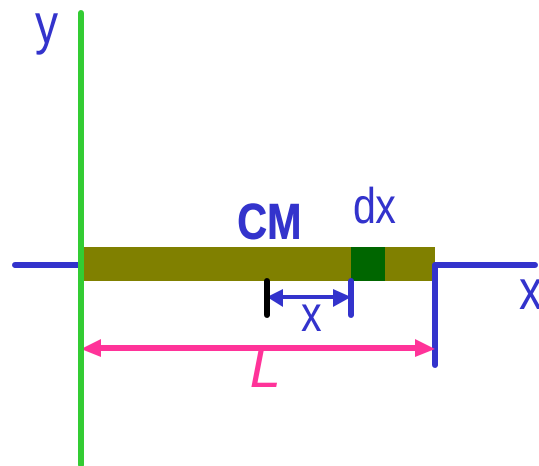
What does this theorem tell you?

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Moment of inertia of any object about any arbitrary axis are the same as the sum of moment of inertia for a rotation about the CM and that of the CM about the rotation axis.

Example for Parallel Axis Theorem

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis that goes through one end of the rod, using parallel-axis theorem.



The line density of the rod is

$$l = \frac{M}{L}$$

so the masslet is

$$dm = l dx = \frac{M}{L} dx$$

The moment of inertia about the CM

$$\begin{aligned} I_{CM} &= \int r^2 dm = \int_{-L/2}^{L/2} \frac{x^2 M}{L} dx = \frac{M}{L} \left[\frac{1}{3} x^3 \right]_{-L/2}^{L/2} \\ &= \frac{M}{3L} \left[\left(\frac{L}{2} \right)^3 - \left(-\frac{L}{2} \right)^3 \right] = \frac{M}{3L} \left(\frac{L^3}{4} \right) = \frac{ML^2}{12} \end{aligned}$$

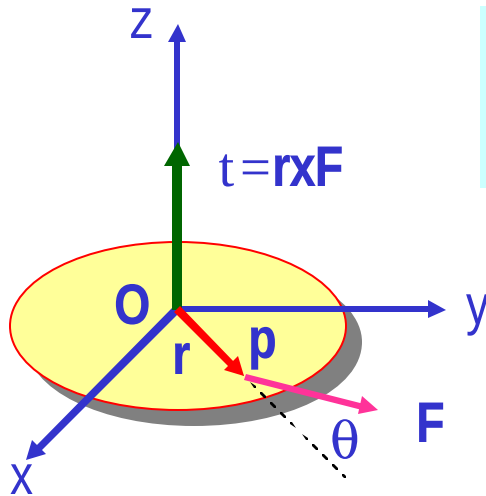
Using the parallel axis theorem

$$I = I_{CM} + D^2 M = \frac{ML^2}{12} + \left(\frac{L}{2} \right)^2 M = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

The result is the same as using the definition of moment of inertia.

Parallel-axis theorem is useful to compute moment of inertia of a rotation of a rigid object with complicated shape about an arbitrary axis

Torque and Vector Product



Let's consider a disk fixed onto the origin O and the force F exerts on the point p. What happens?

The disk will start rotating counter clockwise about the Z axis

The magnitude of torque given to the disk by the force F is

$$t = Fr \sin f$$

But torque is a vector quantity, what is the direction?

How is torque expressed mathematically?

$$\vec{t} \equiv \vec{r} \times \vec{F}$$

What is the direction?

The direction of the torque follows the right-hand rule!!

The above quantity is called
Vector product or Cross product

$$\vec{C} \equiv \vec{A} \times \vec{B}$$

$$|\vec{C}| = |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin q$$

What is the result of a vector product?

Another vector

What is another vector operation we've learned?

Scalar product

$$C \equiv \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos q$$

Result? A scalar

17

Wednesday, Oct. 29, 2003



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Properties of Vector Product

Vector Product is Non-commutative

What does this mean?

If the order of operation changes the result changes

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

Following the right-hand rule, the direction changes

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

Vector Product of two parallel vectors is 0.

$$|\vec{C}| = |\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin \theta \neq |\vec{A}||\vec{B}| \sin 0 = 0$$

Thus,

$$\vec{A} \times \vec{A} = 0$$

If two vectors are perpendicular to each other

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin \theta = |\vec{A}||\vec{B}| \sin 90^\circ = |\vec{A}||\vec{B}| = AB$$

Vector product follows distribution law

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

The derivative of a Vector product with respect to a scalar variable is

$$\frac{d(\vec{A} \times \vec{B})}{dt} = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

More Properties of Vector Product

The relationship between unit vectors, $\vec{i}$, $\vec{j}$ and $\vec{k}$

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$$

$$\vec{i} \times \vec{j} = -\vec{j} \times \vec{i} = \vec{k}$$

$$\vec{j} \times \vec{k} = -\vec{k} \times \vec{j} = \vec{i}$$

$$\vec{k} \times \vec{i} = -\vec{i} \times \vec{k} = \vec{j}$$

Vector product of two vectors can be expressed in the following determinant form

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{i} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \vec{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \vec{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= (A_y B_z - A_z B_y) \vec{i} - (A_x B_z - A_z B_x) \vec{j} + (A_x B_y - A_y B_x) \vec{k}$$



Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Quantities	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = \int r^2 dm$
Length of motion	Distance L	Angle q (Radian)
Speed	$v = \frac{dr}{dt}$	$w = \frac{dq}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$\alpha = \frac{dw}{dt}$
Force	Force $F = ma$	Torque $\tau = Ia$
Work	Work $W = \int_{x_i}^{x_f} F dx$	Work $W = \int_{q_i}^{q_f} \tau dq$
Power	$P = \vec{F} \cdot \vec{v}$	$P = \tau w$
Momentum	$\vec{p} = m \vec{v}$	$\vec{L} = I \vec{w}$
Kinetic Energy	Kinetic $K = \frac{1}{2} mv^2$	Rotational $K_R = \frac{1}{2} I w^2$