

PHYS 1443 – Section 003

Lecture #18

Monday, Nov. 10, 2002

Dr. Jaehoon Yu

1. Torque and Vector Products
2. Rotational Kinetic Energy
3. Work, Power and Energy in Rotation
4. Angular Momentum
5. Angular Momentum and Torque
6. Conservation of Angular Momentum

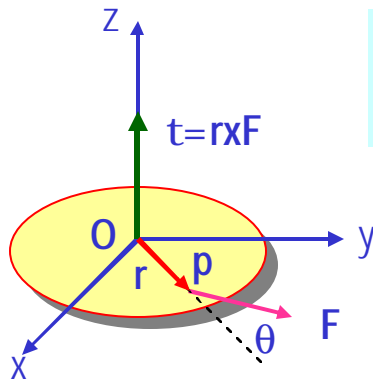


Announcements

- 2nd term exam changes
 - Problem #7, the correct answer is c, 0.6.
 - ➔ Those who answered b and c get credit
 - Problem #21, there is no correct answer
 - ➔ All of you received credit for this problem
- Class average: 52.1 (term 1: 53)
 - The two exams are identical in average



Torque and Vector Product



Let's consider a disk fixed onto the origin O and the force \mathbf{F} exerts on the point p. What happens?

The disk will start rotating counter clockwise about the Z axis

The magnitude of torque given to the disk by the force \mathbf{F} is

$$t = Fr \sin \theta$$

But torque is a vector quantity, what is the direction?
How is torque expressed mathematically?

$$\vec{t} \equiv \vec{r} \times \vec{F}$$

What is the direction?

The direction of the torque follows the right-hand rule!!

The above quantity is called
Vector product or Cross product

$$\vec{C} \equiv \vec{A} \times \vec{B}$$

$$|\vec{C}| = |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta$$

What is the result of a vector product?

Another vector

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What is another vector operation we've learned?

Scalar product

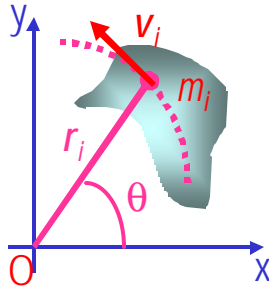
$$C \equiv \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta$$

Result? A scalar



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Rotational Kinetic Energy



What do you think the kinetic energy of a rigid object that is undergoing a circular motion is?

Kinetic energy of a masslet, m_i , moving at a tangential speed, v_i , is $K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 \omega^2$

Since a rigid body is a collection of masslets, the total kinetic energy of the rigid object is

$$K_R = \sum_i K_i = \frac{1}{2} \sum_i m_i r_i^2 \omega^2 = \frac{1}{2} \left(\sum_i m_i r_i^2 \right) \omega^2$$

Since moment of Inertia, I , is defined as

$$I = \sum_i m_i r_i^2$$

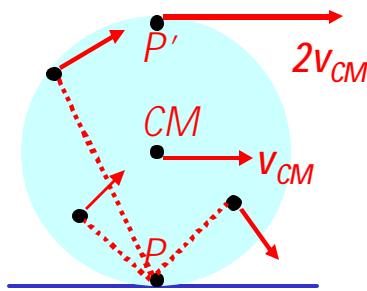
The above expression is simplified as

$$K_R = \frac{1}{2} I \omega^2$$

Total Kinetic Energy of a Rolling Body

What do you think the total kinetic energy of the rolling cylinder is?

Since it is a rotational motion about the point P, we can write the total kinetic energy



$$K = \frac{1}{2} I_P \omega^2$$

Where, I_P , is the moment of inertia about the point P.

Using the parallel axis theorem, we can rewrite

$$K = \frac{1}{2} I_P \omega^2 = \frac{1}{2} (I_{CM} + MR^2) \omega^2 = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} MR^2 \omega^2$$

Since $v_{CM} = R\omega$, the above relationship can be rewritten as

$$K = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M v_{CM}^2$$

What does this equation mean?

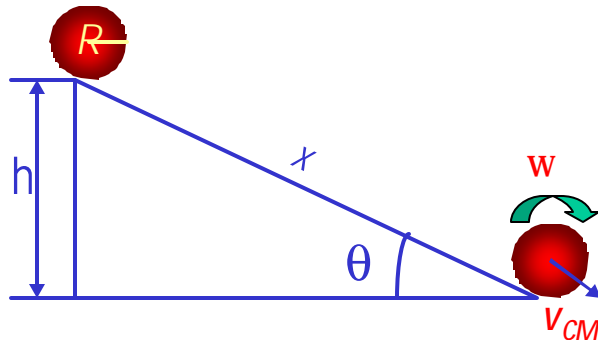
Rotational kinetic energy about the CM

Translational Kinetic energy of the CM

Total kinetic energy of a rolling motion is the sum of the rotational kinetic energy about the CM

And the translational kinetic of the CM

Kinetic Energy of a Rolling Sphere



Let's consider a sphere with radius R rolling down a hill without slipping.

Since $v_{CM} = R\omega$

$$\begin{aligned} K &= \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M R^2 \omega^2 \\ &= \frac{1}{2} I_{CM} \left(\frac{v_{CM}}{R} \right)^2 + \frac{1}{2} M v_{CM}^2 \\ &= \frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}^2 \end{aligned}$$

What is the speed of the CM in terms of known quantities and how do you find this out?

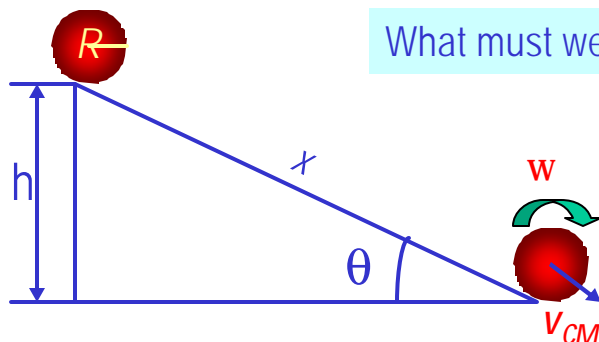
Since the kinetic energy at the bottom of the hill must be equal to the potential energy at the top of the hill

$$K = \frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}^2 = Mgh$$

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM} / MR^2}}$$

Example for Rolling Kinetic Energy

For solid sphere as shown in the figure, calculate the linear speed of the CM at the bottom of the hill and the magnitude of linear acceleration of the CM.



What must we know first?

The moment of inertia the sphere with respect to the CM!!

$$I_{CM} = \int r^2 dm = \frac{2}{5} MR^2$$

Thus using the formula in the previous slide

$$v_{CM} = \sqrt{\frac{2gh}{1 + I_{CM} / MR^2}} = \sqrt{\frac{2gh}{1 + 2/5}} = \sqrt{\frac{10}{7} gh}$$

Since $h = x \sin \theta$,
one obtains

$$v_{CM}^2 = \frac{10}{7} gx \sin \theta$$

Using kinematic
relationship

$$v_{CM}^2 = 2a_{CM}x$$

The linear acceleration
of the CM is

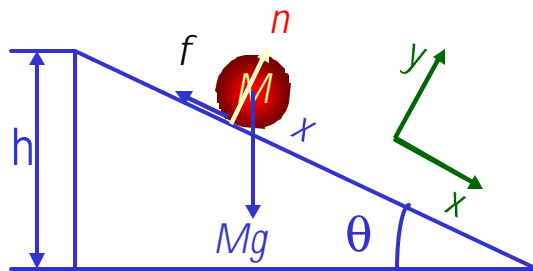
$$a_{CM} = \frac{v_{CM}^2}{2x} = \frac{5}{7} g \sin \theta$$

What do you see?

Linear acceleration of a sphere does
not depend on anything but g and θ .

Example for Rolling Kinetic Energy

For solid sphere as shown in the figure, calculate the linear speed of the CM at the bottom of the hill and the magnitude of linear acceleration of the CM. Solve this problem using Newton's second law, the dynamic method.



What are the forces involved in this motion?

Gravitational Force, Frictional Force, Normal Force

Newton's second law applied to the CM gives

$$\begin{aligned}\sum F_x &= Mg \sin \theta - f = Ma_{CM} \\ \sum F_y &= n - Mg \cos \theta = 0\end{aligned}$$

Since the forces Mg and n go through the CM, their moment arm is 0 and do not contribute to torque, while the static friction f causes torque $\tau_{CM} = fR = I_{CM}a$

We know that

$$I_{CM} = \frac{2}{5}MR^2$$

$$a_{CM} = Ra$$

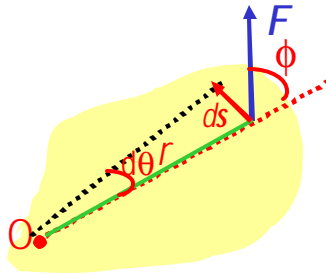
We obtain

Substituting f in dynamic equations

$$f = \frac{I_{CM}a}{R} = \frac{\frac{2}{5}MR^2}{R} \left(\frac{a_{CM}}{R} \right) = \frac{2}{5}Ma_{CM}$$

$$Mg \sin \theta = \frac{7}{5}Ma_{CM} \quad a_{CM} = \frac{5}{7}g \sin \theta$$

Work, Power, and Energy in Rotation



Let's consider a motion of a rigid body with a single external force \mathbf{F} exerting on the point P, moving the object by $d\mathbf{s}$.
The work done by the force \mathbf{F} as the object rotates through the infinitesimal distance $ds=r d\theta$ is

$$dW = \vec{F} \cdot d\vec{s} = (F \sin \phi) r d\theta$$

What is $F \sin \phi$?

The tangential component of force \mathbf{F} .

What is the work done by radial component $F \cos \phi$?

Zero, because it is perpendicular to the displacement.

Since the magnitude of torque is $rF \sin \phi$,

$$dW = (rF \sin \phi) d\theta = \tau d\theta$$

The rate of work, or power becomes

$$P = \frac{dW}{dt} = \frac{\tau d\theta}{dt} = \tau \omega$$

How was the power defined in linear motion?

The rotational work done by an external force equals the change in rotational energy.

$$\sum \tau = I \alpha = I \left(\frac{d\omega}{dt} \right) = I \left(\frac{d\omega}{d\theta} \right) \left(\frac{d\theta}{dt} \right) = I \omega \left(\frac{d\omega}{d\theta} \right)$$

The work put in by the external force then

$$dW = \sum \tau d\theta = I \omega d\omega$$

$$W = \int_{\theta_i}^{\theta_f} \sum \tau d\theta = \int_{\omega_i}^{\omega_f} I \omega d\omega = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2$$

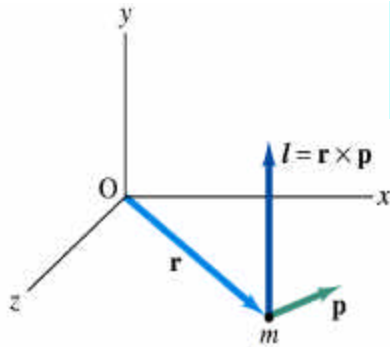
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Angular Momentum of a Particle

If you grab onto a pole while running, your body will rotate about the pole, gaining angular momentum. We've used linear momentum to solve physical problems with linear motions, angular momentum will do the same for rotational motions.



Let's consider a point-like object (particle) with mass m located at the vector location \mathbf{r} and moving with linear velocity \mathbf{v}

The instantaneous angular momentum \mathbf{L} of this particle relative to origin O is

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

What is the unit and dimension of angular momentum? $\text{kg} \cdot \text{m}^2 / \text{s}^2$ [ML^2T^{-2}]

Note that \mathbf{L} depends on origin O. Why? Because \mathbf{r} changes

What else do you learn? The direction of \mathbf{L} is +z

Since \mathbf{p} is $m\mathbf{v}$, the magnitude of \mathbf{L} becomes $L = mvr \sin \theta$

What do you learn from this?

If the direction of linear velocity points to the origin of rotation, the particle does not have any angular momentum.

If the linear velocity is perpendicular to position vector, the particle moves exactly the same way as a point on a rim.

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Angular Momentum and Torque

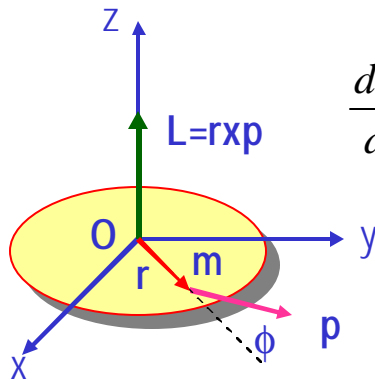
Can you remember how net force exerting on a particle and the change of its linear momentum are related?

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

Total external forces exerting on a particle is the same as the change of its linear momentum.

The same analogy works in rotational motion between torque and angular momentum.

Net torque acting on a particle is $\sum \vec{\tau} = \vec{r} \times \sum \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$



$$\frac{d\vec{L}}{dt} = \frac{d(\vec{r} \times \vec{p})}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{0} + \vec{r} \times \frac{d\vec{p}}{dt}$$

Why does this work?

Because \vec{v} is parallel to the linear momentum

Thus the torque-angular momentum relationship

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt}$$

The net torque acting on a particle is the same as the time rate change of its angular momentum

Angular Momentum of a System of Particles

The total angular momentum of a system of particles about some point is the vector sum of the angular momenta of the individual particles

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \dots + \vec{L}_n = \sum \vec{L}_i$$

Since the individual angular momentum can change, the total angular momentum of the system can change.

Both internal and external forces can provide torque to individual particles.

However, the internal forces do not generate net torque due to Newton's third law.

Let's consider a two particle system where the two exert forces on each other.

Since these forces are action and reaction forces with directions lie on the line connecting the two particles, the vector sum of the torque from these two becomes 0.

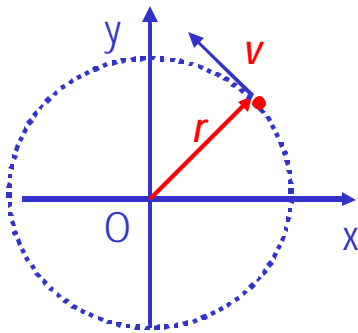
Thus the time rate change of the angular momentum of a system of particles is equal to the net external torque acting on the system

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$



Example for Angular Momentum

A particle of mass m is moving in the xy plane in a circular path of radius r and linear velocity v about the origin O . Find the magnitude and direction of angular momentum with respect to O .



Using the definition of angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m \vec{v} = m \vec{r} \times \vec{v}$$

Since both the vectors, \vec{r} and \vec{v} , are on x - y plane and using right-hand rule, the direction of the angular momentum vector is $+z$ (coming out of the screen)

The magnitude of the angular momentum is $|\vec{L}| = |\vec{r} \times \vec{v}| = mrv \sin \theta = mrv \sin 90^\circ = mrv$

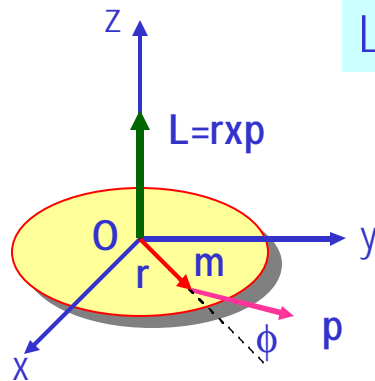
So the angular momentum vector can be expressed as $\vec{L} = mrv \vec{k}$

Find the angular momentum in terms of angular velocity $\vec{\omega}$.

Using the relationship between linear and angular speed

$$\vec{L} = mrv \vec{k} = mr^2 \vec{\omega} \vec{k} = mr^2 \vec{\omega} = I \vec{\omega}$$

Angular Momentum of a Rotating Rigid Body



Let's consider a rigid body rotating about a fixed axis

Each particle of the object rotates in the xy plane about the z -axis at the same angular speed, ω

Magnitude of the angular momentum of a particle of mass m_i about origin O is $m_i v_i r_i$ $L_i = m_i r_i v_i = m_i r_i^2 \omega$

Summing over all particle's angular momentum about z axis

$$L_z = \sum_i L_i = \sum_i (m_i r_i^2 \omega)$$

What do you see?

$$L_z = \sum_i (m_i r_i^2) \omega = I \omega$$

Since I is constant for a rigid body

$$\frac{dL_z}{dt} = I \frac{d\omega}{dt} = I \alpha$$

α is angular acceleration

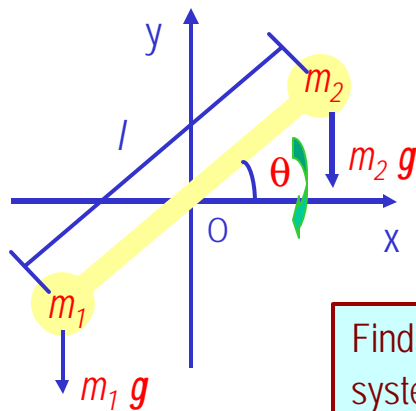
Thus the torque-angular momentum relationship becomes

$$\sum \tau_{ext} = \frac{dL_z}{dt} = I \alpha$$

Thus the net external torque acting on a rigid body rotating about a fixed axis is equal to the moment of inertia about that axis multiplied by the object's angular acceleration with respect to that axis.

Example for Rigid Body Angular Momentum

A rigid rod of mass M and length l pivoted without friction at its center. Two particles of mass m_1 and m_2 are connected to its ends. The combination rotates in a vertical plane with an angular speed of ω . Find an expression for the magnitude of the angular momentum.



The moment of inertia of this system is

$$I = I_{rod} + I_{m_1} + I_{m_2} = \frac{1}{12} M l^2 + \frac{1}{4} m_1 l^2 + \frac{1}{4} m_2 l^2$$

$$= \frac{l^2}{4} \left(\frac{1}{3} M + m_1 + m_2 \right)$$

$$L = I \omega = \frac{\omega l^2}{4} \left(\frac{1}{3} M + m_1 + m_2 \right)$$

Find an expression for the magnitude of the angular acceleration of the system when the rod makes an angle θ with the horizon.

If $m_1 = m_2$, no angular momentum because net torque is 0.

If $\theta = \pm \pi/2$, at equilibrium so no angular momentum.

First compute net external torque

$$\tau_1 = m_1 g \frac{l}{2} \cos \theta \quad \tau_2 = -m_2 g \frac{l}{2} \cos \theta$$

$$\tau_{ext} = \tau_1 + \tau_2 = \frac{gl \cos \theta (m_1 - m_2)}{2}$$

Thus α becomes

$$\alpha = \frac{\sum \tau_{ext}}{I} = \frac{\frac{1}{2} (m_1 - m_2) gl \cos \theta}{\frac{l^2}{4} \left(\frac{1}{3} M + m_1 + m_2 \right)} = \frac{2(m_1 - m_2) \cos \theta}{\left(\frac{1}{3} M + m_1 + m_2 \right)} g / l$$

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Conservation of Angular Momentum

Remember under what condition the linear momentum is conserved?

Linear momentum is conserved when the net external force is 0. $\sum \vec{F} = 0 = \frac{d\vec{p}}{dt}$
 $\vec{p} = \text{const}$

By the same token, the angular momentum of a system is constant in both magnitude and direction, if the resultant external torque acting on the system is 0.

$$\sum \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt} = 0$$
$$\vec{L} = \text{const}$$

What does this mean?

Angular momentum of the system before and after a certain change is the same.

$$\vec{L}_i = \vec{L}_f = \text{constant}$$

Three important conservation laws for isolated system that does not get affected by external forces

$$\begin{cases} K_i + U_i = K_f + U_f & \text{Mechanical Energy} \\ \vec{p}_i = \vec{p}_f & \text{Linear Momentum} \\ \vec{L}_i = \vec{L}_f & \text{Angular Momentum} \end{cases}$$



Example for Angular Momentum Conservation

A star rotates with a period of 30 days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of $1.0 \times 10^4 \text{ km}$, collapses into a neutron star of radius 3.0 km . Determine the period of rotation of the neutron star.

What is your guess about the answer?

The period will be significantly shorter, because its radius got smaller.

Let's make some assumptions:

1. There is no torque acting on it
2. The shape remains spherical
3. Its mass remains constant

Using angular momentum conservation

$$L_i = L_f$$

$$I_i \omega_i = I_f \omega_f$$

$$\omega = \frac{2\pi}{T}$$

The angular speed of the star with the period T is

Thus

$$\omega_f = \frac{I_i \omega_i}{I_f} = \frac{mr_i^2}{mr_f^2} \frac{2\pi}{T_i}$$

$$T_f = \frac{2\pi}{\omega_f} = \left(\frac{r_f^2}{r_i^2} \right) T_i = \left(\frac{3.0}{1.0 \times 10^4} \right)^2 \times 30 \text{ days} = 2.7 \times 10^{-6} \text{ days} = 0.23 \text{ s}$$

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Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Quantities	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = \int r^2 dm$
Length of motion	Distance L	Angle q (Radian)
Speed	$v = \frac{dr}{dt}$	$w = \frac{dq}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$\alpha = \frac{dw}{dt}$
Force	Force $F = ma$	Torque $\tau = I\alpha$
Work	Work $W = \int_{x_i}^{x_f} F dx$	Work $W = \int_{q_i}^{q_f} \tau dq$
Power	$P = \vec{F} \cdot \vec{v}$	$P = \tau w$
Momentum	$\vec{p} = m \vec{v}$	$\vec{L} = I \vec{w}$
Kinetic Energy	Kinetic $K = \frac{1}{2} m v^2$	Rotational $K_R = \frac{1}{2} I w^2$

