PHYS 1443 – Section 003
Lecture #21

Wednesday, Nov. 19, 2003
Dr. Mystery Lecturer

1. Fluid Dymanics : Flow rate and Continuity Equation
2. Bernoulli's Equation
3. Simple Harmonic Motion
4. Simple Block-Spring System
5. Energy of the Simple Harmonic Oscillator

Today's Homework is #11 due on Wednesday, Nov. 26, 2003!!

Next Wednesday's class is cancelled but there will be homework!!

Flow Rate and the Equation of Continuity

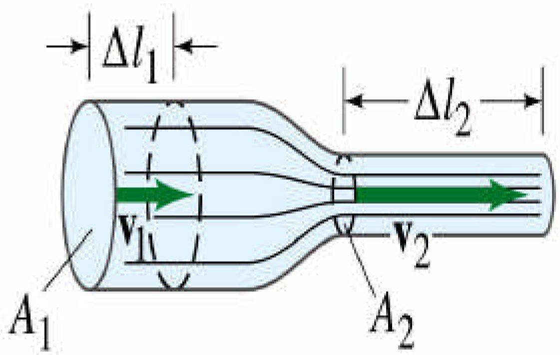
Study of fluid in motion: Fluid Dynamics

If the fluid is water: Water dynamics?? Hydro-dynamics

Two main types of flow

- **Streamline or Laminar flow**: Each particle of the fluid follows a smooth path, a streamline
- **Turbulent flow**: Erratic, small, whirlpool-like circles called eddy current or eddies which absorbs a lot of energy

Flow rate: the mass of fluid that passes a given point per unit time $\Delta m/\Delta t$



$$\frac{\Delta m_1}{\Delta t} = \frac{r_1 \Delta V_1}{\Delta t} = \frac{r_1 A_1 \Delta l_1}{\Delta t} = r_1 A_1 v_1$$

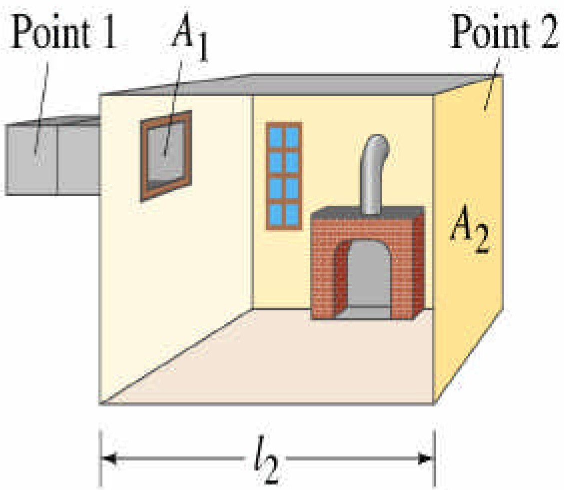
since the total flow must be conserved

$$\frac{\Delta m_1}{\Delta t} = \frac{\Delta m_2}{\Delta t} \Rightarrow r_1 A_1 v_1 = r_2 A_2 v_2$$

Equation of Continuity

Example for Equation of Continuity

How large must a heating duct be if air moving at 3.0m/s along it can replenish the air every 15 minutes in a room of 300m^3 volume? Assume the air's density remains constant.



Using equation of continuity

$$r_1 A_1 v_1 = r_2 A_2 v_2$$

Since the air density is constant

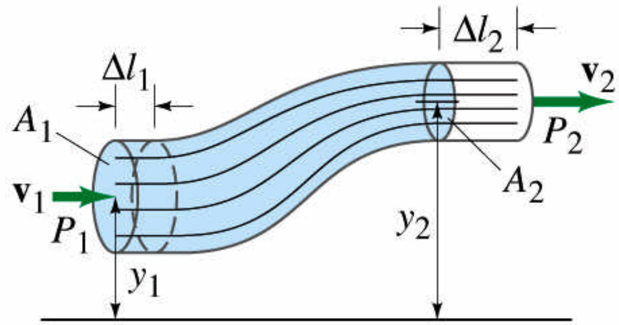
$$A_1 v_1 = A_2 v_2$$

Now let's call the room as the large section of the duct

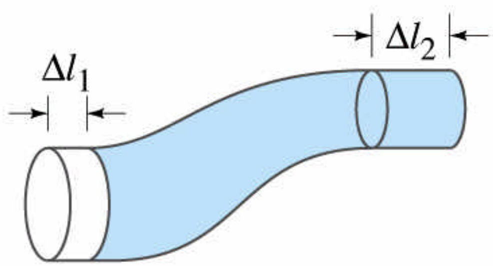
$$A_1 = \frac{A_2 v_2}{v_1} = \frac{A_2 l_2 / t}{v_1} = \frac{V_2}{v_1 \cdot t} = \frac{300}{3.0 \times 900} = 0.11 m^2$$

Bernoulli's Equation

Bernoulli's Principle: Where the velocity of fluid is high, the pressure is low, and where the velocity is low, the pressure is high.



(a)



(b)

Amount of work done by the force, F_1, that exerts pressure, P_1, at point 1

$$W_1 = F_1 \Delta l_1 = P_1 A_1 \Delta l_1$$

Amount of work done on the other section of the fluid is

$$W_2 = -P_2 A_2 \Delta l_2$$

Work done by the gravitational force to move the fluid mass, m, from y_1 to y_2 is

$$W_3 = -mg\left(y_2 - y_1\right)$$

Bernoulli's Equation cont'd

The net work done on the fluid is

$$W = W_1 + W_2 + W_3 \quad = P_1 A_1 \Delta l_1 - P_2 A_2 \Delta l_2 - mgy_2 + mgy_1$$

From the work-energy principle

$$\frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = P_1 A_1 \Delta l_1 - P_2 A_2 \Delta l_2 - mgy_2 + mgy_1$$

Since mass, m, is contained in the volume that flowed in the motion

$$A_1 \Delta l_1 = A_2 \Delta l_2 \qquad \text{and} \qquad m = r A_1 \Delta l_1 = r A_2 \Delta l_2$$

Thus,

$$\frac{1}{2} r A_2 \Delta l_2 v_2^2 - \frac{1}{2} r A_1 \Delta l_1 v_1^2$$

$$= P_1 A_1 \Delta l_1 - P_2 A_2 \Delta l_2 - r A_2 \Delta l_2 g y_2 + r A_1 \Delta l_1 g y_1$$

Bernoulli's Equation cont'd

Since

$$\frac{1}{2} r A_2 \Delta l_2 v_2^2 - \frac{1}{2} r A_1 \Delta l_1 v_1^2 = P_1 A_1 \Delta l_1 - P_2 A_2 \Delta l_2 - r A_2 \Delta l_2 g y_2 + r A_1 \Delta l_1 g y_1$$

We obtain

$$\frac{1}{2} r v_2^2 - \frac{1}{2} r v_1^2 = P_1 - P_2 - r g y_2 + r g y_1$$

Re-organize

$$P_1 + \frac{1}{2} r v_1^2 + r g y_1 = P_2 + \frac{1}{2} r v_2^2 + r g y_2$$

Bernoulli's Equation

Thus, for any two points in the flow

$$P_1 + \frac{1}{2} r v_1^2 + r g y_1 = const.$$

For static fluid

$$P_2 = P_1 + r g \left(y_1 - y_2 \right) = P_1 + r g h$$

Pascal's Law

For the same heights

$$P_2 = P_1 + \frac{1}{2} r \left(v_1^2 - v_2^2 \right)$$

The pressure at the faster section of the fluid is smaller than slower section.

Example for Bernoulli's Equation

Water circulates throughout a house in a hot-water heating system. If the water is pumped at a speed of 0.5m/s through a 4.0cm diameter pipe in the basement under a pressure of 3.0atm, what will be the flow speed and pressure in a 2.6cm diameter pipe on the second 5.0m above? Assume the pipes do not divide into branches.

Using the equation of continuity, flow speed on the second floor is

$$v_2 = \frac{A_1 v_1}{A_2} = \frac{p\, r_1^{\,2} v_1}{p\, r_2^{\,2}} = 0.5 \times \left(\frac{0.020}{0.013}\right)^2 = 1.2 m/s$$

Using Bernoulli's equation, the pressure in the pipe on the second floor is

$$P_2 = P_1 + \frac{1}{2} r \left(v_1^2 - v_2^2\right) + r\, g \left(y_1 - y_2\right)$$
$$= 3.0\times 10^5 + \frac{1}{2} 1\times 10^3 \left(0.5^2 - 1.2^2\right) + 1\times 10^3 \times 9.8 \times (-5)$$
$$= 2.5\times 10^5\, N/m^2$$

Simple Harmonic Motion

What do you think a harmonic motion is?

Motion that occurs by the force that depends on displacement, and the force is always directed toward the system's equilibrium position.

What is a system that has such characteristics?

A system consists of a mass and a spring

When a spring is stretched from its equilibrium position by a length x, the force acting on the mass is

$$F = -kx$$

It's negative, because the force resists against the change of length, directed toward the equilibrium position.

From Newton's second law $F = ma = -kx$ we obtain $a = -\frac{k}{m}x$

This is a second order differential equation that can be solved but it is beyond the scope of this class.

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Condition for simple harmonic motion

What do you observe from this equation?

Acceleration is proportional to displacement from the equilibrium

Acceleration is opposite direction to displacement

Equation of Simple Harmonic Motion

The solution for the 2nd order differential equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$x = A\cos(wt + f)$$

Amplitude

Phase

Angular Frequency

Phase constant

Generalized expression of a simple harmonic motion

Let's think about the meaning of this equation of motion

What happens when t=0 and ϕ=0?

$$x = A\cos(0+0) = A$$

What is ϕ if x is not A at t=0?

$$x = A\cos(f) = x'$$

$$f = \cos^{-1}(x')$$

What are the maximum/minimum possible values of x?

A/-A

An oscillation is fully characterized by its:

- Amplitude
- Period or frequency
- Phase constant

More on Equation of Simple Harmonic Motion

What is the time for full cycle of oscillation?

Since after a full cycle the position must be the same

$$x = A\cos\left(w\left(t+T\right)+f\right) = A\cos\left(wt+2p+f\right)$$

The period $T = \frac{2p}{w}$ One of the properties of an oscillatory motion

How many full cycles of oscillation does this undergo per unit time?

$f = \frac{1}{T} = \frac{w}{2p}$ Frequency

What is the unit?

1/s=Hz

Let's now think about the object's speed and acceleration. $x = A\cos\left(wt+f\right)$

Speed at any given time $v = \frac{dx}{dt} = -wA\sin\left(wt+f\right)$ Max speed $v_{max} = wA$

Acceleration at any given time $a = \frac{dv}{dt} = -w^2 A\cos\left(wt+f\right) = -w^2 x$

Max acceleration $a_{max} = w^2 A$

What do we learn about acceleration?

Acceleration is reverse direction to displacement

Acceleration and speed are π/2 off phase:

When *v* is maximum, *a* is at its minimum

Simple Harmonic Motion continued

Phase constant determines the starting position of a simple harmonic motion.

$$x = A\cos(wt + f)$$ At t=0 $$x\big|_{t=0} = A\cos f$$

This constant is important when there are more than one harmonic oscillation involved in the motion and to determine the overall effect of the composite motion

Let's determine phase constant and amplitude

At t=0

$$x_i = A\cos f \qquad v_i = -wA\sin f$$

By taking the ratio, one can obtain the phase constant

$$\boxed{f = \tan^{-1}\left(-\frac{v_i}{wx_i}\right)}$$

By squaring the two equation and adding them together, one can obtain the amplitude

$$x_i^2 = A^2\cos^2 f$$

$$v_i^2 = w^2 A^2 \sin^2 f$$

$$A^2\left(\cos^2 f + \sin^2 f\right) = A^2 = x_i^2 + \left(\frac{v_i}{w}\right)^2$$

$$\boxed{A = \sqrt{x_i^2 + \left(\frac{v_i}{w}\right)^2}}$$

Example for Simple Harmonic Motion

An object oscillates with simple harmonic motion along the x-axis. Its displacement from the origin varies with time according to the equation; $x = (4.00m)\cos\left(pt + \frac{p}{4}\right)$ where t is in seconds and the angles is in the parentheses are in radians. a) Determine the amplitude, frequency, and period of the motion.

From the equation of motion: $x = A\cos(wt + f) = (4.00m)\cos\left(pt + \frac{p}{4}\right)$

The amplitude, A, is $A = 4.00m$ The angular frequency, ω, is $w = p$

Therefore, frequency and period are $T = \frac{2p}{w} = \frac{2p}{p} = 2s$ $f = \frac{1}{T} = \frac{w}{2p} = \frac{p}{2p} = \frac{1}{2}s^{-1}$

b)Calculate the velocity and acceleration of the object at any time t.

Taking the first derivative on the equation of motion, the velocity is

$$v = \frac{dx}{dt} = -(4.00 \times p)\sin\left(pt + \frac{p}{4}\right)m/s$$

By the same token, taking the second derivative of equation of motion, the acceleration, a, is

$$a = \frac{d^2x}{dt^2} = -(4.00 \times p^2)\cos\left(pt + \frac{p}{4}\right)m/s^2$$

Simple Block-Spring System

A block attached at the end of a spring on a frictionless surface experiences acceleration when the spring is displaced from an equilibrium position.

$$a = -\frac{k}{m}x$$

This becomes a second order differential equation $\frac{d^2x}{dt^2} = -\frac{k}{m}x$ If we denote $w^2 = \frac{k}{m}$

The resulting differential equation becomes

$$\frac{d^2x}{dt^2} = -w^2 x$$

Since this satisfies condition for simple harmonic motion, we can take the solution

$$\boxed{x = A\cos(wt + f)}$$

Does this solution satisfy the differential equation?

Let's take derivatives with respect to time $\frac{dx}{dt} = A\frac{d}{dt}(\cos(wt + f)) = -wA\sin(wt + f)$

Now the second order derivative becomes

$$\frac{d^2x}{dt^2} = -wA\frac{d}{dt}(\sin(wt + f)) = -w^2 A\cos(wt + f) = -w^2 x$$

Whenever the force acting on a particle is linearly proportional to the displacement from some equilibrium position and is in the opposite direction, the particle moves in simple harmonic motion.

More Simple Block-Spring System

How do the period and frequency of this harmonic motion look?

Since the angular frequency ω is $w = \sqrt{\frac{k}{m}}$

The period, T, becomes $T = \frac{2p}{w} = 2p\sqrt{\frac{m}{k}}$

So the frequency is $f = \frac{1}{T} = \frac{w}{2p} = \frac{1}{2p}\sqrt{\frac{k}{m}}$

What can we learn from these?

- Frequency and period do not depend on amplitude
- Period is inversely proportional to spring constant and proportional to mass

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Special case #1

Let's consider that the spring is stretched to distance A and th e block is let go from rest, giving 0 initial speed; x_i=A, v_i=0,

$$x = \mathrm{A}\cos wt \quad v = \frac{dx}{dt} = -w\mathrm{A}\sin wt \quad a = \frac{d^2x}{dt^2} = -w^2\mathrm{A}\cos wt \quad a_i = -w^2\mathrm{A} = -kA/m$$

This equation of motion satisfies all the conditions. So it is the solution for this motion.

Special case #2

Suppose block is given non-zero initial velocity v_i to positive x at the instant it is at the equilibrium, x_i=0

$$f = \tan^{-1}\left(-\frac{v_i}{wx_i}\right) = \tan^{-1}(-\infty) = -\frac{p}{2} \quad x = \mathrm{A}\cos\left(wt - \frac{p}{2}\right) = A\sin(wt)$$

Is this a good solution?