PHYS 1443 – Section 003
Lecture #22

Monday, Nov. 24, 2003
Dr. Jaehoon Yu

1. Simple Block-Spring System
2. Energy of the Simple Harmonic Oscillator
3. Pendulum
   - Simple Pendulum
   - Physical Pendulum
   - Torsion Pendulum
4. Simple Harmonic Motion and Uniform Circular Motion
5. Damped Oscillation

Announcements

- Evaluation today
- Quiz
  - Average: 3.5/6
  - Marked improvements! Keep up the good work!!
- Homework # 12
  - Will be posted tomorrow
  - Due at noon, Wednesday, Dec. 3
- The final exam
  - On Monday, Dec. 8, in the class tentatively
  - Covers: Chap. 10 not covered in Term #2 – whatever we get at by Dec. 3 (chapter 15??)
- Need to talk to me? I will be around tomorrow. Come by my office! But please call me first!

No class Wednesday! Have a safe and happy Thanksgiving!

Simple Block-Spring System

A block attached at the end of a spring on a frictionless surface experiences acceleration when the spring is displaced from an equilibrium position.

$$F_{spring} = ma = -kx$$

$$a = -\frac{k}{m}x$$

This becomes a second order differential equation $\frac{d^2x}{dt^2} = -\frac{k}{m}x$ If we denote $\omega^2 = \frac{k}{m}$

The resulting differential equation becomes

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Since this satisfies condition for simple harmonic motion, we can take the solution

$$\boxed{x = A\cos(\omega t + \phi)}$$

Does this solution satisfy the differential equation?

Let's take derivatives with respect to time $\frac{dx}{dt} = A\frac{d}{dt}(\cos(\omega t + \phi)) = -\omega A\sin(\omega t + \phi)$

Now the second order derivative becomes

$$\frac{d^2x}{dt^2} = -\omega A\frac{d}{dt}(\sin(\omega t + \phi)) = -\omega^2 A\cos(\omega t + \phi) = -\omega^2 x$$

Whenever the force acting on a particle is linearly proportional to the displacement from some equilibrium position and is in the opposite direction, the particle moves in simple harmonic motion.

More Simple Block-Spring System

How do the period and frequency of this harmonic motion look?

Since the angular frequency ω is $\omega = \sqrt{\dfrac{k}{m}}$

The period, T, becomes $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{m}{k}}$

So the frequency is $f = \dfrac{1}{T} = \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$

What can we learn from these?

- Frequency and period do not depend on amplitude
- Period is inversely proportional to spring constant and proportional to mass

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Special case #1

Let's consider that the spring is stretched to distance A and the block is let go from rest, giving 0 initial speed; x_i=A, v_i=0,

$$x = A\cos\omega t \quad v = \frac{dx}{dt} = -\omega A\sin\omega t \quad a = \frac{d^2x}{dt^2} = -\omega^2 A\cos\omega t \quad a_i = -\omega^2 A = -kA/m$$

This equation of motion satisfies all the conditions. So it is the solution for this motion.

Special case #2

Suppose block is given non-zero initial velocity v_i to positive x at the instant it is at the equilibrium, x_i=0

$$\phi = \tan^{-1}\left(-\frac{v_i}{\omega x_i}\right) = \tan^{-1}(-\infty) = -\frac{\pi}{2} \quad x = A\cos\left(\omega t - \frac{\pi}{2}\right) = A\sin(\omega t)$$

Is this a good solution?

Example for Spring Block System

A car with a mass of 1300kg is constructed so that its frame is supported by four springs. Each spring has a force constant of 20,000N/m. If two people riding in the car have a combined mass of 160kg, find the frequency of vibration of the car after it is driven over a pothole in the road.

Let's assume that mass is evenly distributed to all four springs.

The total mass of the system is 1460kg.

Therefore each spring supports 365kg each.

From the frequency relationship based on Hook's law

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Thus the frequency for vibration of each spring is

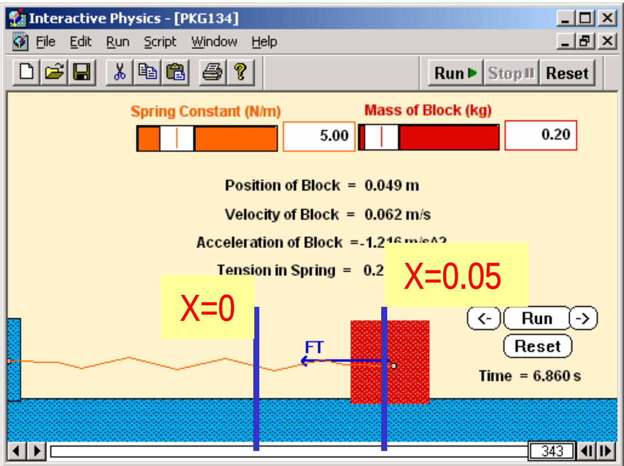
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{20000}{365}} = 1.18\,s^{-1} = 1.18\,Hz$$

How long does it take for the car to complete two full vibrations?

The period is $T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} = 0.849\,s$ For two cycles $2T = 1.70\,s$

Example for Spring Block System

A block with a mass of 200g is connected to a light spring for which the force constant is 5.00 N/m and is free to oscillate on a horizontal, frictionless surface. The block is displaced 5.00 cm from equilibrium and released from reset. Find the period of its motion.



From the Hook's law, we obtain

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.00}{0.20}} = 5.00\ s^{-1}$$

As we know, period does not depend on the amplitude or phase constant of the oscillation, therefore the period, T, is simply

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5.00} = 1.26\ s$$

Determine the maximum speed of the block.

From the general expression of the simple harmonic motion, the speed is

$$v_{\max} = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$= \omega A = 5.00 \times 0.05 = 0.25 m/s$$

Energy of the Simple Harmonic Oscillator

How do you think the mechanical energy of the harmonic oscillator look without friction?

Kinetic energy of a harmonic oscillator is

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$$

The elastic potential energy stored in the spring

$$PE = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$$

Therefore the total mechanical energy of the harmonic oscillator is

$$E = KE + PE = \frac{1}{2}\left[m\omega^2 A^2 \sin^2(\omega t + \phi) + kA^2 \cos^2(\omega t + \phi)\right]$$

Since $\omega = \sqrt{k/m}$

$$E = KE + PE = \frac{1}{2}\left[kA^2 \sin^2(\omega t + \phi) + kA^2 \cos^2(\omega t + \phi)\right] = \frac{1}{2}kA^2$$

Total mechanical energy of a simple harmonic oscillator is a constant of a motion and is proportional to the square of the amplitude

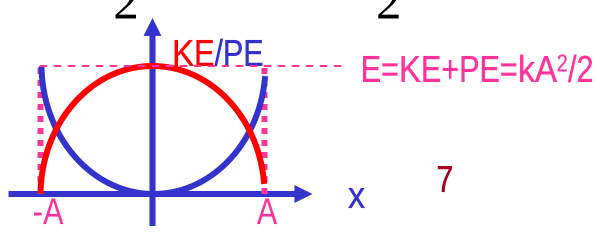
Maximum KE is when PE=0

$$KE_{\max} = \frac{1}{2}mv^2_{\max} = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) = \frac{1}{2}m\omega^2 A^2 = \frac{1}{2}kA^2$$

One can obtain speed

$$E = KE + PE = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

$$v = \pm\sqrt{k/m\left(A^2 - x^2\right)} = \pm\omega\sqrt{A^2 - x^2}$$



Example for Energy of Simple Harmonic Oscillator

A 0.500kg cube connected to a light spring for which the force constant is 20.0 N/m oscillates on a horizontal, frictionless track. a) Calculate the total energy of the system and the maximum speed of the cube if the amplitude of the motion is 3.00 cm.

From the problem statement, A and k are

$$k = 20.0N/m$$
$$A = 3.00cm = 0.03m$$

The total energy of the cube is

$$E = KE + PE = \frac{1}{2}kA^2 = \frac{1}{2}(20.0)\times(0.03)^2 = 9.00\times10^{-3}\,J$$

Maximum speed occurs when kinetic energy is the same as the total energy

$$KE_{\max} = \frac{1}{2}mv_{\max}^2 = E = \frac{1}{2}kA^2$$
$$v_{\max} = A\sqrt{\frac{k}{m}} = 0.03\sqrt{\frac{20.0}{0.500}} = 0.190m/s$$

b) What is the velocity of the cube when the displacement is 2.00 cm.

velocity at any given displacement is

$$v = \sqrt{k/m\left(A^2 - x^2\right)} = \sqrt{20.0\cdot\left(0.03^2 - 0.02^2\right)/0.500} = 0.141m/s$$

c) Compute the kinetic and potential energies of the system when the displacement is 2.00 cm.

Kinetic energy, KE

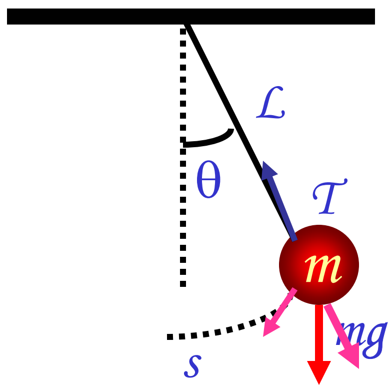
$$KE = \frac{1}{2}mv^2 = \frac{1}{2}0.500\times(0.141)^2 = 4.97\times10^{-3}\,J$$

Potential energy, PE

$$PE = \frac{1}{2}kx^2 = \frac{1}{2}20.0\times(0.02)^2 = 4.00\times10^{-3}\,J$$

The Pendulum

A simple pendulum also performs periodic motion.



The net force exerted on the bob is

$$\sum F_r = T - mg\cos\theta_A = 0$$

$$\sum F_t = -mg\sin\theta_A = ma = m\frac{d^2 s}{dt^2}$$

Since the arc length, s, is $s = L\theta$

$$\frac{d^2 s}{dt^2} = L\frac{d^2\theta}{dt^2} = -g\sin\theta \quad \text{results} \Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Again became a second degree differential equation, satisfying conditions for simple harmonic motion

If θ is very small, sinθ~θ $\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta = -\omega^2\theta$ giving angular frequency $\omega = \sqrt{\frac{g}{L}}$

The period for this motion is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$

The period only depends on the length of the string and the gravitational acceleration

Example for Pendulum

Christian Huygens (1629-1695), the greatest clock maker in history, suggested that an international unit of length could be defined as the length of a simple pendulum having a period of exactly 1s. How much shorter would out length unit be had this suggestion been followed?

Since the period of a simple pendulum motion is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

The length of the pendulum in terms of T is

$$L = \frac{T^2 g}{4\pi^2}$$

Thus the length of the pendulum when T=1s is

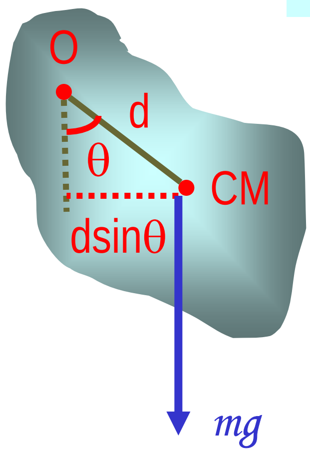
$$L = \frac{T^2 g}{4\pi^2} = \frac{1 \times 9.8}{4\pi^2} = 0.248\,m$$

Therefore the difference in length with respect to the current definition of 1m is

$$\Delta L = 1 - L = 1 - 0.248 = 0.752\,m$$

Physical Pendulum

Physical pendulum is an object that oscillates about a fixed axis which does not go through the object's center of mass.



Consider a rigid body pivoted at a point O that is a distance d from the CM.

The magnitude of the net torque provided by the gravity is

$$\sum \tau = -mgd \sin\theta$$

Then $$\sum \tau = I\alpha = I\frac{d^2\theta}{dt^2} = -mgd \sin\theta$$

Therefore, one can rewrite $$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\sin\theta \approx -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta$$

Thus, the angular frequency ω is $$\omega = \sqrt{\frac{mgd}{I}}$$

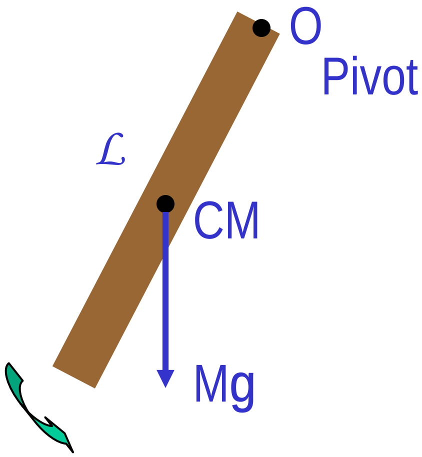
And the period for this motion is $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

By measuring the period of physical pendulum, one can measure moment of inertia.

Does this work for simple pendulum?

Example for Physical Pendulum

A uniform rod of mass M and length L is pivoted about one end and oscillates in a vertical plane. Find the period of oscillation if the amplitude of the motion is small.



Moment of inertia of a uniform rod, rotating about the axis at one end is $I = \frac{1}{3}ML^2$

The distance d from the pivot to the CM is L/2, therefore the period of this physical pendulum is

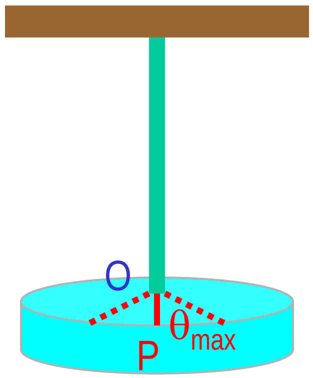
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{2ML^2}{3MgL}} = 2\pi\sqrt{\frac{2L}{3g}}$$

Calculate the period of a meter stick that is pivot about one end and is oscillating in a vertical plane.

Since L=1m, the period is $T = 2\pi\sqrt{\frac{2L}{3g}} = 2\pi\sqrt{\frac{2}{3\cdot 9.8}} = 1.64s$

So the frequency is $f = \frac{1}{T} = 0.61s^{-1}$

Torsion Pendulum



When a rigid body is suspended by a wire to a fixed support at the top and the body is twisted through some small angle θ, the twisted wire can exert a restoring torque on the body that is proportional to the angular displacement.

The torque acting on the body due to the wire is

$$\tau = -\kappa\theta$$

κ is the torsion constant of the wire

Applying the Newton's second law of rotational motion

$$\sum \tau = I\alpha = I\frac{d^2\theta}{dt^2} = -\kappa\theta$$

Then, again the equation becomes

$$\frac{d^2\theta}{dt^2} = -\left(\frac{\kappa}{I}\right)\theta = -\omega^2\theta$$

Thus, the angular frequency ω is

$$\omega = \sqrt{\frac{\kappa}{I}}$$

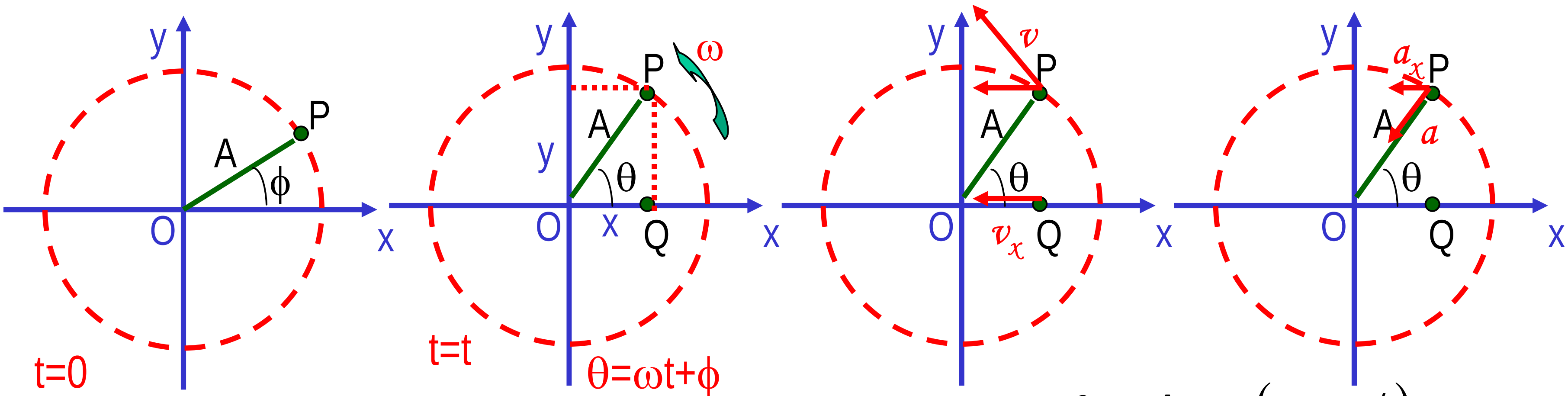
And the period for this motion is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\kappa}}$$

This result works as long as the elastic limit of the wire is not exceeded

Simple Harmonic and Uniform Circular Motions

Uniform circular motion can be understood as a superposition of two simple harmonic motions in x and y axis.



When the particle rotates at a uniform angular speed ω, x and y coordinate position become

$$x = A\cos\theta = A\cos(\omega t + \phi)$$

$$y = A\sin\theta = A\sin(\omega t + \phi)$$

Since the linear velocity in a uniform circular motion is Aω, the velocity components are

$$v_x = -v\sin\theta = -A\omega\sin(\omega t + \phi)$$

$$v_y = +v\cos\theta = A\omega\cos(\omega t + \phi)$$

Since the radial acceleration in a uniform circular motion is $v^2/A = \omega^2 A$, the components are

$$a_x = -a\cos\theta = -A\omega^2\cos(\omega t + \phi)$$

$$a_y = -a\sin\theta = -A\omega^2\sin(\omega t + \phi)$$

Example for Uniform Circular Motion

A particle rotates counterclockwise in a circle of radius 3.00m with a constant angular speed of 8.00 rad/s. At t=0, the particle has an x coordinate of 2.00m and is moving to the right. A) Determine the x coordinate as a function of time.

Since the radius is 3.00m, the amplitude of oscillation in x direction is 3.00m. And the angular frequency is 8.00rad/s. Therefore the equation of motion in x direction is

$$x = A\cos\theta = (3.00m)\cos(8.00t + \phi)$$

Since x=2.00, when t=0 $\quad 2.00 = (3.00m)\cos\phi; \quad \phi = \cos^{-1}\left(\frac{2.00}{3.00}\right) = 48.2^\circ$

However, since the particle was moving to the right ϕ=-48.2°,

$$x = (3.00m)\cos(8.00t - 48.2^\circ)$$

Find the x components of the particle's velocity and acceleration at any time t.

Using the displcement

$$v_x = \frac{dx}{dt} = -(3.00 \cdot 8.00)\sin(8.00t - 48.2) = (-24.0m/s)\sin(8.00t - 48.2^\circ)$$

Likewise, from velocity

$$a_x = \frac{dv}{dt} = (-24.0 \cdot 8.00)\cos(8.00t - 48.2) = (-192m/s^2)\cos(8.00t - 48.2^\circ)$$