PHYS 1443 – Section 003
Lecture #4

Wednesday, Sept. 1, 2004
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1. One Dimensional Motion
   - Acceleration
   - Motion under constant acceleration
   - Free Fall
2. Motion in Two Dimensions
   - Vector Properties and Operations
   - Motion under constant acceleration
   - Projectile Motion

Displacement, Velocity and Speed

Displacement $$\boxed{\Delta x \equiv x_f - x_i}$$

Average velocity $$\boxed{v_x \equiv \frac{x_f - x_i}{t_f - t_i} = \frac{\Delta x}{\Delta t}}$$

Average speed $$\boxed{v \equiv \frac{\text{Total Distance Traveled}}{\text{Total Time Spent}}}$$

Instantaneous velocity $$\boxed{v_x = \lim_{\text{Ät} \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}}$$

Instantaneous speed $$\boxed{\left|v_x\right| = \left|\lim_{\text{Ät} \to 0} \frac{\Delta x}{\Delta t}\right| = \left|\frac{dx}{dt}\right|}$$

Acceleration

Change of velocity in time (what kind of quantity is this?)

- Average acceleration:

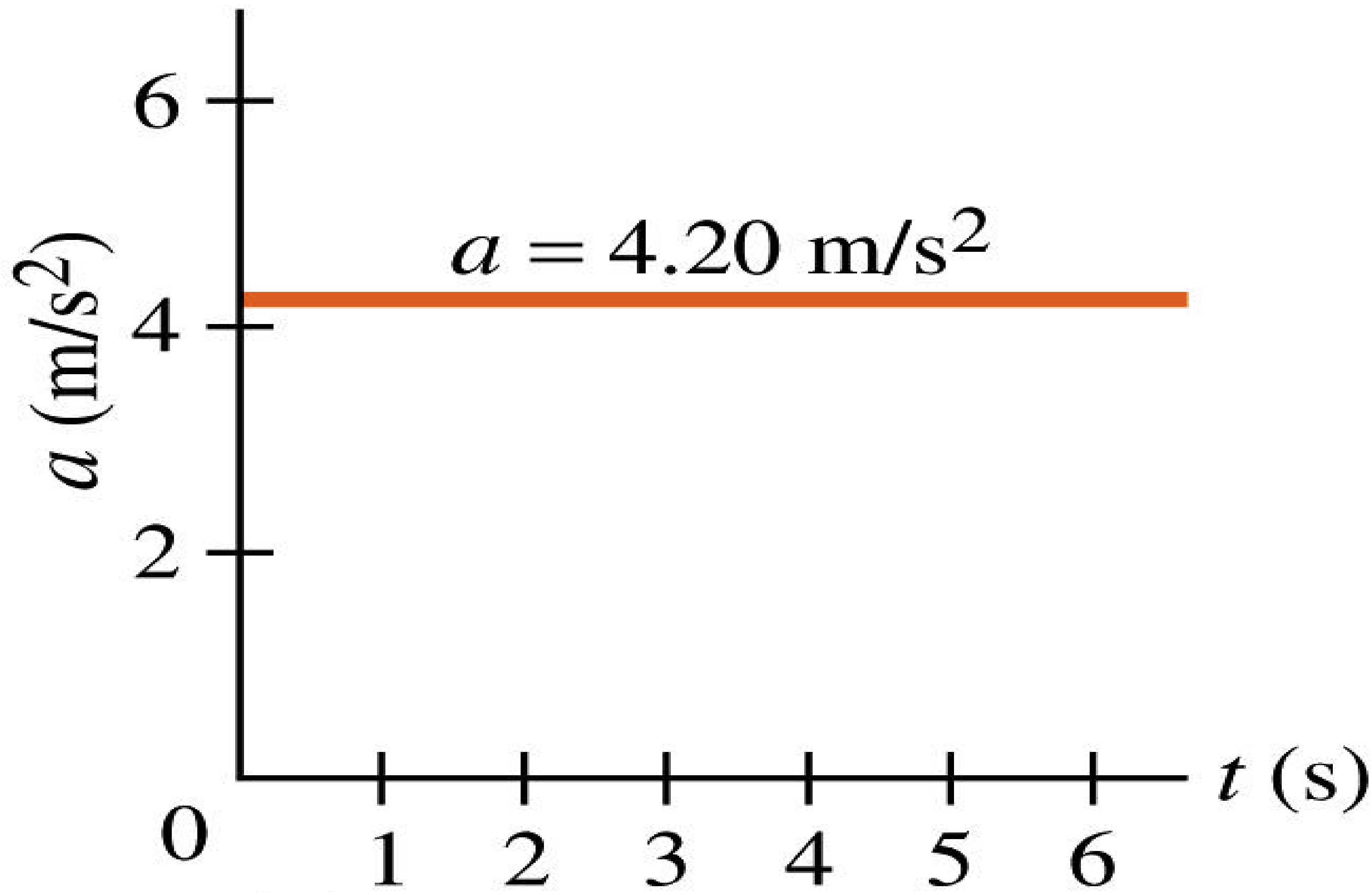
$$a_x \equiv \frac{v_{xf} - v_{xi}}{t_f - t_i} = \frac{\Delta v_x}{\Delta t} \quad \text{analogs to} \quad v_x \equiv \frac{x_f - x_i}{t_f - t_i} = \frac{\Delta x}{\Delta t}$$

- Instantaneous acceleration:

$$a_x \equiv \lim_{\text{Ät} \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \quad \text{analogs to} \quad \boxed{v_x \equiv \lim_{\text{Ät} \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}}$$

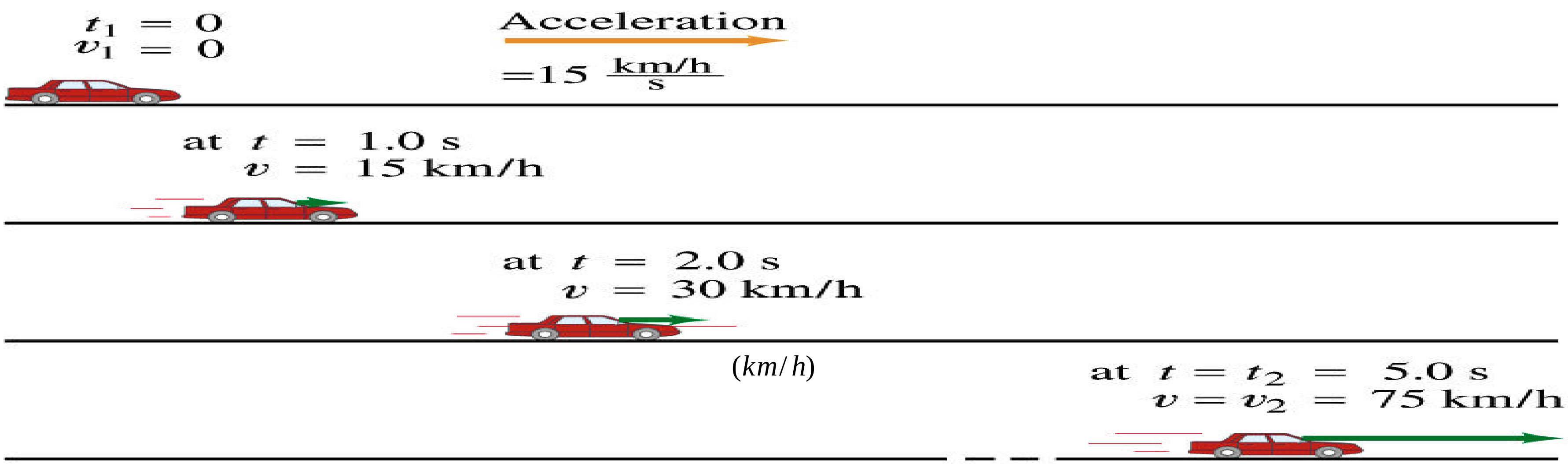
- In calculus terms: A slope (derivative) of velocity with respect to time or change of slopes of position as a function of time

Acceleration vs Time Plot



Example 2.4

A car accelerates along a straight road from rest to 75km/h in 5.0s.



What is the magnitude of its average acceleration?

$$v_{xi} = 0\ m/s$$

$$v_{xf} = \frac{75000m}{3600s} = 21\ m/s$$

$$\bar{a}_x = \frac{v_{xf} - v_{xi}}{t_f - t_i} = \frac{\Delta v_x}{\Delta t} = \frac{21-0}{5.0} = \frac{21}{5.0} = 4.2(m/s^2)$$

$$= \frac{4.2\times(3600)^2}{1000} = 5.4\times10^4(km/h^2)$$

Example for Acceleration

- Velocity, v_x, is express in: $v_x(t) = \left(40 - 5t^2\right) m/s$
- Find average acceleration in time interval, t=0 to t=2.0s

$$v_{xi}(t_i = 0) = 40(m/s)$$

$$v_{xf}(t_f = 2.0) = \left(40 - 5 \times 2.0^2\right) = 20(m/s)$$

$$a_x = \frac{v_{xf} - v_{xi}}{t_f - t_i} = \frac{\Delta v_x}{\Delta t} = \frac{20 - 40}{2.0 - 0} = -10(m/s^2)$$

- Find instantaneous acceleration at any time t and t=2.0s

Instantaneous Acceleration at any time

$$a_x(t) \equiv \frac{dv_x}{dt} = \frac{d}{dt}\left(40 - 5t^2\right) = -10t$$

Instantaneous Acceleration at any time t=2.0s

$$a_x(t = 2.0) = -10 \times (2.0) = -20(m/s^2)$$

Meanings of Acceleration

- When an object is moving in a constant velocity ($v=v_0$), there is no acceleration ($a=0$)
  - Is there net acceleration when an object is not moving? **No!**
- When an object is moving faster as time goes on, ($v=v(t)$), acceleration is positive ($a>0$)
  - Incorrect, since the object might be moving in negative direction initially
- When an object is moving slower as time goes on, ($v=v(t)$), acceleration is negative ($a<0$)
  - Incorrect, since the object might be moving in negative direction initially
- In all cases, velocity is positive, unless the direction of the movement changes.
  - Incorrect, since the object might be moving in negative direction initially
- Is there acceleration if an object moves in a constant speed but changes direction?

The answer is YES!!

Example 2.7

A particle is moving on a straight line so that its position as a function of time is given by the equation $x=(2.10m/s^2)t^2+2.8m$.

(a) Compute the average acceleration during the time interval from t_1=3.00s to t_2=5.00s.

$$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \frac{d}{dt}\left(2.10t^2 + 2.80\right) = 4.20t$$

$$v_{xi} = v_x\left(t = 3.00s\right) = 4.20 \times 3.00 = 12.6\left(m/s\right)$$

$$v_{xf} = v_x\left(t = 5.00s\right) = 4.20 \times 5.00 = 21.0\left(m/s\right)$$

$$\bar{a}_x = \frac{\Delta v_x}{\Delta t} = \frac{21.0 - 12.6}{5.00 - 3.00} = \frac{8.40}{2.00} = 4.2\left(m/s^2\right)$$

(b) Compute the particle's instantaneous acceleration as a function of time.

$$a_x \equiv \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt} = \frac{d}{dt}\left(4.20t\right) = 4.20\left(m/s^2\right)$$

What does this mean?

The acceleration of this particle is independent of time.

This particle is moving under a constant acceleration.

One Dimensional Motion

- Let's start with the simplest case: acceleration is a constant ($a=a_0$)
- Using definitions of average acceleration and velocity, we can derive equations of motion (description of motion, velocity and position as a function of time)

$$\bar{a}_x = a_x = \frac{v_{xf} - v_{xi}}{t_f - t_i}$$ (If $t_f=t$ and $t_i=0$) $$a_x = \frac{v_{xf} - v_{xi}}{t} \Rightarrow v_{xf} = v_{xi} + a_x t$$

For constant acceleration, average velocity is a simple numeric average

$$\bar{v}_x = \frac{v_{xi} + v_{xf}}{2} = \frac{2v_{xi} + a_x t}{2} = v_{xi} + \frac{1}{2} a_x t$$

$$\bar{v}_x = \frac{x_f - x_i}{t_f - t_i}$$ (If $t_f=t$ and $t_i=0$) $$\bar{v}_x = \frac{x_f - x_i}{t} \Rightarrow x_f = x_i + \bar{v}_x t$$

Resulting Equation of Motion becomes

$$x_f = x_i + \bar{v}_x t = x_i + v_{xi} t + \frac{1}{2} a_x t^2$$

One Dimensional Motion cont'd

Average velocity $\bar{v}_x = \frac{v_{xi} + v_{xf}}{2}$ $\quad x_f = x_i + \bar{v}_x t = x_i + \left(\frac{v_{xi} + v_{xf}}{2}\right)t$

Since $a_x = \frac{v_{xf} - v_{xi}}{t}$ Solving for t $t = \frac{v_{xf} - x_{xi}}{a_x}$

Substituting t in the above equation,

$$x_f = x_i + \left(\frac{v_{xf} + v_{xi}}{2}\right)\left(\frac{v_{xf} - v_{xi}}{a_x}\right) = x_i + \frac{v_{xf}^2 - v_{xi}^2}{2a_x}$$

Resulting in

$$v_{xf}^2 = v_{xi}^2 + 2a_x\left(x_f - x_i\right)$$

Kinematic Equations of Motion on a Straight Line Under Constant Acceleration

$$\boxed{v_{xf}(t) = v_{xi} + a_x t}$$

Velocity as a function of time

$$\boxed{x_f - x_i = \frac{1}{2}\bar{v}_x t = \frac{1}{2}\left(v_{xf} + v_{xi}\right)t}$$

Displacement as a function of velocities and time

$$\boxed{x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2}$$

Displacement as a function of time, velocity, and acceleration

$$\boxed{v_{xf}^2 = v_{xi}^2 + 2a_x\left(x_f - x_i\right)}$$

Velocity as a function of Displacement and acceleration

You may use different forms of Kinetic equations, depending on the information given to you for specific physical problems!!

How do we solve a problem using a kinematic formula for constant acceleration?

- Identify what information is given in the problem.
  - Initial and final velocity?
  - Acceleration?
  - Distance?
  - Time?
- Identify what the problem wants.
- Identify which formula is appropriate and easiest to solve for what the problem wants.
  - Frequently multiple formula can give you the answer for the quantity you are looking for. ➔ Do not just use any formula but use the one that can be easiest to solve.
- Solve the equation for the quantity wanted

Example 2.11

Suppose you want to design an air-bag system that can protect the driver in a head-on collision at a speed 100km/s (~60miles/hr). Estimate how fast the air-bag must inflate to effectively protect the driver. Assume the car crumples upon impact over a distance of about 1m. How does the use of a seat belt help the driver?

How long does it take for the car to come to a full stop? ⇒ As long as it takes for it to crumple.

The initial speed of the car is $$v_{xi} = 100km/h = \frac{100000m}{3600s} = 28m/s$$

We also know that $v_{xf} = 0m/s$ and $x_f - x_i = 1m$

Using the kinematic formula $$v_{xf}^2 = v_{xi}^2 + 2a_x\left(x_f - x_i\right)$$

The acceleration is $$a_x = \frac{v_{xf}^2 - v_{xi}^2}{2\left(x_f - x_i\right)} = \frac{0 - \left(28m/s\right)^2}{2 \times 1m} = -390m/s^2$$

Thus the time for air-bag to deploy is $$t = \frac{v_{xf} - v_{xi}}{a} = \frac{0 - 28m/s}{-390m/s^2} = 0.07s$$

Free Fall

- Falling motion is a motion under the influence of gravitational pull (gravity) only; Which direction is a freely falling object moving?
  - A motion under constant acceleration
  - All kinematic formula we learned can be used to solve for falling motions.
- Gravitational acceleration is inversely proportional to the distance between the object and the center of the earth
- The gravitational acceleration is $g=9.80m/s^2$ on the surface of the earth, most of the time.
- The direction of gravitational acceleration is **ALWAYS** toward the center of the earth, which we normally call (-y); where up and down direction are indicated as the variable "y"
- Thus the correct denotation of gravitational acceleration on the surface of the earth is $g=-9.80m/s^2$

Example for Using 1D Kinematic Equations on a Falling object

Stone was thrown straight upward at t=0 with +20.0m/s initial velocity on the roof of a 50.0m high building,

What is the acceleration in this motion? $\boxed{g=-9.80m/s^2}$

(a) Find the time the stone reaches at the maximum height.

What is so special about the maximum height? V=0

$$v_f = v_{yi} + a_y t = +20.0 - 9.80t = 0.00m/s$$

Solve for t

$$t = \frac{20.0}{9.80} = 2.04s$$

(b) Find the maximum height.

$$y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2 = 50.0 + 20 \times 2.04 + \frac{1}{2} \times (-9.80) \times (2.04)^2$$

$$= 50.0 + 20.4 = 70.4(m)$$

Example of a Falling Object cnt'd

(c) Find the time the stone reaches back to its original height.

$$t = 2.04 \times 2 = 4.08s$$

(d) Find the velocity of the stone when it reaches its original height.

$$v_{yf} = v_{yi} + a_y t = 20.0 + (-9.80) \times 4.08 = -20.0(m/s)$$

(e) Find the velocity and position of the stone at t=5.00s.

Velocity

$$v_{yf} = v_{yi} + a_y t = 20.0 + (-9.80) \times 5.00 = -29.0(m/s)$$

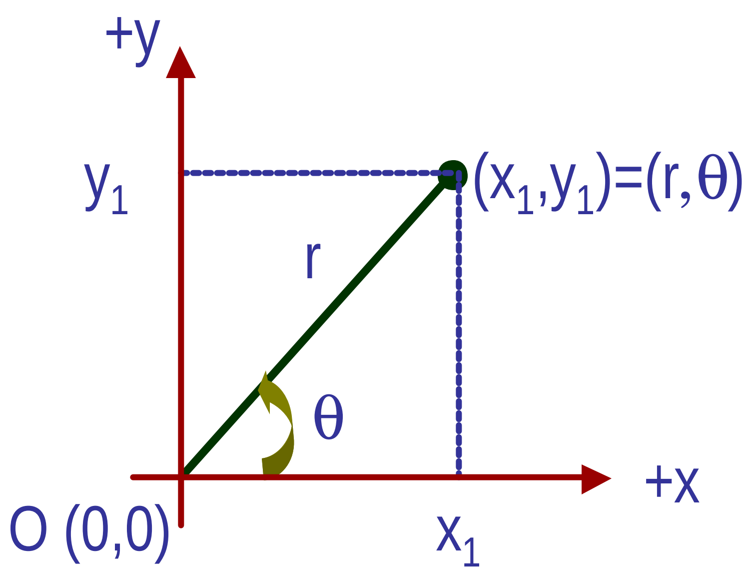
Position

$$y_f = y_i + v_{yi} t + \frac{1}{2} a_y t^2$$

$$= 50.0 + 20.0 \times 5.00 + \frac{1}{2} \times (-9.80) \times (5.00)^2 = +27.5(m)$$

Coordinate Systems

- Makes it easy to express locations or positions
- Two commonly used systems, depending on convenience
  - Cartesian (Rectangular) Coordinate System
    - Coordinates are expressed in (x,y)
  - Polar Coordinate System
    - Coordinates are expressed in (r,θ)
- Vectors become a lot easier to express and compute



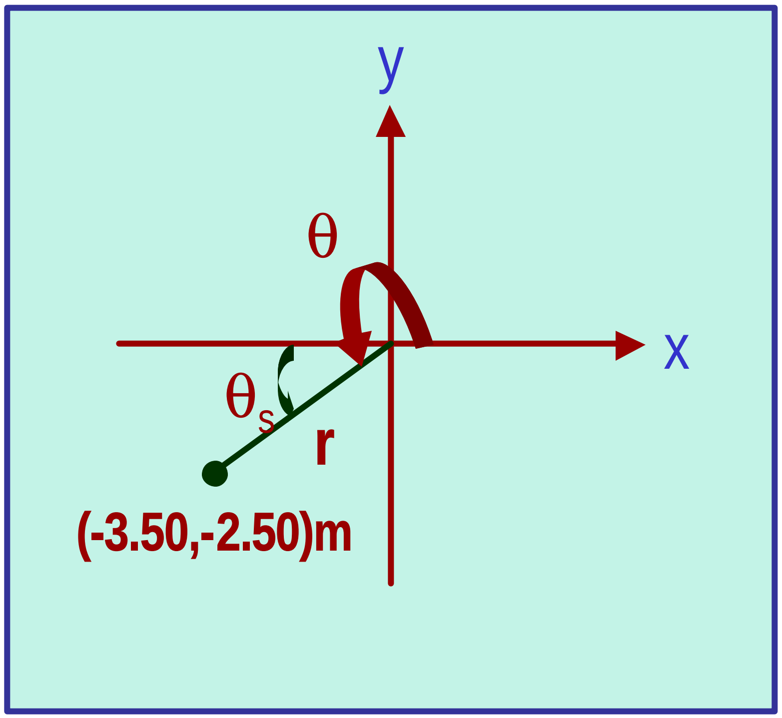
How are Cartesian and Polar coordinates related?

$$x_1 = r\cos q \qquad r = \sqrt{\left(x_1^2 + y_1^2\right)}$$

$$y_1 = r\sin q \qquad \tan q = \frac{y_1}{x_1}$$

Example

Cartesian Coordinate of a point in the xy plane are (x,y)= (-3.50,-2.50)m. Find the polar coordinates of this point.



$$r = \sqrt{\left(x^2 + y^2\right)}$$

$$= \sqrt{\left((-3.50)^2 + (-2.50)^2\right)}$$

$$= \sqrt{18.5} = 4.30(m)$$

$$\theta = 180 + \theta_s$$

$$\tan\theta_s = \frac{-2.50}{-3.50} = \frac{5}{7}$$

$$\theta_s = \tan^{-1}\left(\frac{5}{7}\right) = 35.5^{\circ}$$

$$\therefore \theta = 180 + \theta_s = 180^{\circ} + 35.5^{\circ} = 216^{\circ}$$

Vector and Scalar

Vector quantities have both magnitude (size) and direction

Force, gravitational pull, momentum

Normally denoted in **BOLD** letters, ***F***, or a letter with arrow on top $\vec{F}$

Their sizes or magnitudes are denoted with normal letters, *F*, or absolute values: $\left|\vec{F}\right|$ or $\left|F\right|$

Scalar quantities have magnitude only

Can be completely specified with a value and its unit

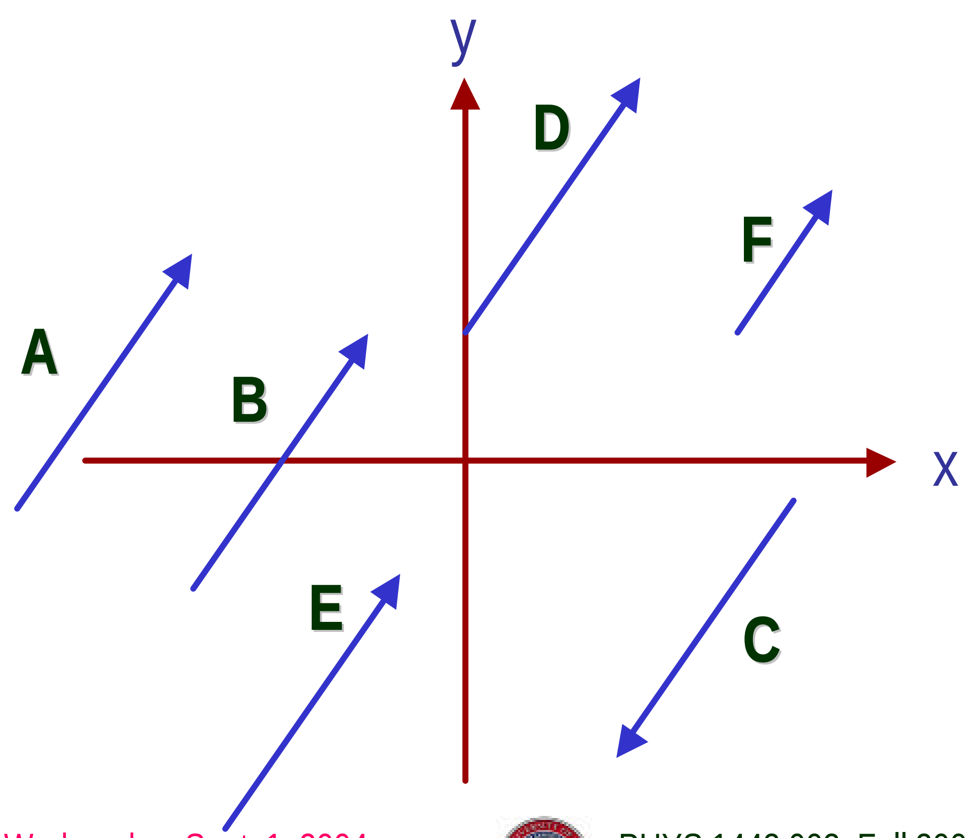
Energy, heat, mass, weight

Normally denoted in normal letters, *E*

Both have units!!!

Properties of Vectors

- Two vectors are the same if their sizes and the directions are the same, no matter where they are on a coordinate system.



Which ones are the same vectors?

A=B=E=D

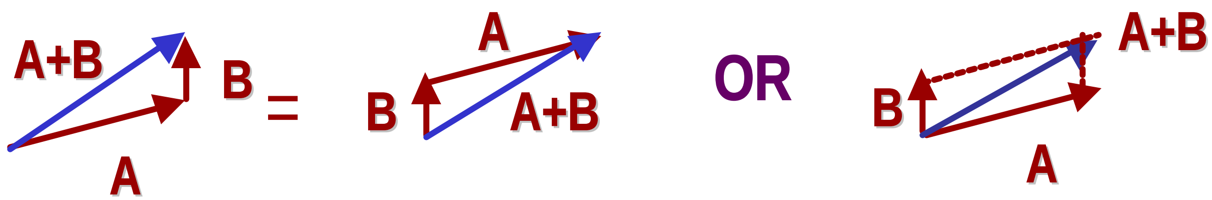
Why aren't the others?

C: The same magnitude but opposite direction: **C=-A:** A negative vector

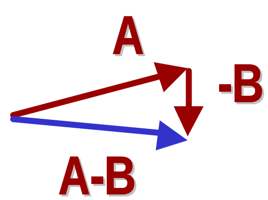
F: The same direction but different magnitude

Vector Operations

- Addition:
  - Triangular Method: One can add vectors by connecting the head of one vector to the tail of the other (head-to-tail)
  - Parallelogram method: Connect the tails of the two vectors and extend
  - Addition is commutative: Changing order of operation does not affect the results **A+B=B+A**, **A+B+C+D+E=E+C+A+B+D**



- Subtraction:
  - The same as adding a negative vector:**A** - **B** = **A** + (-**B**)



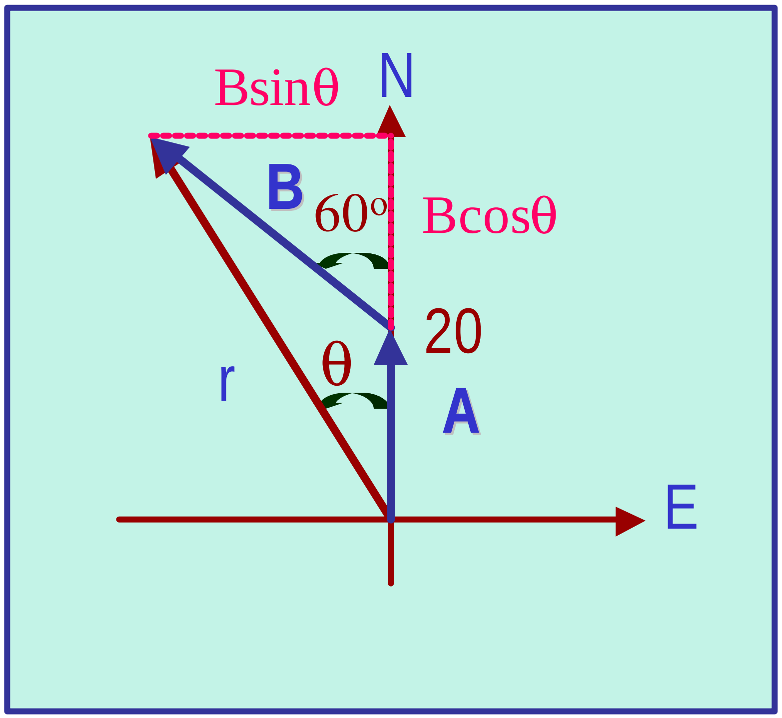
Since subtraction is the equivalent to adding a negative vector, subtraction is also commutative!!!

- Multiplication by a scalar is increasing the magnitude **A, B**=2**A**

$$\boxed{|B| = 2|A|}$$

Example of Vector Addition

A car travels 20.0km due north followed by 35.0km in a direction 60.0° west of north. Find the magnitude and direction of resultant displacement.



$$r = \sqrt{(A + B\cos q)^2 + (B\sin q)^2}$$

$$= \sqrt{A^2 + B^2\left(\cos^2 q + \sin^2 q\right) + 2AB\cos q}$$

$$= \sqrt{A^2 + B^2 + 2AB\cos q}$$

$$= \sqrt{(20.0)^2 + (35.0)^2 + 2 \times 20.0 \times 35.0 \cos 60}$$

$$= \sqrt{2325} = 48.2(km)$$

$$q = \tan^{-1}\frac{\left|\vec{B}\right| \sin 60}{\left|\vec{A}\right| + \left|\vec{B}\right| \cos 60}$$

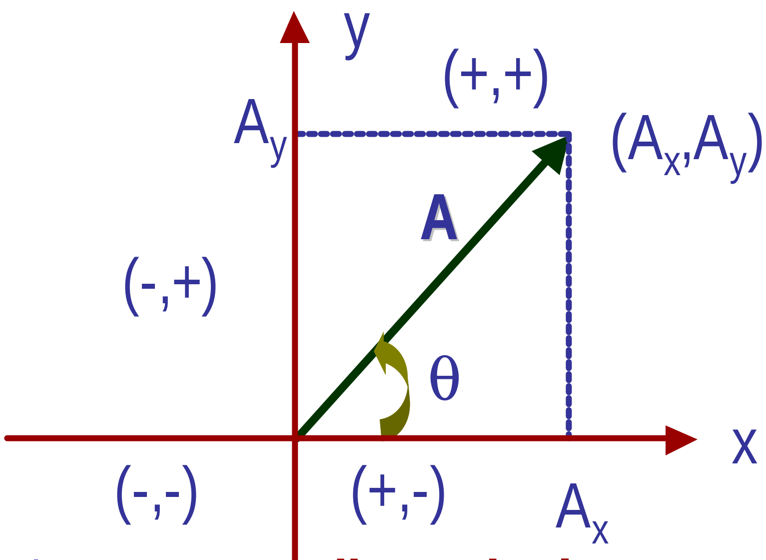
$$= \tan^{-1}\frac{35.0 \sin 60}{20.0 + 35.0 \cos 60}$$

$$= \tan^{-1}\frac{30.3}{37.5} = 38.9^{\circ} \text{ to W wrt N}$$

Find other ways to solve this problem…

Components and Unit Vectors

Coordinate systems are useful in expressing vectors in their components



$$A_x = \left|\vec{A}\right| \cos\theta$$

$$A_y = \left|\vec{A}\right| \sin\theta$$

} Components

$$\left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2}$$

} Magnitude

$$\left|\vec{A}\right| = \sqrt{\left(\left|\vec{A}\right|\cos\theta\right)^2 + \left(\left|\vec{A}\right|\sin\theta\right)^2}$$

$$= \sqrt{\left|\vec{A}\right|^2\left(\cos^2\theta + \sin^2\theta\right)} = \left|\vec{A}\right|$$

- Unit vectors are **dimensionless** vectors whose **magnitude are exactly 1**
  - Unit vectors are usually expressed in **i, j, k** or $\vec{i}\,,\ \vec{j}\,,\ \vec{k}$
  - Vectors can be expressed using components and unit vectors

So the above vector **A** can be written as

$$\vec{A} = A_x\vec{i} + A_y\vec{j} = \left|\vec{A}\right|\cos\theta\,\vec{i} + \left|\vec{A}\right|\sin\theta\,\vec{j}$$

Examples of Vector Operations

Find the resultant vector which is the sum of **A**=(2.0**i**+2.0**j**) and **B** =(2.0**i**-4.0**j**)

$$\vec{C} = \vec{A} + \vec{B} = \left(2.0\vec{i} + 2.0\vec{j}\right) + \left(2.0\vec{i} - 4.0\vec{j}\right)$$

$$= (2.0+2.0)\vec{i} + (2.0-4.0)\vec{j} = 4.0\vec{i} - 2.0\vec{j}\,(m)$$

$$\left|\vec{C}\right| = \sqrt{(4.0)^2 + (-2.0)^2}$$

$$= \sqrt{16+4.0} = \sqrt{20} = 4.5(m)$$

$$\theta = \tan^{-1}\frac{C_y}{C_x} = \tan^{-1}\frac{-2.0}{4.0} = -27^o$$

Find the resultant displacement of three consecutive displacements: $\mathbf{d}_1$=(15**i**+30**j** +12**k**)cm, $\mathbf{d}_2$=(23**i**+14**j** -5.0**k**)cm, and $\mathbf{d}_3$=(-13**i**+15**j**)cm

$$\vec{D} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3 = \left(15\vec{i} + 30\vec{j} + 12\vec{k}\right) + \left(23\vec{i} + 14\vec{j} - 5.0\vec{k}\right) + \left(-13\vec{i} + 15\vec{j}\right)$$

$$= (15+23-13)\vec{i} + (30+14+15)\vec{j} + (12-5.0)\vec{k} = 25\vec{i} + 59\vec{j} + 7.0\vec{k}\,(cm)$$

Magnitude

$$\left|\vec{D}\right| = \sqrt{(25)^2 + (59)^2 + (7.0)^2} = 65(cm)$$

Displacement, Velocity, and Acceleration in 2-dim

- Displacement: $$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$
- Average Velocity: $$\vec{v} \equiv \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$$
- Instantaneous Velocity: $$\vec{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$
- Average Acceleration $$\vec{a} \equiv \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$
- Instantaneous Acceleration: $$\vec{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$$

How is each of these quantities defined in 1-D?

2-dim Motion Under Constant Acceleration

- Position vectors in x-y plane: $\vec{r}_i = x_i\vec{i} + y_i\vec{j}$ $\quad \vec{r}_f = x_f\vec{i} + y_f\vec{j}$
- Velocity vectors in x-y plane: $\vec{v}_i = v_{xi}\vec{i} + v_{yi}\vec{j}$ $\quad \vec{v}_f = v_{xf}\vec{i} + v_{yf}\vec{j}$

Velocity vectors in terms of acceleration vector

$$v_{xf} = v_{xi} + a_x t \qquad v_{yf} = v_{yi} + a_y t$$

$$\vec{v}_f = \left(v_{xi} + a_x t\right)\vec{i} + \left(v_{yi} + a_y t\right)\vec{j} = \vec{v}_i + \vec{a}t$$

- How are the position vectors written in acceleration vectors?

$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2 \qquad y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$

$$\vec{r}_f = x_f\vec{i} + y_f\vec{j} = \left(x_i + v_{xi}t + \frac{1}{2}a_x t^2\right)\vec{i} + \left(y_i + v_{yi}t + \frac{1}{2}a_y t^2\right)\vec{j}$$

$$= \left(x_i\vec{i} + y_i\vec{j}\right) + \left(v_{xi}\vec{i} + v_{yi}\vec{j}\right)t + \frac{1}{2}\left(a_x\vec{i} + a_y\vec{j}\right)t^2 = \vec{r}_i + \vec{v}_i t + \frac{1}{2}\vec{a}t^2$$

Example for 2-D Kinematic Equations

A particle starts at origin when t=0 with an initial velocity **v**=(20**i**-15**j**)m/s. The particle moves in the xy plane with $a_x=4.0m/s^2$. Determine the components of velocity vector at any time, t.

$$v_{xf}=v_{xi}+a_x t=20+4.0t\left(m/s\right) \quad v_{yf}=v_{yi}+a_y t=-15+0t=-15\left(m/s\right)$$

Velocity vector

$$\vec{v}(t)=v_x\left(t\right)\vec{i}+v_y\left(t\right)\vec{j}=\left(20+4.0t\right)\vec{i}-15\vec{j}(m/s)$$

Compute the velocity and speed of the particle at t=5.0 s.

$$\vec{v}_{t=5}=v_{x,t=5}\vec{i}+v_{y,t=5}\vec{j}=\left(20+4.0\times5.0\right)\vec{i}-15\vec{j}=\left(40\vec{i}-15\vec{j}\right)\ m/s$$

$$speed=\left|\vec{v}\right|=\sqrt{\left(v_x\right)^2+\left(v_y\right)^2}=\sqrt{\left(40\right)^2+\left(-15\right)^2}=43m/s$$

Example for 2-D Kinematic Eq. Cnt'd

Angle of the Velocity vector

$$q = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{-15}{40}\right) = \tan^{-1}\left(\frac{-3}{8}\right) = -21^{o}$$

Determine the *x* and *y* components of the particle at t=5.0 s.

$$x_f = v_{xi}t + \frac{1}{2}a_x t^2 = 20\times5 + \frac{1}{2}\times4\times5^2 = 150(m)$$

$$y_f = v_{yi}t = -15\times5 = -75\,(m)$$

Can you write down the position vector at t=5.0s?

$$\vec{r}_f = x_f\vec{i} + y_f\vec{j} = 150\vec{i} - 75\vec{j}\,(m)$$