

PHYS 1443 – Section 003

Lecture #11

Monday, Oct. 4, 2004

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1. Newton's Law of Universal Gravitation
2. Kepler's Laws
3. Motion in Accelerated Frames



Newton's Law of Universal Gravitation

People have been very curious about stars in the sky, making observations for a long time. But the data people collected have not been explained until Newton has discovered the law of gravitation.

Every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

How would you write this law mathematically?

$$F_g \propto \frac{m_1 m_2}{r_{12}^2} \quad \xrightarrow{\text{With } G} \quad F_g = G \frac{m_1 m_2}{r_{12}^2}$$

G is the universal gravitational constant, and its value is

$$G = 6.673 \times 10^{-11} \quad \text{Unit?} \quad \text{N} \cdot \text{m}^2 / \text{kg}^2$$

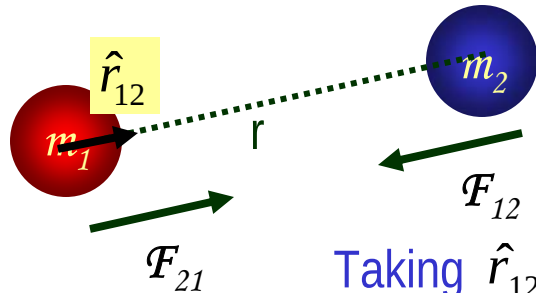
This constant is not given by the theory but must be measured by experiments.

This form of forces is known as the inverse-square law, because the magnitude of the force is inversely proportional to the square of the distances between the objects.



More on Law of Universal Gravitation

Consider two particles exerting gravitational forces to each other.



Two objects exert gravitational force on each other following Newton's 3rd law.

Taking $\hat{r}_{12}$ as the unit vector, we can write the force m_2 experiences as

$$\vec{F}_{12} = -G \frac{m_1 m_2}{r^2} \hat{r}_{12}$$

What do you think the negative sign mean?

It means that the force exerted on the particle 2 by particle 1 is an attractive force, pulling #2 toward #1.

Gravitational force is a field force: Forces act on object without physical contact between the objects at all times, independent of medium between them.

The gravitational force exerted by a finite size, spherically symmetric mass distribution on an object outside of it is the same as when the entire mass of the distributions is concentrated at the center of the object.

What do you think the gravitational force on the surface of the earth look?

$$F_g = G \frac{M_E m}{R_E^2}$$

Example for Gravitation

Using the fact that $g=9.80\text{m/s}^2$ at the Earth's surface, find the average density of the Earth.

Since the gravitational acceleration is

$$g = G \frac{M_E}{R_E^2} = 6.67 \times 10^{-11} \frac{M_E}{R_E^2}$$

Solving for M_E

$$M_E = \frac{R_E^2 g}{G}$$

Therefore the
density of the
Earth is

$$\rho = \frac{M_E}{V_E} = \frac{\frac{R_E^2 g}{G}}{\frac{4\pi}{3} R_E^3} = \frac{3g}{4\pi G R_E}$$

$$= \frac{3 \times 9.80}{4\pi \times 6.67 \times 10^{-11} \times 6.37 \times 10^6} = 5.50 \times 10^3 \text{ kg/m}^3$$



Free Fall Acceleration & Gravitational Force

Weight of an object with mass m is mg . Using the force exerting on a particle of mass m on the surface of the Earth, one can obtain

$$mg = G \frac{M_E m}{R_E^2}$$

$$g = G \frac{M_E}{R_E^2}$$

What would the gravitational acceleration be if the object is at an altitude h above the surface of the Earth?

$$F_g = mg' = G \frac{M_E m}{r^2} = G \frac{M_E m}{(R_E + h)^2}$$

$$g' = G \frac{M_E}{(R_E + h)^2}$$

Distance from the center of the Earth to the object at the altitude h .

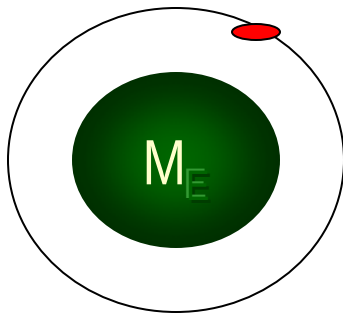
What do these tell us about the gravitational acceleration?

- The gravitational acceleration is independent of the mass of the object
- The gravitational acceleration decreases as the altitude increases
- If the distance from the surface of the Earth gets infinitely large, the weight of the object approaches 0.



Example for Gravitational Force

The international space station is designed to operate at an altitude of 350km. When completed, it will have a weight (measured on the surface of the Earth) of $4.22 \times 10^6 \text{ N}$. What is its weight when in its orbit?



The total weight of the station on the surface of the Earth is

$$F_{GE} = mg = G \frac{M_E m}{R_E^2} = 4.22 \times 10^6 \text{ N}$$

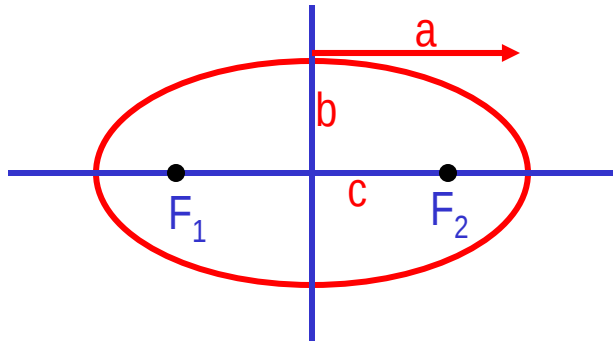
Since the orbit is at 350km above the surface of the Earth, the gravitational force at that height is

$$F_O = mg = G \frac{M_E m}{(R_E + h)^2} = \frac{R_E^2}{(R_E + h)^2} F_{GE}$$

Therefore the weight in the orbit is

$$F_O = \frac{R_E^2}{(R_E + h)^2} F_{GE} = \frac{(6.37 \times 10^6)^2}{(6.37 \times 10^6 + 3.50 \times 10^5)^2} \times 4.22 \times 10^6 = 3.80 \times 10^6 \text{ N}$$

Kepler's Laws & Ellipse



Ellipses have two different axis, major (long) and minor (short) axis, and two focal points, F_1 & F_2

a is the length of a semi-major axis

b is the length of a semi-minor axis

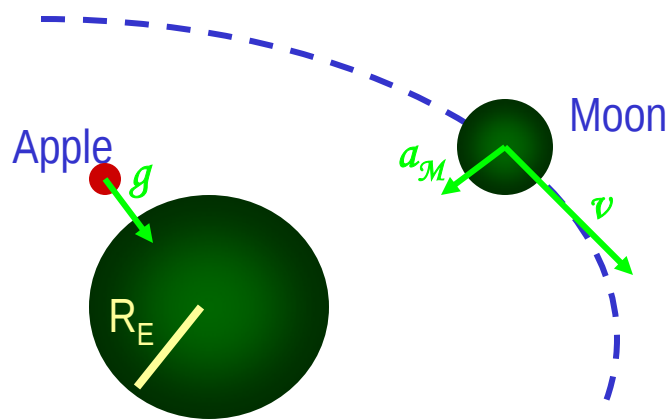
Kepler lived in Germany and discovered the law's governing planets' movement some 70 years before Newton, by analyzing data.

1. All planets move in elliptical orbits with the Sun at one focal point.
2. The radius vector drawn from the Sun to a planet sweeps out equal area in equal time intervals. (*Angular momentum conservation*)
3. The square of the orbital period of any planet is proportional to the cube of the semi-major axis of the elliptical orbit.

Newton's laws explain the cause of the above laws. Kepler's third law is a direct consequence of law of gravitation being inverse square law.

The Law of Gravity and Motions of Planets

- Newton assumed that the law of gravitation applies the same whether it is on the Moon or the apple on the surface of the Earth.
- The interacting bodies are assumed to be point like particles.



Newton predicted that the ratio of the Moon's acceleration a_M to the apple's acceleration g would be

$$\frac{a_M}{g} = \frac{(1/r_M)^2}{(1/R_E)^2} = \left(\frac{R_E}{r_M}\right)^2 = \left(\frac{6.37 \times 10^6}{3.84 \times 10^8}\right)^2 = 2.75 \times 10^{-4}$$

Therefore the centripetal acceleration of the Moon, a_M , is

$$a_M = 2.75 \times 10^{-4} \times 9.80 = 2.70 \times 10^{-3} \text{ m/s}^2$$

Newton also calculated the Moon's orbital acceleration a_M from the knowledge of its distance from the Earth and its orbital period, $T=27.32 \text{ days}=2.36 \times 10^6 \text{ s}$

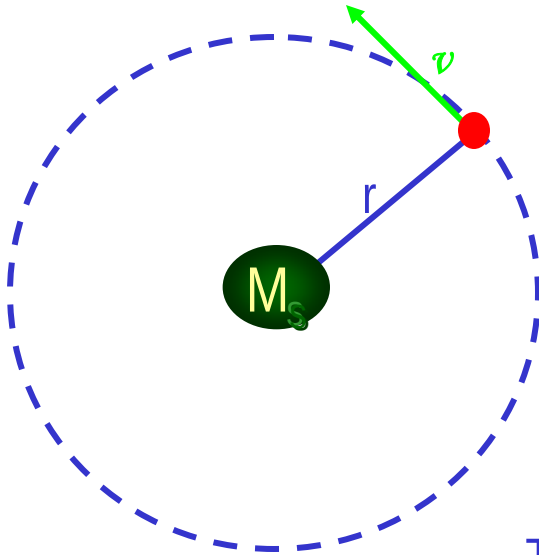
$$a_M = \frac{v^2}{r_M} = \frac{(2\pi r_M / T)^2}{r_M} = \frac{4\pi^2 r_M}{T^2} = \frac{4\pi^2 \times 3.84 \times 10^8}{(2.36 \times 10^6)^2} = 2.72 \times 10^{-3} \text{ m/s}^2 \approx \frac{9.80}{(60)^2}$$

This means that the Moon's distance is about 60 times that of the Earth's radius, and its acceleration is reduced by the square of the ratio. This proves that the inverse square law is valid.



Kepler's Third Law

It is crucial to show that Kepler's third law can be predicted from the inverse square law for circular orbits.



Since the gravitational force exerted by the Sun is radially directed toward the Sun to keep the planet on a near circular path, we can apply Newton's second law

$$\frac{GM_s M_p}{r^2} = \frac{M_p v^2}{r}$$

Since the orbital speed, v , of the planet with period T is $v = \frac{2\pi r}{T}$

The above can be written $\frac{GM_s M_p}{r^2} = \frac{M_p (2\pi r / T)^2}{r}$

Solving for T one can obtain $T^2 = \left(\frac{4\pi^2}{GM_s} \right) r^3 = K_s r^3$ and $K_s = \left(\frac{4\pi^2}{GM_s} \right) = 2.97 \times 10^{-19} \text{ s}^2 / \text{m}^3$

This is Kepler's third law. It's also valid for ellipse for r being the length of the semi-major axis. The constant K_s is independent of mass of the planet.

Example of Kepler's Third Law

Calculate the mass of the Sun using the fact that the period of the Earth's orbit around the Sun is 3.16×10^7 s, and its distance from the Sun is 1.496×10^{11} m.

Using Kepler's third law.

$$T^2 = \left(\frac{4\pi^2}{GM_s} \right) r^3 = K_s r^3$$

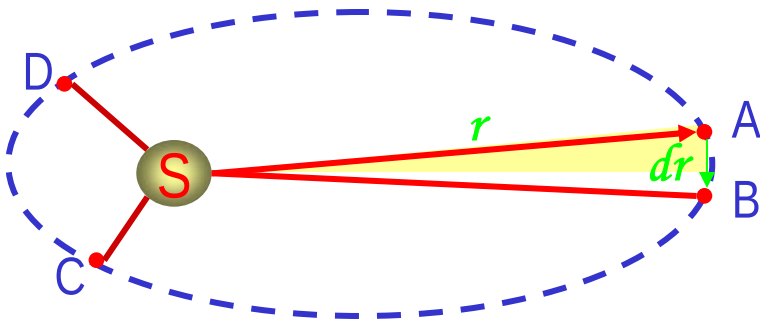
The mass of the Sun, M_s , is

$$M_s = \left(\frac{4\pi^2}{GT^2} \right) r^3$$
$$= \left(\frac{4\pi^2}{6.67 \times 10^{-11} \times (3.16 \times 10^7)^2} \right) \times (1.496 \times 10^{11})^3$$
$$= 1.99 \times 10^{30} \text{ kg}$$



Kepler's Second Law and Angular Momentum Conservation

Consider a planet of mass M_p moving around the Sun in an elliptical orbit.



Since the gravitational force acting on the planet is always toward radial direction, it is a *central force*. Therefore the torque acting on the planet by this force is always 0.

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times F\hat{r} = 0$$

Since torque is the time rate change of angular momentum $\mathcal{L}$, the angular momentum is constant.

$$\vec{\tau} = \frac{d\vec{L}}{dt} = 0 \quad \vec{L} = \text{const}$$

Because the gravitational force exerted on a planet by the Sun results in no torque, the angular momentum L of the planet is constant.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times M_p \vec{v} = M_p \vec{r} \times \vec{v} = \text{const}$$

Since the area swept by the motion of the planet is

$$dA = \frac{1}{2} |\vec{r} \times d\vec{r}| = \frac{1}{2} |\vec{r} \times \vec{v} dt| = \frac{L}{2M_p} dt \quad \Rightarrow \quad \frac{dA}{dt} = \frac{L}{2M_p} = \text{const}$$

This is Kepler's second law which states that the radius vector from the Sun to a planet sweeps out equal areas in equal time intervals.

Motion in Accelerated Frames

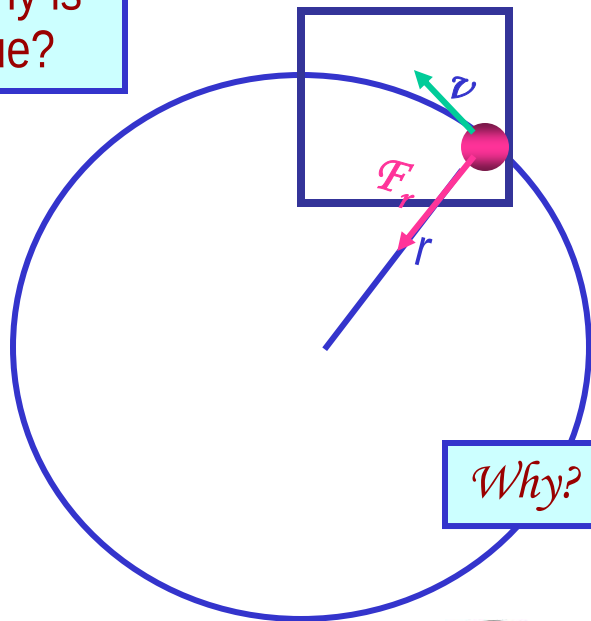
Newton's laws are valid only when observations are made in an inertial frame of reference. What happens in a non-inertial frame?

Fictitious forces are needed to apply Newton's second law in an accelerated frame.

This force does not exist when the observations are made in an inertial reference frame.

What does this mean and why is this true?

Let's consider a free ball inside a box under uniform circular motion.



How does this motion look like in an inertial frame (or frame outside a box)?

We see that the box has a radial force exerted on it but none on the ball directly

How does this motion look like in the box?

The ball is tumbled over to the wall of the box and feels that it is getting force that pushes it toward the wall.

Why?

According to Newton's first law, the ball wants to continue on its original movement but since the box is turning, the ball feels like it is being pushed toward the wall relative to everything else in the box.

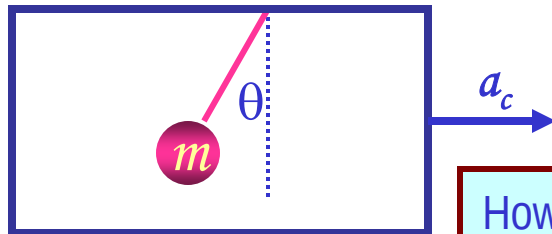
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Example of Motion in Accelerated Frames

A ball of mass m is hung by a cord to the ceiling of a boxcar that is moving with an acceleration a . What do the inertial observer at rest and the non-inertial observer traveling inside the car conclude? How do they differ?

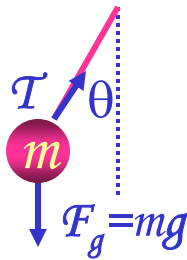


This is how the ball looks like no matter which frame you are in.

How do the free-body diagrams look for two frames?

How do the motions interpreted in these two frames? Any differences?

*Inertial
Frame*



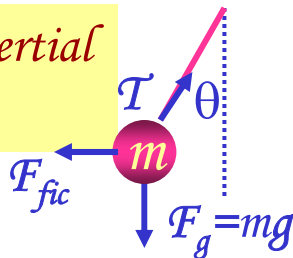
$$\begin{aligned}\sum \vec{F} &= \vec{F}_g + \vec{T} \\ \sum F_x &= ma_x = ma_c = T \sin \theta \\ \sum F_y &= T \cos \theta - mg = 0\end{aligned}$$

$$T = \frac{mg}{\cos \theta} \quad a_c = g \tan \theta$$

For an inertial frame observer, the forces being exerted on the ball are only T and F_g . The acceleration of the ball is the same as that of the box car and is provided by the x component of the tension force.

In the non-inertial frame observer, the forces being exerted on the ball are T , F_g , and F_{fic} . For some reason the ball is under a force, F_{fic} , that provides acceleration to the ball.

*Non-Inertial
Frame*



$$\begin{aligned}\sum \vec{F} &= \vec{F}_g + \vec{T} + \vec{F}_{fic} \\ \sum F_x &= T \sin \theta - F_{fic} = 0 \quad F_{fic} = ma_{fic} = T \sin \theta \\ \sum F_y &= T \cos \theta - mg = 0\end{aligned}$$

$$T = \frac{mg}{\cos \theta} \quad a_{fic} = g \tan \theta$$

While the mathematical expression of the acceleration of the ball is identical to that of inertial frame observer's, the cause of the force is dramatically different.

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