PHYS 1443 – Section 003
Lecture #25

Wednesday, Dec. 1, 2004

Dr. Jaehoon Yu

1. Review
2. Problem solving session

Homework #12 is due midnight, Friday, Dec. 3, 2004!!

Final Exam, Monday, Dec. 6!!

Announcements

- Final Exam
  - Date: Monday, Dec. 6
  - Time: 11:00am – 12:30pm
  - Location: SH103
  - Covers: CH 10 – CH 14

Rotational Kinematics

The first type of motion we have learned in linear kinematics was under a constant acceleration. We will learn about the rotational motion under constant angular acceleration, because these are the simplest motions in both cases.

Just like the case in linear motion, one can obtain

Angular Speed under constant angular acceleration:

$$\omega_f = \omega_i + \alpha t$$

Angular displacement under constant angular acceleration:

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

One can also obtain

$$\omega_f^2 = \omega_i^2 + 2\alpha\left(\theta_f - \theta_i\right)$$

Angular Displacement, Velocity, and Acceleration

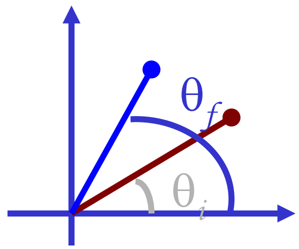
Using what we have learned in the previous slide, how would you define the angular displacement?

$$\Delta\theta = \theta_f - \theta_i$$

How about the average angular speed?

Unit? rad/s

$$\overline{\omega} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$



And the instantaneous angular speed?

Unit? rad/s

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

By the same token, the average angular acceleration

Unit? rad/s^2

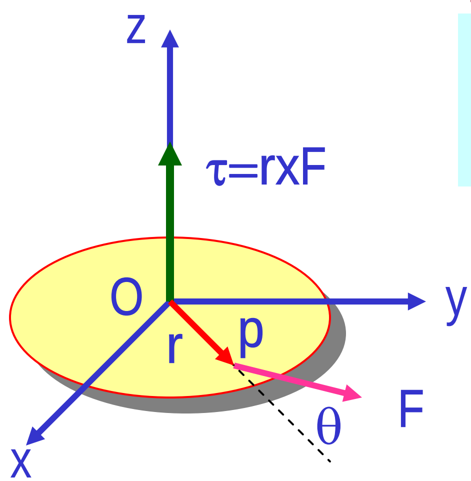
$$\overline{\alpha} \equiv \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

And the instantaneous angular acceleration? Unit? rad/s^2

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

When rotating about a fixed axis, every particle on a rigid object rotates through the same angle and has the same angular speed and angular acceleration.

Torque and Vector Product



Let's consider a disk fixed onto the origin O and the force $\mathcal{F}$ exerts on the point p. What happens?

The disk will start rotating counter clockwise about the Z axis

The magnitude of torque given to the disk by the force $\mathcal{F}$ is

$$\tau = Fr\sin\phi$$

But torque is a vector quantity, what is the direction?
How is torque expressed mathematically?

$$\boxed{\vec{\tau} \equiv \vec{r} \times \vec{F}}$$

What is the direction? The direction of the torque follows the right-hand rule!!

The above quantity is called
Vector product or Cross product

$$\vec{C} \equiv \vec{A} \times \vec{B}$$

$$\boxed{\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta}$$

What is the result of a vector product?

Another vector

What is another vector operation we've learned?

Scalar product

$$C \equiv \vec{A} \cdot \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$$

Result? A scalar

More Properties of Vector Product

The relationship between unit vectors, $\vec{i}, \vec{j}$ and $\vec{k}$

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$$

$$\vec{i} \times \vec{j} = -\vec{j} \times \vec{i} = \vec{k}$$

$$\vec{j} \times \vec{k} = -\vec{k} \times \vec{j} = \vec{i}$$

$$\vec{k} \times \vec{i} = -\vec{i} \times \vec{k} = \vec{j}$$

Vector product of two vectors can be expressed in the following determinant form

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{i} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \vec{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \vec{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= \left(A_y B_z - A_z B_y\right)\vec{i} - \left(A_x B_z - A_z B_x\right)\vec{j} + \left(A_x B_y - A_y B_x\right)\vec{k}$$

Moment of Inertia

Rotational Inertia: Measure of resistance of an object to changes in its rotational motion. Equivalent to mass in linear motion.

For a group of particles $$I \equiv \sum_i m_i r_i^2$$

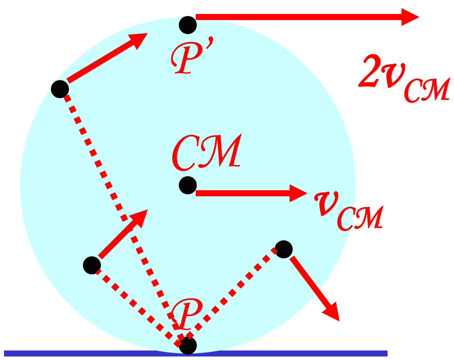
For a rigid body $$I \equiv \int r^2 dm$$

What are the dimension and unit of Moment of Inertia? $\left[ML^2\right]$ $kg \cdot m^2$

Determining Moment of Inertia is extremely important for computing equilibrium of a rigid body, such as a building.

Total Kinetic Energy of a Rolling Body

What do you think the total kinetic energy of the rolling cylinder is?



Since it is a rotational motion about the point P, we can write the total kinetic energy

$$K = \frac{1}{2} I_P \omega^2$$

Where, I_P, is the moment of inertia about the point P.

Using the parallel axis theorem, we can rewrite

$$K = \frac{1}{2} I_P \omega^2 = \frac{1}{2}\left(I_{CM} + MR^2\right)\omega^2 = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} MR^2\omega^2$$

Since $v_{CM}=R\omega$, the above relationship can be rewritten as

$$K = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}^2$$

Rotational kinetic energy about the CM

Translational Kinetic energy of the CM

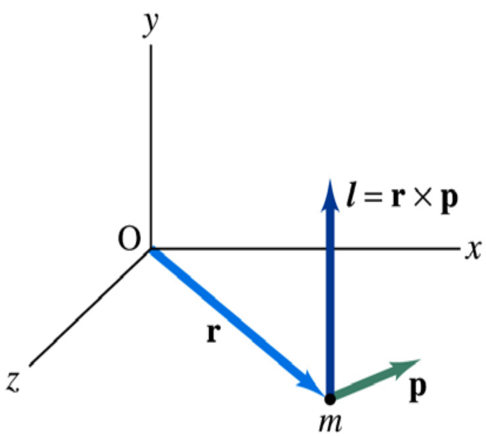
What does this equation mean?

Total kinetic energy of a rolling motion is the sum of the rotational kinetic energy about the CM

And the translational kinetic of the CM

Angular Momentum of a Particle

If you grab onto a pole while running, your body will rotate about the pole, gaining angular momentum. We've used linear momentum to solve physical problems with linear motions, angular momentum will do the same for rotational motions.



Let's consider a point-like object (particle) with mass m located at the vector location r and moving with linear velocity v

The angular momentum $\mathcal{L}$ of this particle relative to the origin O is

$$\boxed{\vec{L} \equiv \vec{r} \times \vec{p}}$$

What is the unit and dimension of angular momentum? $kg\cdot m^2/s^2$ $[ML^2T^{-2}]$

Note that $\mathcal{L}$ depends on origin O. Why? Because r changes

What else do you learn? The direction of $\mathcal{L}$ is +z

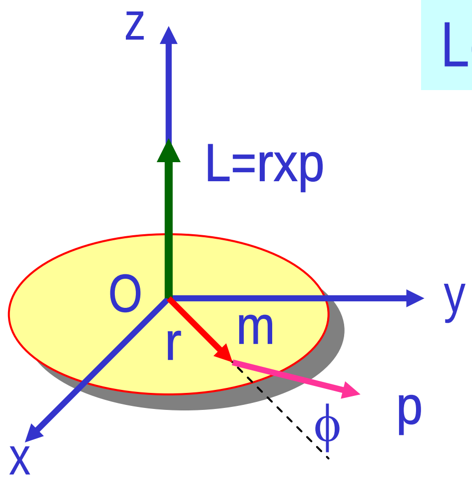
Since p is mv, the magnitude of $\mathcal{L}$ becomes $L = mvr\sin\phi$

What do you learn from this?

If the direction of linear velocity points to the origin of rotation, the particle does not have any angular momentum.

If the linear velocity is perpendicular to position vector, the particle moves exactly the same way as a point on a rim.

Angular Momentum of a Rotating Rigid Body



Let's consider a rigid body rotating about a fixed axis

Each particle of the object rotates in the xy plane about the z-axis at the same angular speed, ω

Magnitude of the angular momentum of a particle of mass m_i about origin O is $m_i v_i r_i$

$$L_i = m_i r_i v_i = m_i r_i^2 \omega$$

Summing over all particle's angular momentum about z axis

$$L_z = \sum_i L_i = \sum_i \left(m_i r_i^2 \omega \right)$$

What do you see?

$$L_z = \sum_i \left(m_i r_i^2 \right) \omega = I\omega$$

Since I is constant for a rigid body

$$\frac{dL_z}{dt} = I \frac{d\omega}{dt} = I\alpha$$

α is angular acceleration

Thus the torque-angular momentum relationship becomes

$$\boxed{\sum \tau_{ext} = \frac{dL_z}{dt} = I\alpha}$$

Thus the net external torque acting on a rigid body rotating about a fixed axis is equal to the moment of inertia about that axis multiplied by the object's angular acceleration with respect to that axis.

Conservation of Angular Momentum

Remember under what condition the linear momentum is conserved?

Linear momentum is conserved when the net external force is 0.

$$\sum \vec{F} = 0 = \frac{d\vec{p}}{dt}$$

$$\vec{p} = const$$

By the same token, the angular momentum of a system is constant in both magnitude and direction, if the resultant external torque acting on the system is 0.

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt} = 0$$

$$\vec{L} = const$$

What does this mean?

Angular momentum of the system before and after a certain change is the same.

$$\vec{L}_i = \vec{L}_f = \text{constant}$$

Three important conservation laws for isolated system that does not get affected by external forces

$K_i + U_i = K_f + U_f$	Mechanical Energy
$\vec{p}_i = \vec{p}_f$	Linear Momentum
$\vec{L}_i = \vec{L}_f$	Angular Momentum

Similarity Between Linear and Rotational Motions

All physical quantities in linear and rotational motions show striking similarity.

Quantities	Linear	Rotational
Mass	Mass M	Moment of Inertia $I = mr^2$
Length of motion	Distance L	Angle θ (Radian)
Speed	$v = \frac{\Delta r}{\Delta t}$	$\omega = \frac{\Delta \theta}{\Delta t}$
Acceleration	$a = \frac{\Delta v}{\Delta t}$	$\alpha = \frac{\Delta \omega}{\Delta t}$
Force	Force $F = ma$	Torque $\tau = I\alpha$
Work	Work $W = Fd\cos\theta$	Work $W = \tau\theta$
Power	$P = \vec{F} \cdot \vec{v}$	$P = \tau\omega$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{\omega}$
Kinetic Energy	Kinetic $K = \frac{1}{2}mv^2$	Rotational $K_R = \frac{1}{2}I\omega^2$

Conditions for Equilibrium

What do you think does the term "An object is at its equilibrium" mean?

The object is either at rest (Static Equilibrium) or its center of mass is moving with a constant velocity (Dynamic Equilibrium).

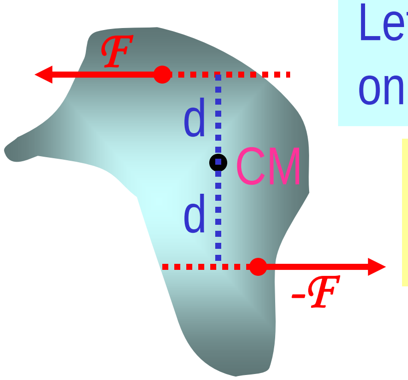
When do you think an object is at its equilibrium?

Translational Equilibrium: Equilibrium in linear motion

$$\sum \vec{F} = 0$$

Is this it?

The above condition is sufficient for a point-like particle to be at its static equilibrium. However for object with size this is not sufficient. One more condition is needed. What is it?



Let's consider two forces equal magnitude but opposite direction acting on a rigid object as shown in the figure. What do you think will happen?

The object will rotate about the CM. The net torque acting on the object about any axis must be 0.

$$\sum \vec{\tau} = 0$$

For an object to be at its *static equilibrium*, the object should not have linear or angular speed.

$$v_{CM} = 0 \quad \omega = 0$$

How do we solve equilibrium problems?

1. Identify all the forces and their directions and locations
2. Draw a free-body diagram with forces indicated on it
3. Write down vector force equation for each x and y component with proper signs
4. Select a rotational axis for torque calculations ➔ Selecting the axis such that the torque of one of the unknown forces become 0.
5. Write down torque equation with proper signs
6. Solve the equations for unknown quantities

Fluid and Pressure

What are the three states of matter?

Solid, Liquid, and Gas

How do you distinguish them?

By the time it takes for a particular substance to change its shape in reaction to external forces.

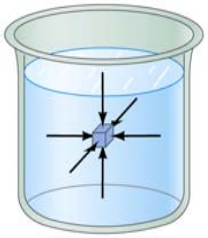
What is a fluid?

A collection of molecules that are randomly arranged and loosely bound by forces between them or by the external container.

We will first learn about mechanics of fluid at rest, *fluid statics*.

In what way do you think fluid exerts stress on the object submerged in it?

Fluid cannot exert shearing or tensile stress. Thus, the only force the fluid exerts on an object immersed in it is the forces perpendicular to the surfaces of the object.



This force by the fluid on an object usually is expressed in the form of the force on a unit area at the given depth, the pressure, defined as

$$P \equiv \frac{F}{A}$$

Expression of pressure for an infinitesimal area dA by the force dF is

$$P = \frac{dF}{dA}$$

Note that pressure is a scalar quantity because it's the magnitude of the force on a surface area A.

What is the unit and dimension of pressure?

Unit: N/m^2

Dim.: $[M][L^{-1}][T^{-2}]$

Special SI unit for pressure is Pascal

$$1Pa \equiv 1N/m^2$$

Pascal's Principle and Hydraulics

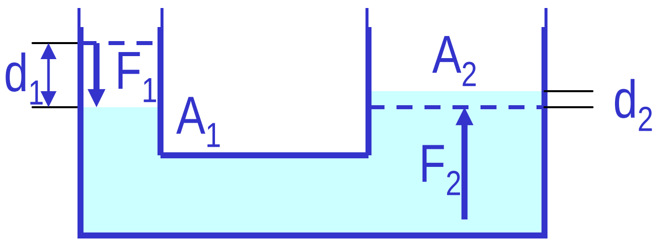
A change in the pressure applied to a fluid is transmitted undiminished to every point of the fluid and to the walls of the container.

$$P = P_0 + \rho g h$$

What happens if P_0 is changed?

The resultant pressure P at any given depth h increases as much as the change in P_0.

This is the principle behind hydraulic pressure. How?



Since the pressure change caused by the the force F_1 applied on to the area A_1 is transmitted to the F_2 on an area A_2.

$$P = \frac{F_1}{A_1} = \frac{F_2}{A_2}$$

Therefore, the resultant force F_2 is

$$F_2 = \frac{A_2}{A_1} F_1$$

In other words, the force gets multiplied by the ratio of the areas A_2/A_1 and is transmitted to the force F_2 on the surface.

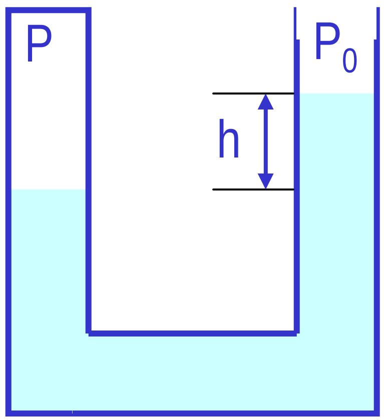
This seems to violate some kind of conservation law, doesn't it?

No, the actual displaced volume of the fluid is the same. And the work done by the forces are still the same.

$$F_2 = \frac{d_1}{d_2} F_1$$

Absolute and Relative Pressure

How can one measure the pressure?



One can measure the pressure using an open-tube manometer, where one end is connected to the system with unknown pressure P and the other open to air with pressure P_0.

The measured pressure of the system is $$P = P_0 + \rho g h$$

This is called the <u>absolute pressure</u>, because it is the actual value of the system's pressure.

In many cases we measure pressure difference with respect to atmospheric pressure due to changes in P_0 depending on the environment. This is called <u>gauge or relative pressure</u>.

$$P_G = P - P_0 = \rho g h$$

The common barometer which consists of a mercury column with one end closed at vacuum and the other open to the atmosphere was invented by Evangelista Torricelli.

Since the closed end is at vacuum, it does not exert any force. 1 atm is

$$P_0 = \rho g h = (13.595 \times 10^3 \, kg/m^3)(9.80665 \, m/s^2)(0.7600 \, m)$$
$$= 1.013 \times 10^5 \, Pa = 1 \, atm$$

If one measures the tire pressure with a gauge at 220kPa the actual pressure is 101kPa+220kPa=303kPa.

Buoyant Forces and Archimedes' Principle

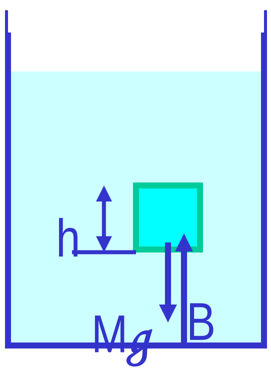
Why is it so hard to put an inflated beach ball under water while a small piece of steel sinks in the water?

The water exerts force on an object immersed in the water.
This force is called Buoyant force.

How does the Buoyant force work?

The magnitude of the buoyant force always equals the weight of the fluid in the volume displaced by the submerged object.

This is called, Archimedes' principle. What does this mean?



Let's consider a cube whose height is h and is filled with fluid and at in its equilibrium so that its weight Mg is balanced by the buoyant force B.

$$B = F_g = Mg$$

The pressure at the bottom of the cube is larger than the top by ρgh.

Therefore,

$$\Delta P = B / A = \rho gh$$

$$B = \Delta PA = \rho ghA = \rho Vg$$

$$B = \rho Vg = Mg = F_g$$

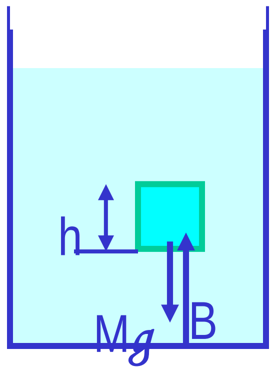
Where Mg is the weight of the fluid.

More Archimedes' Principle

Let's consider buoyant forces in two special cases.

Case 1: Totally submerged object

Let's consider an object of mass M, with density ρ_0, is immersed in the fluid with density ρ_f.



The magnitude of the buoyant force is $B = \rho_f V g$

The weight of the object is $F_g = Mg = \rho_0 V g$

Therefore total force of the system is $F = B - F_g = (\rho_f - \rho_0) V g$

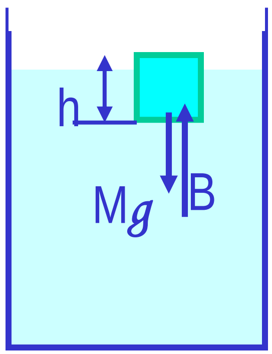
What does this tell you?

The total force applies to different directions depending on the difference of the density between the object and the fluid.

1. If the density of the object is smaller than the density of the fluid, the buoyant force will push the object up to the surface.
2. If the density of the object is larger that the fluid's, the object will sink to the bottom of the fluid.

More Archimedes' Principle

Case 2: Floating object



Let's consider an object of mass M, with density ρ_0, is in static equilibrium floating on the surface of the fluid with density ρ_f, and the volume submerged in the fluid is V_f.

The magnitude of the buoyant force is $B = \rho_f V_f g$

The weight of the object is $F_g = Mg = \rho_0 V_0 g$

Therefore total force of the system is $F = B - F_g = \rho_f V_f g - \rho_0 V_0 g = 0$

Since the system is in static equilibrium $\rho_f V_f g = \rho_0 V_0 g$

$$\frac{\rho_0}{\rho_f} = \frac{V_f}{V_0}$$

What does this tell you?

Since the object is floating its density is always smaller than that of the fluid.

The ratio of the densities between the fluid and the object determines the submerged volume under the surface.

Bernoulli's Equation cont'd

Since

$$\frac{1}{2}\rho \cancel{A_2 \Delta l_2} v_2^2 - \frac{1}{2}\rho \cancel{A_1 \Delta l_1} v_1^2 = P_1 \cancel{A_1 \Delta l_1} - P_2 \cancel{A_2 \Delta l_2} - \rho \cancel{A_2 \Delta l_2} g y_2 + \rho \cancel{A_1 \Delta l_1} g y_1$$

We obtain

$$\frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2 = P_1 - P_2 - \rho g y_2 + \rho g y_1$$

Re-organize

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

Bernoulli's Equation

Thus, for any two points in the flow

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = \ const.$$

Result of Energy conservation!

For static fluid

$$P_2 = P_1 + \rho g \left(y_1 - y_2 \right) = P_1 + \rho g h$$

Pascal's Law

For the same heights

$$P_2 = P_1 + \frac{1}{2}\rho \left(v_1^2 - v_2^2 \right)$$

The pressure at the faster section of the fluid is smaller than slower section.

Simple Harmonic Motion

Motion that occurs by the force that depends on displacement, and the force is always directed toward the system's equilibrium position.

What is a system that has such characteristics?

A system consists of a mass and a spring

When a spring is stretched from its equilibrium position by a length x, the force acting on the mass is

$$F = -kx$$

It's negative, because the force resists against the change of length, directed toward the equilibrium position.

From Newton's second law $F = ma = -kx$ we obtain $a = -\frac{k}{m}x$

This is a second order differential equation that can be solved but it is beyond the scope of this class.

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Condition for simple harmonic motion

What do you observe from this equation?

Acceleration is proportional to displacement from the equilibrium
Acceleration is opposite direction to displacement

This system is doing a simple harmonic motion (SHM).

Equation of Simple Harmonic Motion

The solution for the 2nd order differential equation

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

$$x = A\cos\left(\boxed{\omega t + \phi}\right)$$

Amplitude

Phase

Angular Frequency

Phase constant

Generalized expression of a simple harmonic motion

Let's think about the meaning of this equation of motion

What happens when t=0 and ϕ=0?

$$x = A\cos(0+0) = A$$

What is ϕ if x is not A at t=0?

$$x = A\cos(\phi) = x'$$

$$\phi = \cos^{-1}(x')$$

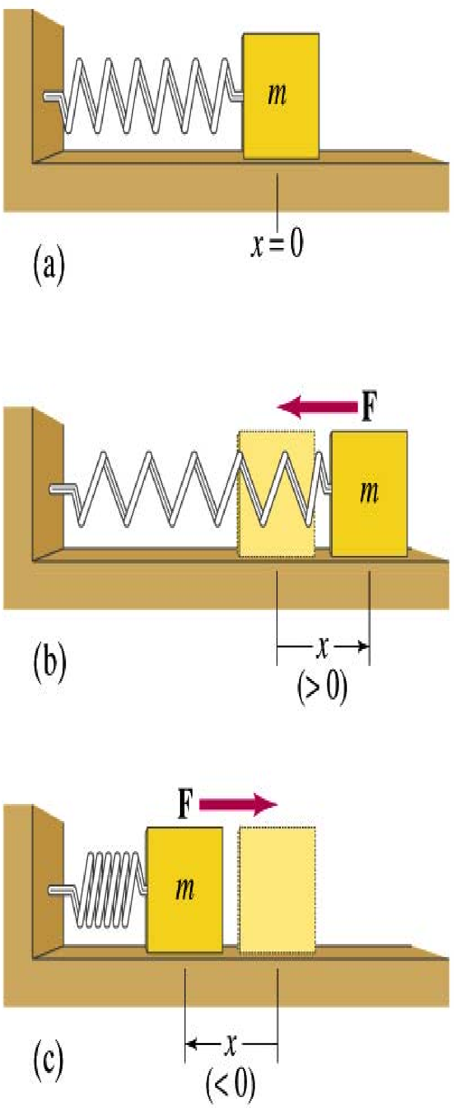
An oscillation is fully characterized by its:

- Amplitude
- Period or frequency
- Phase constant

What are the maximum/minimum possible values of x?

A/-A

Vibration or Oscillation Properties



The maximum displacement from the equilibrium is

Amplitude

One cycle of the oscillation

The complete to-and-fro motion from an initial point

Period of the motion, T

The time it takes to complete one full cycle

Unit? s

Frequency of the motion, f

The number of complete cycles per second

Unit? s^{-1}

Relationship between period and frequency?

$$f = \frac{1}{T} \quad \text{or} \quad T = \frac{1}{f}$$

Simple Block-Spring System

A block attached at the end of a spring on a frictionless surface experiences acceleration when the spring is displaced from an equilibrium position.

$$F_{spring} = ma$$
$$= -kx$$
$$a = -\frac{k}{m}x$$

This becomes a second order differential equation $\frac{d^2x}{dt^2} = -\frac{k}{m}x$ If we denote $\omega^2 = \frac{k}{m}$

The resulting differential equation becomes $\frac{d^2x}{dt^2} = -\omega^2 x$

Since this satisfies condition for simple harmonic motion, we can take the solution $\boxed{x = A\cos(\omega t + \phi)}$

Does this solution satisfy the differential equation?

Let's take derivatives with respect to time $\frac{dx}{dt} = A\frac{d}{dt}(\cos(\omega t + \phi)) = -\omega A\sin(\omega t + \phi)$

Now the second order derivative becomes

$$\frac{d^2x}{dt^2} = -\omega A\frac{d}{dt}(\sin(\omega t + \phi)) = -\omega^2 A\cos(\omega t + \phi) = -\omega^2 x$$

Whenever the force acting on a particle is linearly proportional to the displacement from some equilibrium position and is in the opposite direction, the particle moves in simple harmonic motion.

Energy of the Simple Harmonic Oscillator

How do you think the mechanical energy of the harmonic oscillator look without friction?

Kinetic energy of a harmonic oscillator is

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$$

The elastic potential energy stored in the spring

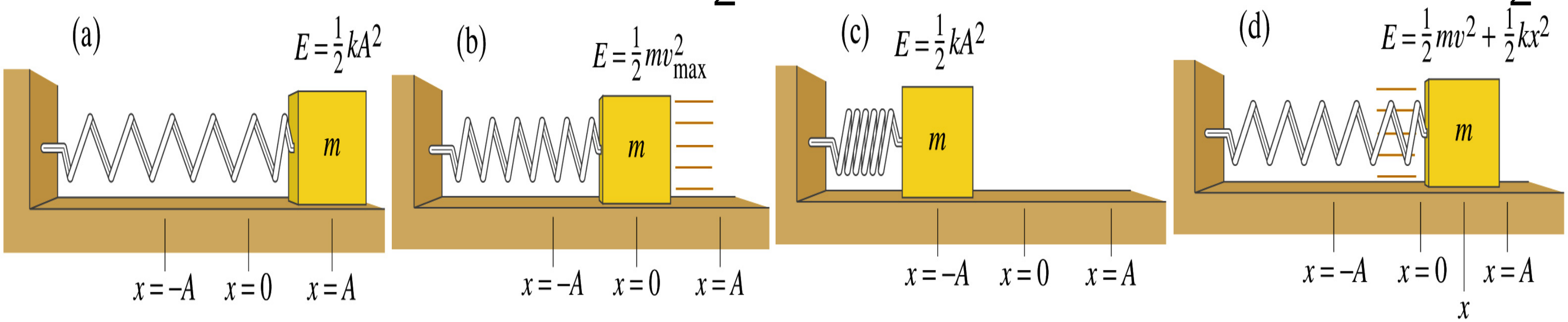
$$PE = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$$

Therefore the total mechanical energy of the harmonic oscillator is

$$E = KE + PE = \frac{1}{2}\left[m\omega^2 A^2 \sin^2(\omega t + \phi) + kA^2 \cos^2(\omega t + \phi)\right]$$

Since $\omega = \sqrt{k/m}$

$$E = KE + PE = \frac{1}{2}\left[kA^2 \sin^2(\omega t + \phi) + kA^2 \cos^2(\omega t + \phi)\right] = \frac{1}{2}kA^2$$



Total mechanical energy of a simple harmonic oscillator is proportional to the square of the amplitude.

Congratulations!!!!

You all have done very well!!!

I certainly had a lot of fun with ya'll!

Good luck with your exams!!!

Happy Holidays!!